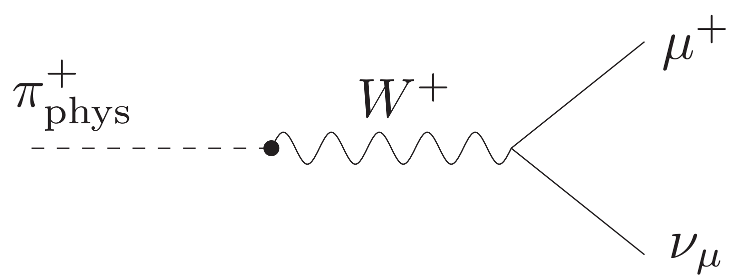
Resolution of the Weinstein paradox

How does the charged pion decay? In Lecture 1, we showed that the Goldstone bosons G^a that are "eaten" by the $W^\pm$ and Z in the Higgs mechanism and the physical pions are orthogonal states, with

$$\langle 0|j^a_\mu|G^b\rangle = i(f_\pi^2+v^2)^{1/2}p_\mu\delta^{ab}\,, \qquad\qquad \langle 0|j^a_\mu|\pi^b\rangle_{\rm phys} = 0\,.$$

However, if you look at old textbooks on the weak interactions, they will insist that the (physical) charged pion decays via



But, the π–W vertex above is proportional to $\langle 0|j^-_\mu|\pi^+\rangle_{\rm phys} = 0$. So how does the charged pion decay? I learned about this paradox from Marvin Weinstein many years ago, so I call this the Weinstein paradox.

The paradox is resolved by noting that the $|\omega^a\rangle$ couple to lepton pairs via the Yukawa interactions,

$$\mathcal{L}_{\text{Yukawa}} = -\frac{\sqrt{2}m_\ell}{v}(\bar{\ell}_R \ell_L \Phi^{0\,*} + \bar{\ell}_R \nu_L \Phi^-) + \text{h.c.}\,,$$

where $\Phi^\pm = \omega^\pm$. However, the ω^a are not quite the true Goldstone bosons of electroweak symmetry breaking, as there is a very small admixture of the physical pion state,

$$|\omega^a\rangle = \frac{1}{\sqrt{f_\pi^2 + v^2}}\left[v|G^a\rangle - f_\pi|\pi^a\rangle_{\text{phys}}\right] \simeq |G^a\rangle - \frac{f_\pi}{v}|\pi^a\rangle_{\text{phys}}\,,$$

in the limit of $v \gg f_\pi$. Plugging this into the Yukawa Lagrangian above [with $\nu_L = \frac{1}{2}(1-\gamma_5)\nu$] and writing $G_F \equiv [v^2\sqrt{2}]^{-1}$ for the Fermi constant yields

$$\mathscr{L}_{\pi_{\text{phys}}\ell^+\nu} = f_\pi G_F m_\ell\,\overline{\ell}(1-\gamma_5)\nu\pi^- + \text{h.c.}\,,$$

which yields the "standard" amplitude for the decay $\pi^+ \to \ell^+\nu$ (including the lepton mass suppression) found in the old textbooks.

Lecture II

Looking Beyond the SM Higgs Boson

Outline

- Hints of the Standard Model demise?
- Effective theory of SM Higgs deviations
- Non-minimal Higgs sectors
- The two-Higgs doublet model (2HDM)
- The approach to the alignment limit

Hints of the Standard Model demise?

What does the discovery of the Higgs boson imply for the Standard Model?

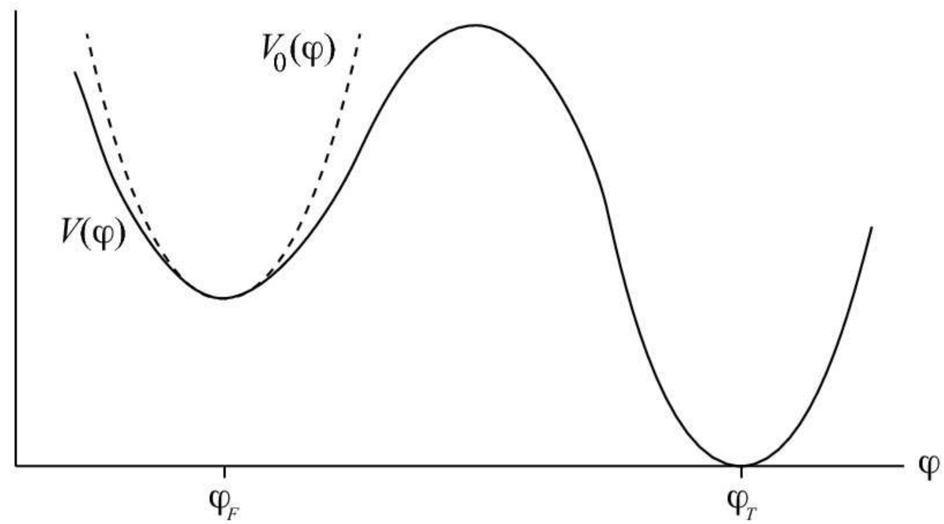
The Standard Model is an effective theory that provides an excellent description of fundamental physics at the electroweak scale.

We know that the Standard Model must break down at the Planck scale, $M_{\rm PL} \sim 10^{19}$ GeV, where the gravitational interactions can no longer be ignored.

But, might the Standard Model persist as a good effective field theory all the way up to the Planck scale?*

*Note: Strictly speaking, neutrinos are massless in the Standard Model. One elegant way to incorporate neutrino masses is the seesaw mechanism, which introduces right handed neutrinos with Majorana masses of order 10^{14} GeV. So, to be more precise, one can ask whether the seesaw-extended Standard Model can persist all the way up to the Planck scale?

The Standard Model of EWSB scalar dynamics was previously analyzed at tree-level. Do we really know that the true minimum (φ_T) of the scalar potential is at $v = 246$ GeV? Maybe this is a false minimum (φ_F).



What about potential minima at very large values of the field, approaching the Planck scale? In this regime, a radiatively-corrected scalar potential could include large logarithmic contributions, e.g. $\ln(\Phi/v)$ for large field values, that can spoil the perturbation analysis. However, such large logarithms can be re-summed using renormalization group (RG) methods.

Without introducing the technical details, let me quote the final result. The one-loop RG-improved scalar potential of the SM is given by

$$V(\Phi) = \mu^2(t)G^2(t)\Phi^\dagger\Phi + \tfrac{1}{2}\lambda(t)G^4(t)(\Phi^\dagger\Phi)^2 \,,$$

with $t = \ln(\Phi^2/M^2)$ and M is an arbitrary scale, typically chosen to be the energy scale of the tree-level potential minimum. The running Higgs self-coupling evolves according to

$$\frac{d\lambda(t)}{dt} = \beta_\lambda[g_i(t), \lambda(t)] \,,$$

where the g_i (which include all relevant gauge and Yukawa couplings) also evolve according to their corresponding β-functions. The evolution of $\mu^2(t)$ can also be determined although it is not of much interest to us here. Finally,

$$G(t) = \exp\left\{ -\int_0^t dt' \, \gamma[g_i(t'), \lambda(t')\right\} \,,$$

where γ is the anomalous dimensions of the scalar field. $G(t)$ is necessarily positive, but its precise value is not critical to this discussion.

Let us return to

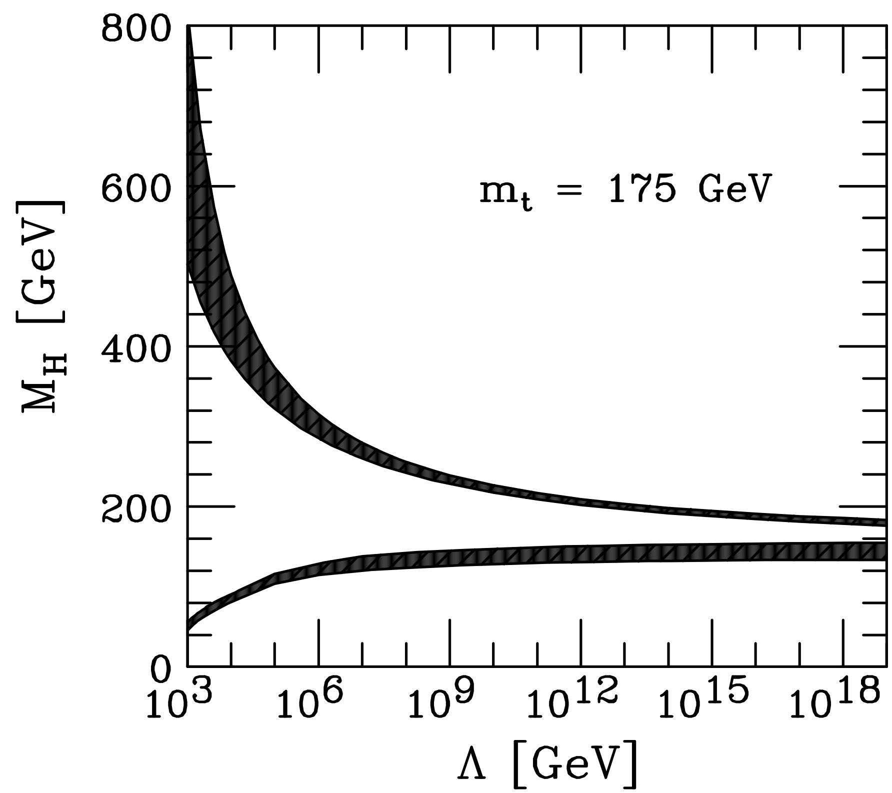
$$V(\Phi) = \mu^2(t)G^2(t)\Phi^\dagger\Phi + \tfrac{1}{2}\lambda(t)G^4(t)(\Phi^\dagger\Phi)^2\,,$$

As t increases, if $\lambda(t)$ is driven to infinity (say at one loop), we say we have encountered a *Landau pole.*

Although the perturbative analysis breaks down once $\lambda \gg 1$, the presence of a Landau pole strongly suggests that the degrees of freedom of the effective theory must be reorganized in some way. The SM does not provide a good description of the physics above this energy scale.

Suppose $\lambda(t)$ is driven negative at some scale. In this case, the minimum of the scalar potential at $v = 246$ GeV is no longer a global minimum.[†] Presumably, this instability is also an indication that the SM does not provide a good description of the physics above this energy scale.

[†]Indeed, the scalar potential could be unbounded from below (bad!) unless the potential turns around again at an even higher energy scale.



Theoretical uncertainties on the lower [Altarelli and Isidori; Casas, Espinosa and Quirós] and upper [Hambye and Riesselmann] Higgs mass bounds as a function of energy scale Λ at which the Standard Model breaks down, assuming $m_t = 175$ GeV and $\alpha_s(m_Z) = 0.118$. The shaded areas above reflect the theoretical uncertainties in the calculations of the Higgs mass bounds.

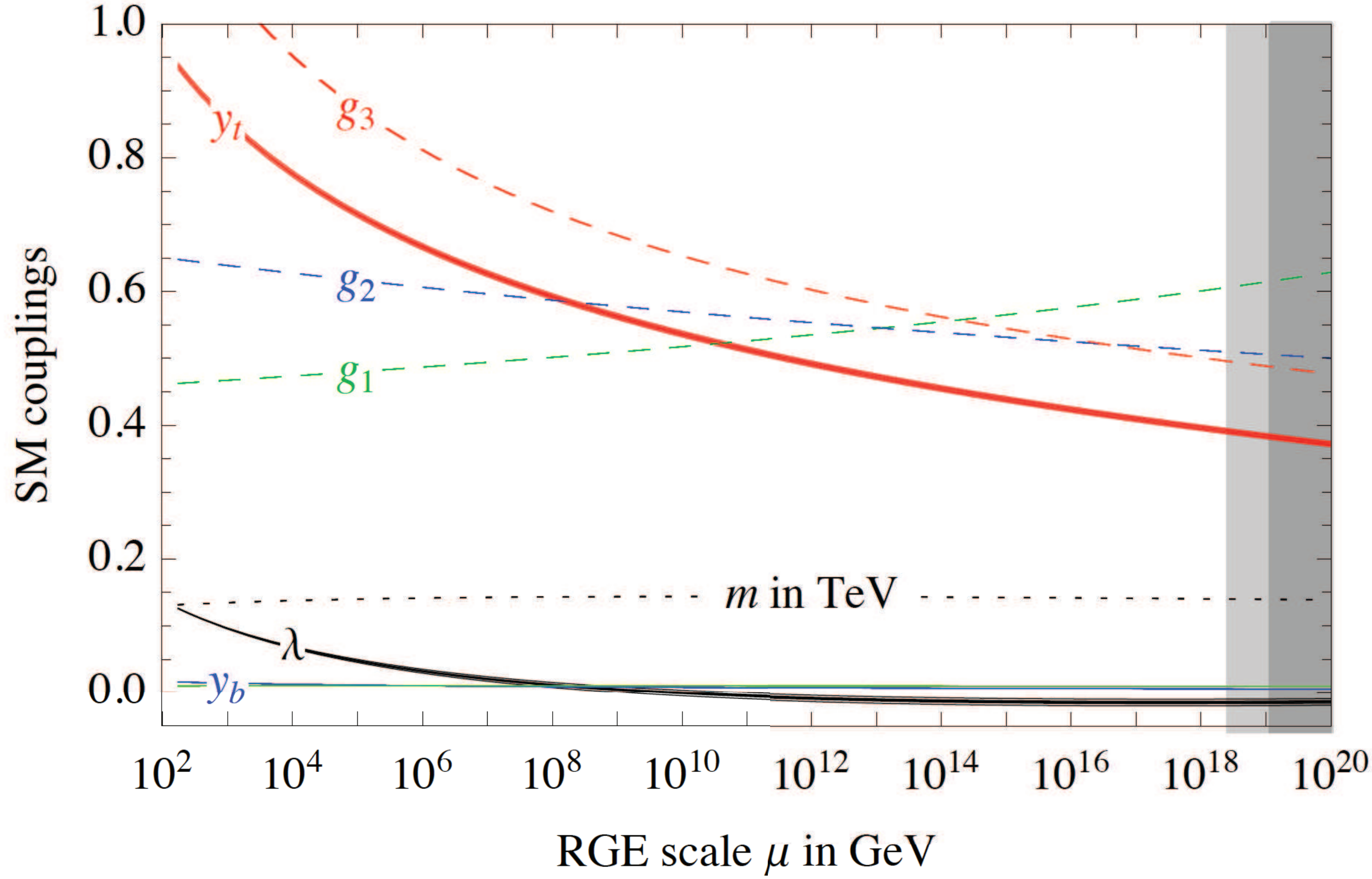
The above plot (nearly 20 years old) implies that for $m_H = 125$ GeV, there is no danger of a Landau pole below the Planck scale. But, we need to worry about the absolute stability of the electroweak vacuum.

Since $m_H^2 = \lambda v^2$, where λ is evaluated at the scale of electroweak symmetry breaking, the value of the Higgs mass provides the low energy boundary condition for the evolution of λ at higher energy scales. Thus, the value of the Higgs mass provides information of the possible demise of the SM at higher energies.

At one-loop, the behavior of $\lambda(t)$ is controlled primarily by:

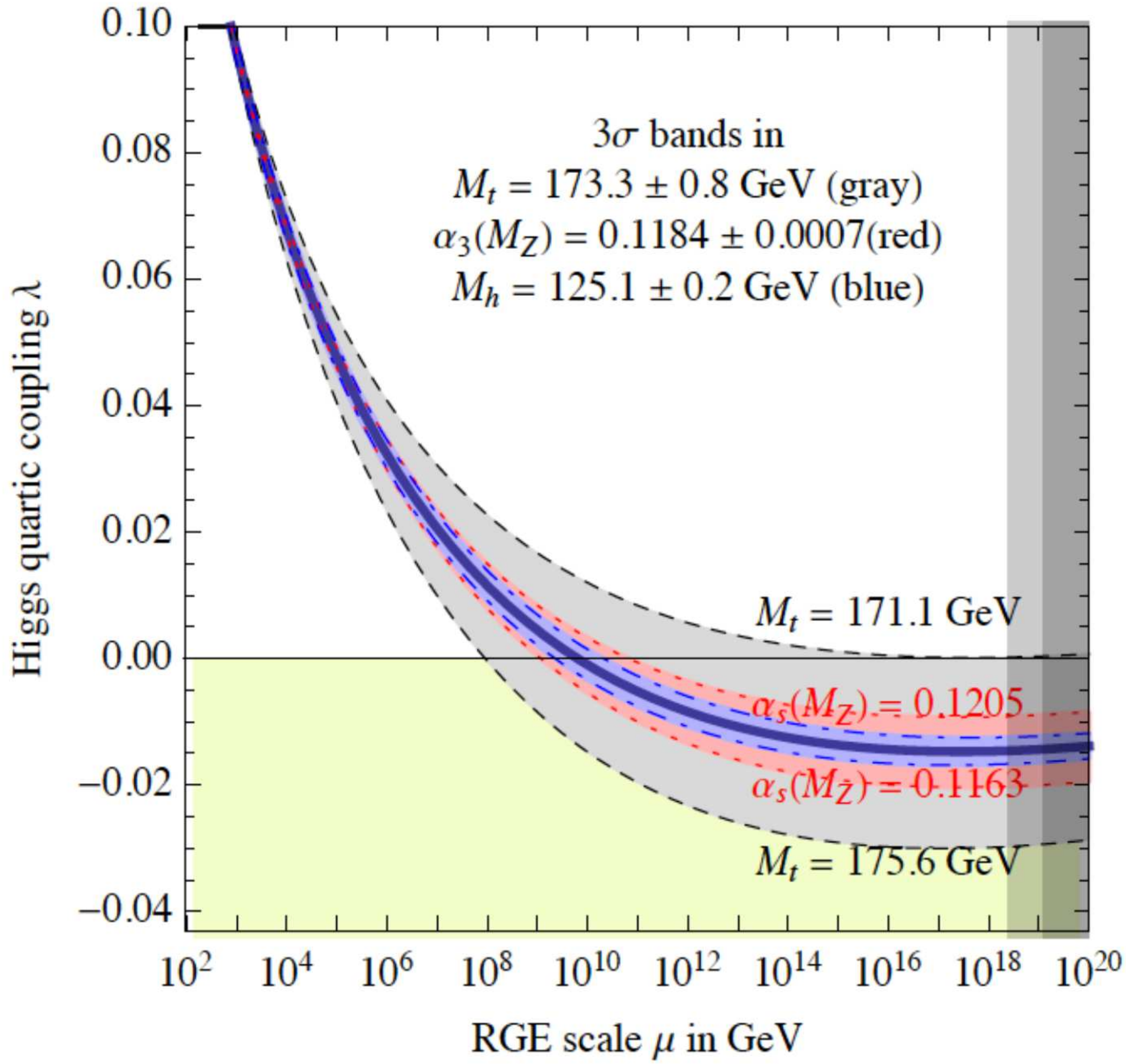
$$
\begin{aligned}
\frac{d\lambda}{dt} &= \frac{3}{16\pi^2}\left\{2\lambda^2 + 2\lambda h_t^2 - 2h_t^4 - \tfrac{1}{2}\lambda(3g^2 + g'^2) + \tfrac{1}{8}\left[2g^4 + (g^2 + g'^2)^2\right]\right\} \\
\frac{dh_t}{dt} &= \frac{1}{16\pi^2}\left[\tfrac{9}{4}h_t^3 - 4g_s^2 h_t - \tfrac{9}{8}g^2 h_t - \tfrac{17}{24}g'^2 h_t\right] , \\
\frac{dg_s^2}{dt} &= -\frac{7g_s^4}{16\pi^2} ,
\end{aligned}
$$

where $\lambda(0) = m_H^2/v$ and $h_t(0) = \sqrt{2}m_t/v$ and the low-energy values of the gauge couplings set the boundary conditions.



Running $\overline{\rm MS}$ couplings of the SM. Three loop RG equations and two-loop thresholds at the electroweak scale are included. Here, the Higgs self-coupling is normalized such that $m_H^2 = 2\lambda v^2$. Taken from D. Buttazzoa et al., JHEP 1312 (2013) 089 [arXiv:1307.3536].

How close does λ get to zero near the Planck scale?



Taken from D. Buttazzoa et al., JHEP 1312 (2013) 089 [arXiv:1307.3536].

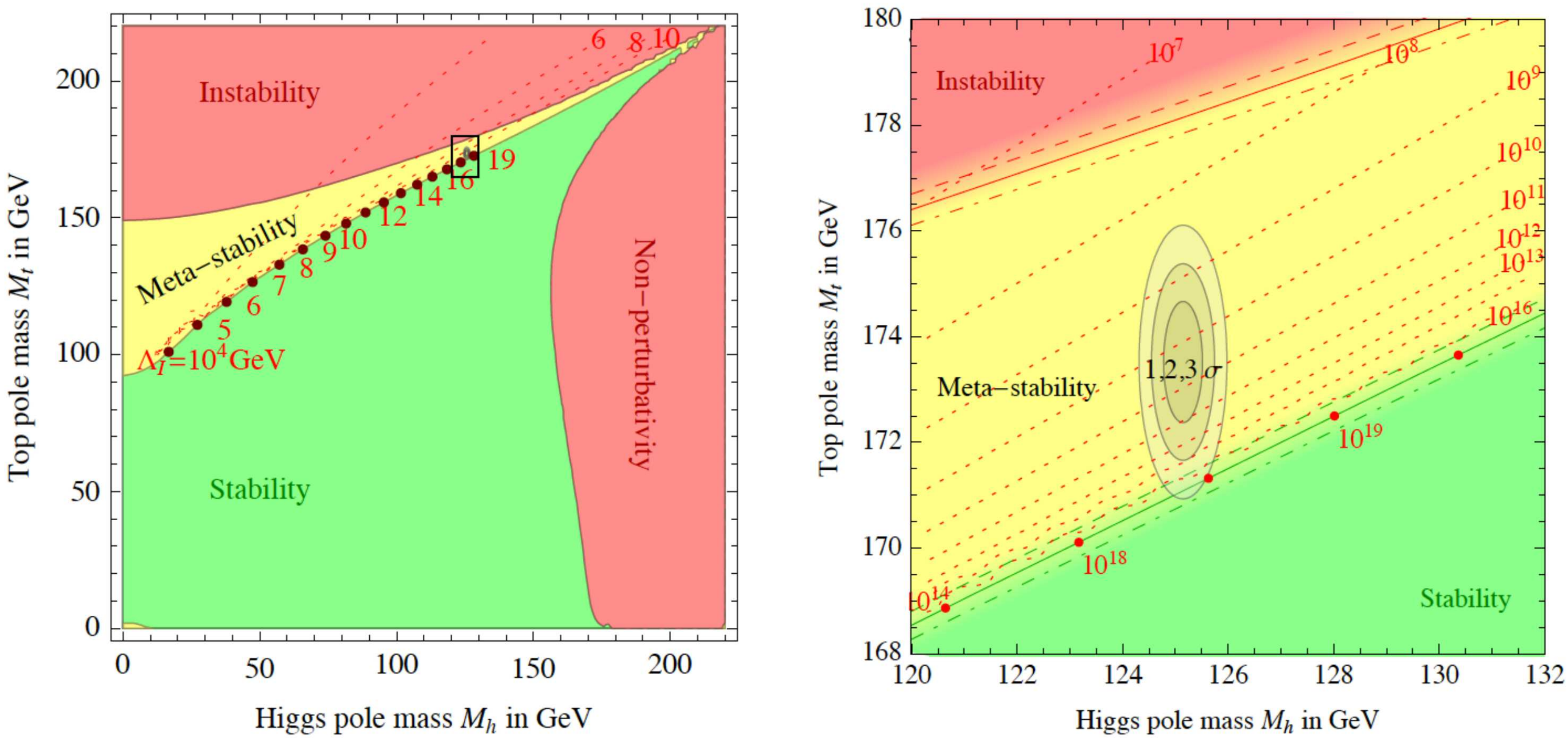
Perhaps one should not worry too much if the electroweak minimum is not absolutely stable (assuming that the true minimum lies at very large field values). Indeed, all we should really impose is that the electroweak vacuum, if metastable, should have a lifetime much longer than the age of the universe. The latter condition would be satisfied if $m_H = 125$ GeV.

If one prefers to impose absolute stability of the SM Higgs potential, then the SM must break down below the Planck scale.[‡]

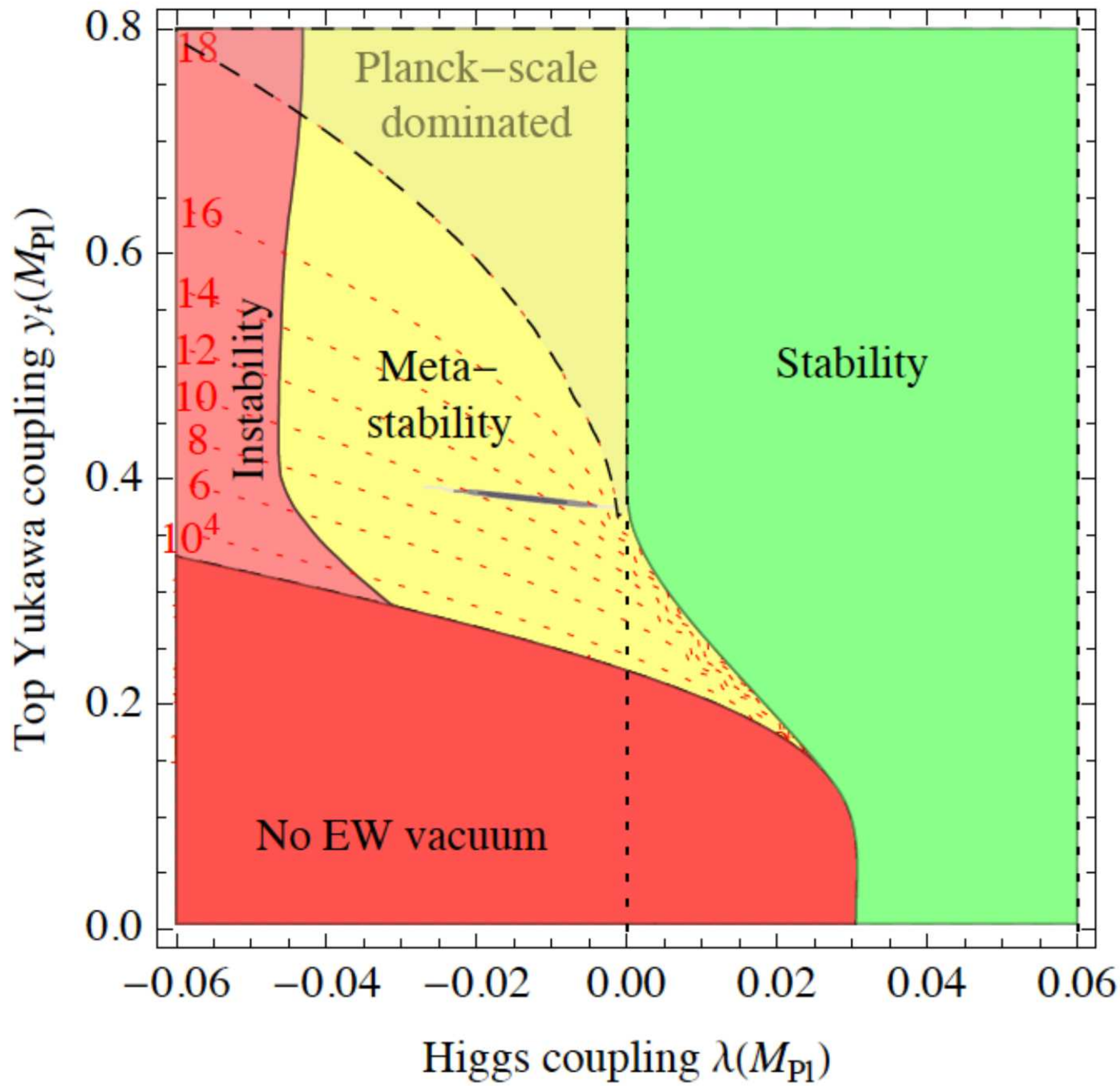
To quote from G. Degrassi et al., arXiv:1205.6497,

> "While [the Higgs self-coupling parameter evaluated] at the Planck scale is remarkably close to zero, absolute stability of the Higgs potential is excluded at 98% CL for $m_H < 126$ GeV."

[‡]There are recent claims in the literature that Planck-scale effects due to physics beyond the SM could have a non-negligible impact on this conclusion.



Taken from D. Buttazzoa et al., JHEP 1312 (2013) 089 [arXiv:1307.3536].



Taken from D. Buttazzoa et al., JHEP 1312 (2013) 089 [arXiv:1307.3536].

New physics beyond the SM can affect the Higgs data

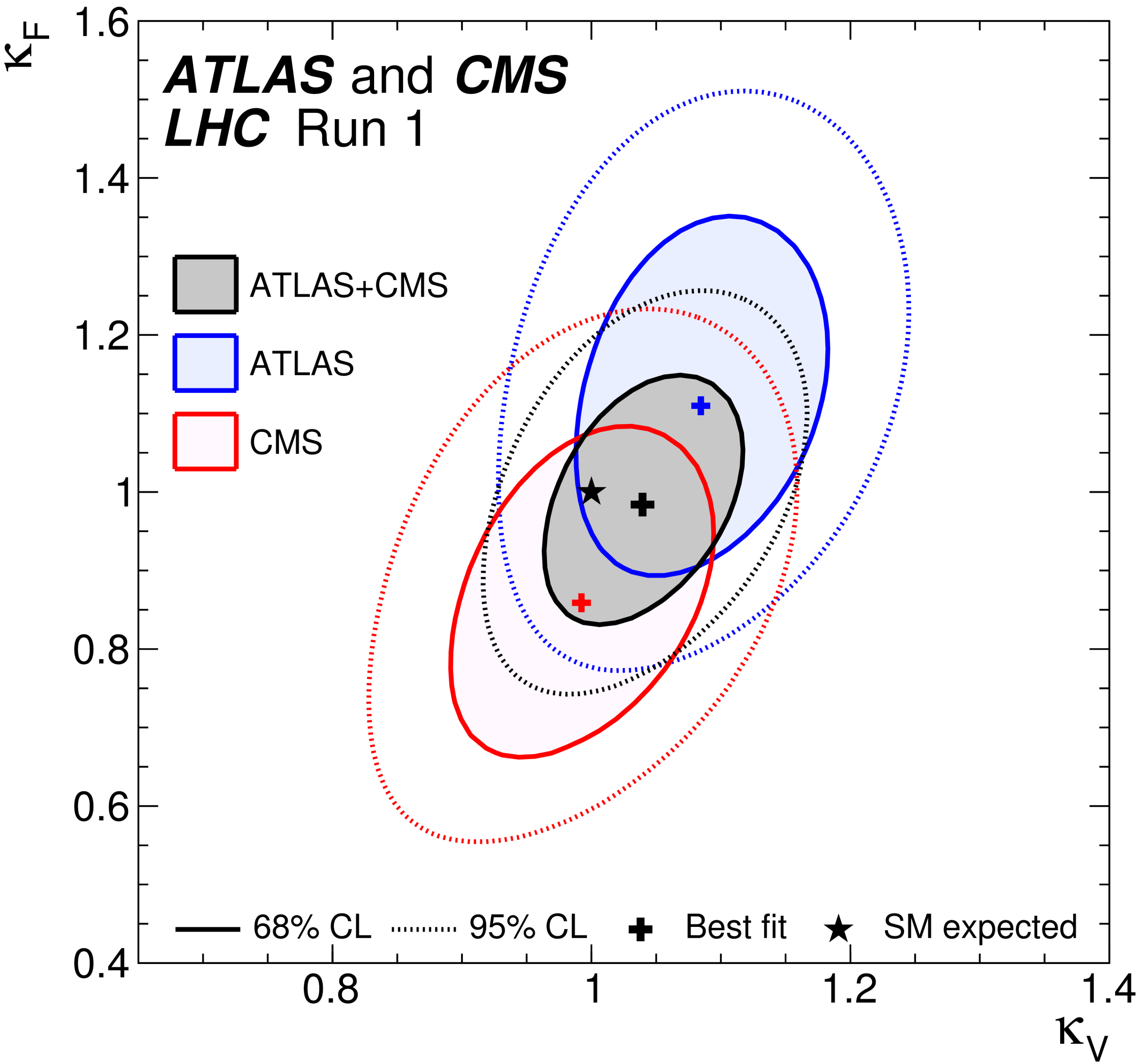
New physics beyond the SM (BSM) may exist near or above the TeV scale. Hints of this BSM physics could be revealed in the Higgs data by generating small deviations in Higgs couplings from their SM values.

The LHC Collaborations employ the so-called κ-framework to parameterize such deviations. Under certain circumstances, the production and decay of the Higgs boson can be factorized, such that $\sigma \cdot \mathrm{B}$ of an individual channel ($i \to H \to f$) can be parameterized as:

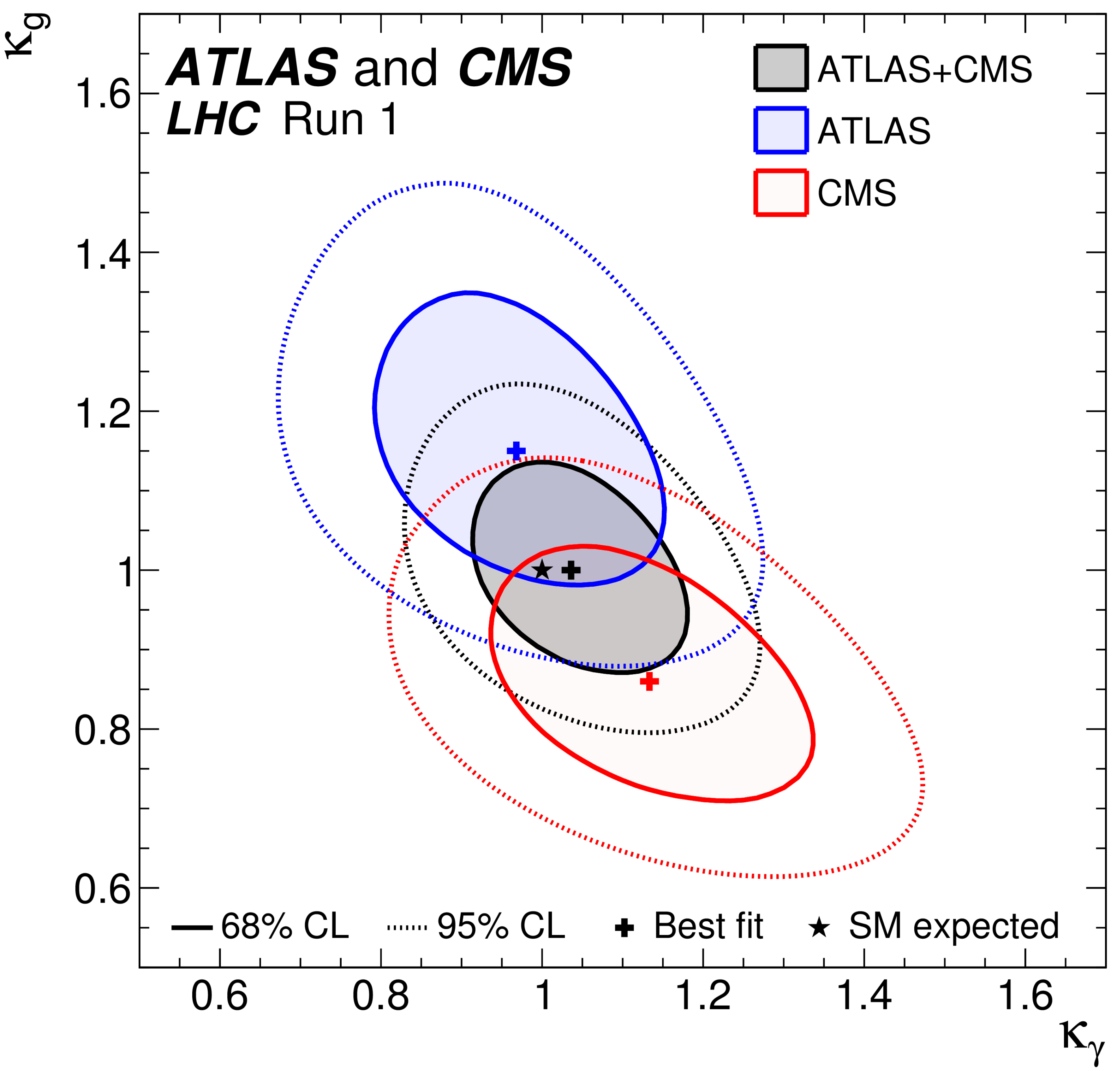
$$\sigma_i \cdot \mathrm{B}^f = \frac{\sigma_i(\kappa) \cdot \Gamma^f(\kappa)}{\Gamma_H},$$

where Γ_H is the total Higgs width, and the coupling modifiers, κ_j, are defined by

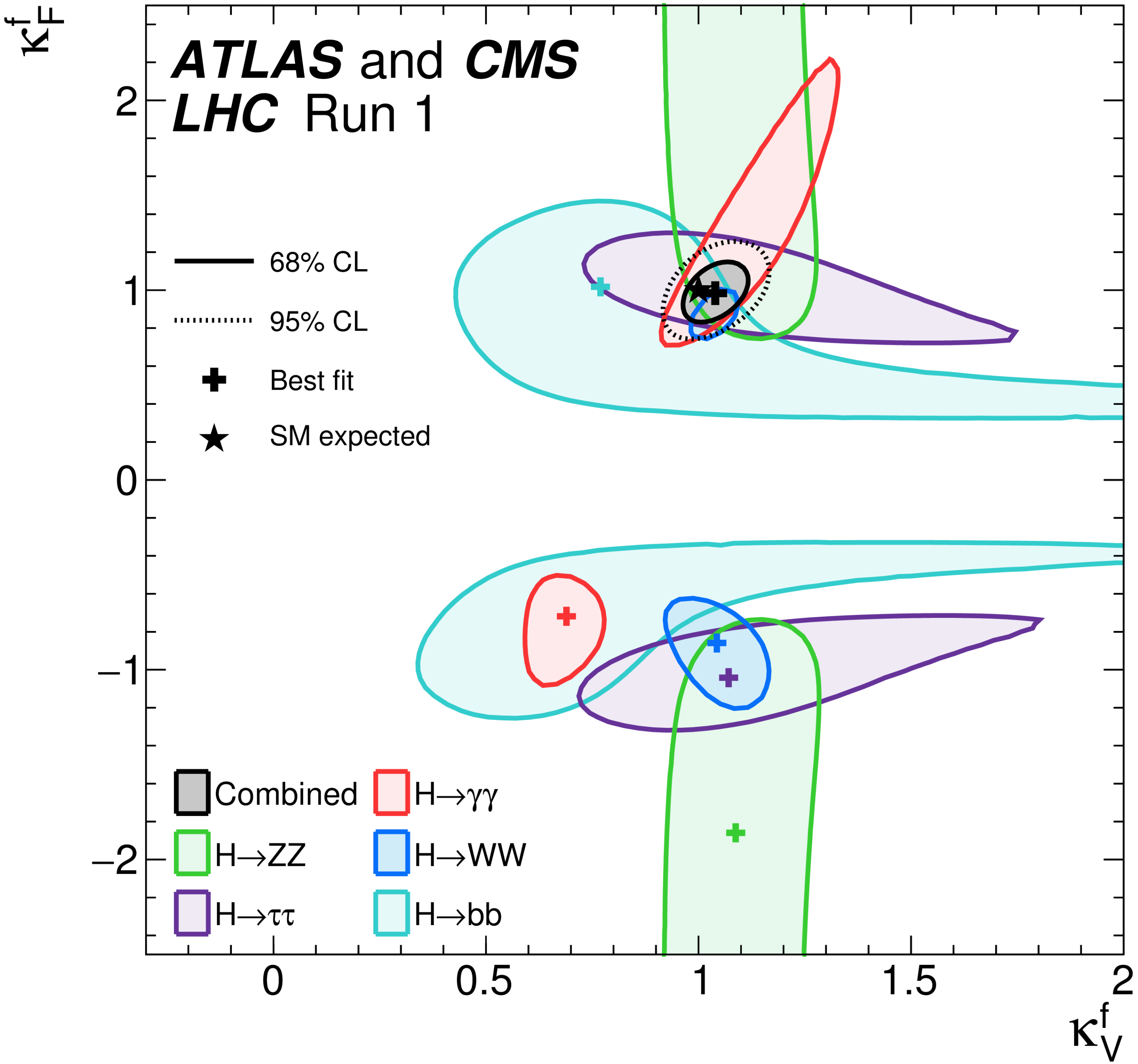
$$\kappa_j^2 = \sigma_j / \sigma_j(\mathrm{SM}), \quad \text{or} \quad \kappa_j^2 = \Gamma^j / \Gamma^j_{\mathrm{SM}}.$$



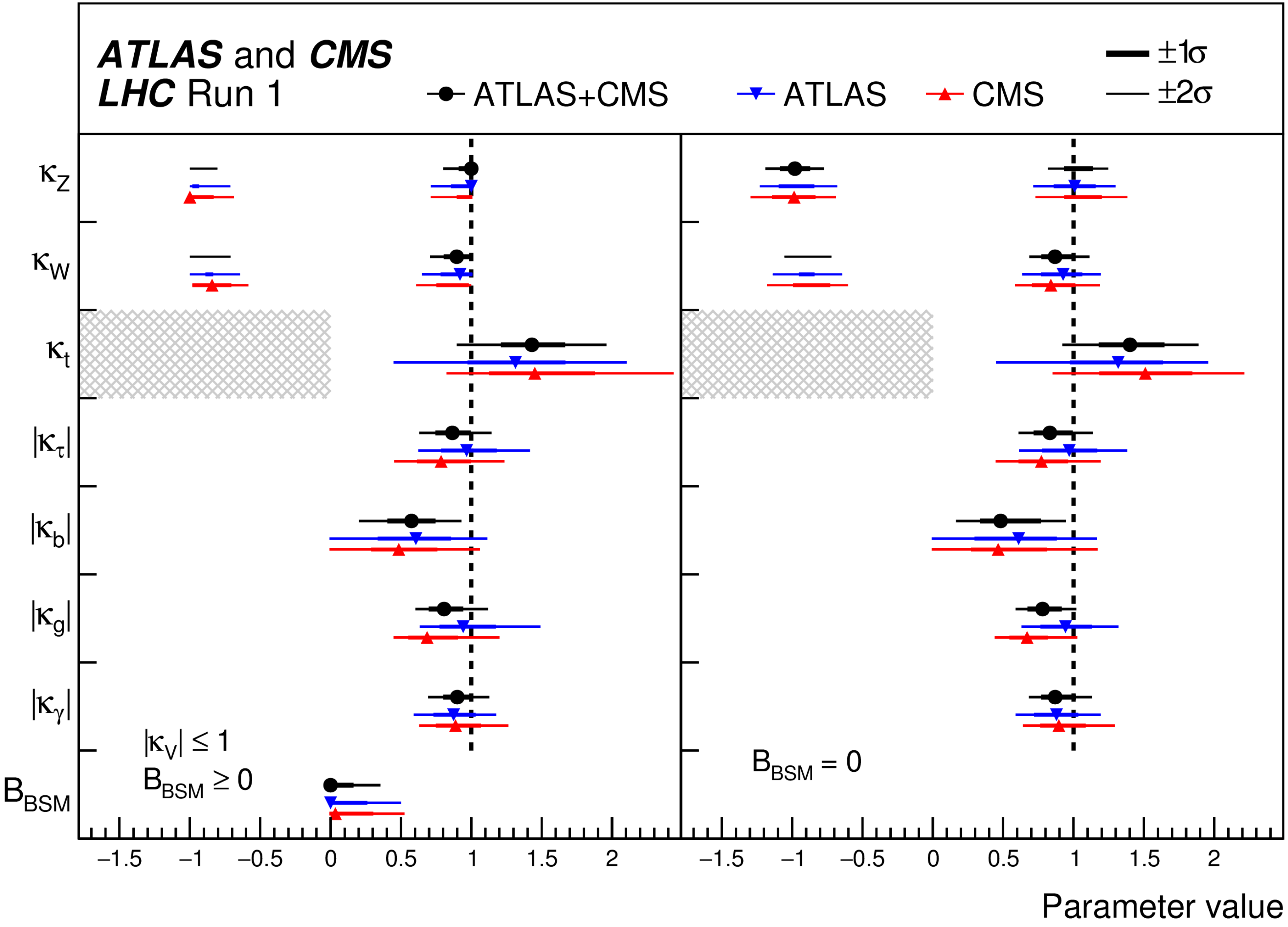
ATLAS and CMS
LHC Run 1
ATLAS+CMS
ATLAS
CMS
68% CL
95% CL
Best fit
SM expected
κ_F
κ_V
1.6
1.4
1.2
1
0.8
0.6
0.4
0.8
1
1.2
1.4



ATLAS and CMS
LHC Run 1
ATLAS+CMS
ATLAS
CMS
68% CL
95% CL
Best fit
SM expected
κ_g
κ_γ
1.6
1.4
1.2
1
0.8
0.6
0.6
0.8
1
1.2
1.4
1.6



ATLAS and CMS
LHC Run 1
68% CL
95% CL
Best fit
SM expected
Combined
H→γγ
H→ZZ
H→WW
H→ττ
H→bb
κ_F^f
κ_V^f



ATLAS and CMS
LHC Run 1
ATLAS+CMS
ATLAS
CMS
±1σ
±2σ
κ_Z
κ_W
κ_t
$|\kappa_\tau|$
$|\kappa_b|$
$|\kappa_g|$
$|\kappa_\gamma|$
B_{BSM}
$|\kappa_V| \leq 1$
$B_{BSM} \geq 0$
$B_{BSM} = 0$
−1.5 −1 −0.5 0 0.5 1 1.5 2
Parameter value

Higgs effective field theory (EFT)

Assuming that the new BSM physics lies at some scale Λ sufficiently above the electroweak scale, one can formally integrate out the unknown physics. The end result is a Lagrangian just involving SM fields of the form,

$$\mathscr{L}_{\rm EFT} = \mathscr{L}_{\rm SM} + \sum_{k=1}^{\infty} \frac{1}{\Lambda^k} \mathscr{L}^{D=4+k} \,,$$

where $\mathscr{L}^D$ is the sum of operators involving SM fields of dimension D. Although not indicated in the notation above, Λ can stand for multiple scales (depending on the operator).

One feature of the SM is that for D odd (and greater than 4), the corresponding operators violate lepton number. For these operators, the relevant scale Λ is very large, and thus too small to be probed at the LHC.

Thus, the first set of non-trivial operators enter at $D = 6$.

A complete classification of the $D = 6$ operators has been achieved. This is not trivial, as there are redundancies and different choices for operator bases. Different bases can be related by using integration by parts, field redefinitions, equations of motion, Fierz transformations, etc.

In total, for the one generation SM, there are 59 independent operators, of which 17 are complex. For the three generation SM, there are 2499 independent operators (although the latter is significantly reduced in theories of minimal flavor violation).

In the analysis of Higgs couplings, many of these operators are not directly relevant. Some of the operator coefficients are highly constrained by the precision electroweak observables. Other coefficients are now becoming constrained in light of the Higgs data.

By analyzing the effective Lagrangian in the unitary gauge, one can make contact with the κ-framework.

Example: Warsaw Basis

H^4D^2 and H^6

O_H	$\left[\partial_\mu (H^\dagger H)\right]^2$
O_T	$\left(H^\dagger \overleftrightarrow{D}_\mu H\right)^2$
O_{6H}	$(H^\dagger H)^3$

f^2H^3

O_e	$-(H^\dagger H - \frac{v^2}{2})\bar{e} H^\dagger \ell$
O_u	$-(H^\dagger H - \frac{v^2}{2})\bar{u} \widetilde{H}^\dagger q$
O_d	$-(H^\dagger H - \frac{v^2}{2})\bar{d} H^\dagger q$

V^3D^3

O_{3G}	$g_s^3 f^{abc} G^a_{\mu\nu} G^b_{\nu\rho} G^c_{\rho\mu}$
$O_{3\widetilde{G}}$	$g_s^3 f^{abc} \widetilde{G}^a_{\mu\nu} G^b_{\nu\rho} G^c_{\rho\mu}$
O_{3W}	$g^3 \epsilon^{ijk} W^i_{\mu\nu} W^j_{\nu\rho} W^k_{\rho\mu}$
$O_{3\widetilde{W}}$	$g^3 \epsilon^{ijk} \widetilde{W}^i_{\mu\nu} W^j_{\nu\rho} W^k_{\rho\mu}$

V^2H^2

O_{GG}	$\frac{g_s^2}{4} H^\dagger H\, G^a_{\mu\nu} G^a_{\mu\nu}$
$O_{\widetilde{GG}}$	$\frac{g_s^2}{4} H^\dagger H\, \widetilde{G}^a_{\mu\nu} G^a_{\mu\nu}$
O_{WW}	$\frac{g^2}{4} H^\dagger H\, W^i_{\mu\nu} W^i_{\mu\nu}$
$O_{\widetilde{WW}}$	$\frac{g^2}{4} H^\dagger H\, \widetilde{W}^i_{\mu\nu} W^i_{\mu\nu}$
O_{BB}	$\frac{g'^2}{4} H^\dagger H\, B_{\mu\nu} B_{\mu\nu}$
$O_{\widetilde{BB}}$	$\frac{g'^2}{4} H^\dagger H\, \widetilde{B}_{\mu\nu} B_{\mu\nu}$
O_{WB}	$g g' H^\dagger \sigma^i H\, W^i_{\mu\nu} B_{\mu\nu}$
$O_{\widetilde{WB}}$	$g g' H^\dagger \sigma^i H\, \widetilde{W}^i_{\mu\nu} B_{\mu\nu}$

f^2H^2D

$O_{H\ell}$	$i\bar{\ell}\gamma_\mu \ell H^\dagger \overleftrightarrow{D}_\mu H$
$O'_{H\ell}$	$i\bar{\ell}\sigma^i \gamma_\mu \ell H^\dagger \sigma^i \overleftrightarrow{D}_\mu H$
O_{He}	$i\bar{e}\gamma_\mu e H^\dagger \overleftrightarrow{D}_\mu H$
O_{Hq}	$i\bar{q}\gamma_\mu q H^\dagger \overleftrightarrow{D}_\mu H$
O'_{Hq}	$i\bar{q}\sigma^i \gamma_\mu q H^\dagger \sigma^i \overleftrightarrow{D}_\mu H$
O_{Hu}	$i\bar{u}\gamma_\mu u H^\dagger \overleftrightarrow{D}_\mu H$
O_{Hd}	$i\bar{d}\gamma_\mu d H^\dagger \overleftrightarrow{D}_\mu H$
O_{Hud}	$i\bar{u}\gamma_\mu d \tilde{H}^\dagger D_\mu H$

f^2VHD

O_{eW}	$g\bar{\ell}\sigma_{\mu\nu} e \sigma^i H W^i_{\mu\nu}$
O_{eB}	$g'\bar{\ell}\sigma_{\mu\nu} e H B_{\mu\nu}$
O_{uG}	$g_s \bar{q}\sigma_{\mu\nu} T^a u \widetilde{H}\, G^a_{\mu\nu}$
O_{uW}	$g\bar{q}\sigma_{\mu\nu} u \sigma^i \widetilde{H}\, W^i_{\mu\nu}$
O_{uB}	$g'\bar{q}\sigma_{\mu\nu} u \widetilde{H}\, B_{\mu\nu}$
O_{dG}	$g_s \bar{q}\sigma_{\mu\nu} T^a d H\, G^a_{\mu\nu}$
O_{dW}	$g\bar{q}\sigma_{\mu\nu} d \sigma^i H\, W^i_{\mu\nu}$
O_{dB}	$g'\bar{q}\sigma_{\mu\nu} d H\, B_{\mu\nu}$

$(\bar{L}L)(\bar{L}L)$ and $(\bar{L}R)(\bar{L}R)$

$O_{\ell\ell}$	$(\bar{\ell}\gamma_\mu \ell)(\bar{\ell}\gamma_\mu \ell)$
O_{qq}	$(\bar{q}\gamma_\mu q)(\bar{q}\gamma_\mu q)$
O'_{qq}	$(\bar{q}\gamma_\mu \sigma^i q)(\bar{q}\gamma_\mu \sigma^i q)$
$O_{\ell q}$	$(\bar{\ell}\gamma_\mu \ell)(\bar{q}\gamma_\mu q)$
$O'_{\ell q}$	$(\bar{\ell}\gamma_\mu \sigma^i \ell)(\bar{q}\gamma_\mu \sigma^i q)$
O_{quqd}	$(\bar{q}^j u)\epsilon_{jk}(\bar{q}^k d)$
O'_{quqd}	$(\bar{q}^j T^a u)\epsilon_{jk}(\bar{q}^k T^a d)$
$O_{\ell equ}$	$(\bar{\ell}^j e)\epsilon_{jk}(\bar{q}^k u)$
$O'_{\ell equ}$	$(\bar{\ell}^j \sigma_{\mu\nu} e)\epsilon_{jk}(\bar{q}^k \sigma^{\mu\nu} u)$
$O_{\ell edq}$	$(\bar{\ell}^j e)(\bar{d} q^j)$

$(\bar{R}R)(\bar{R}R)$

O_{ee}	$(\bar{e}\gamma_\mu e)(\bar{e}\gamma_\mu e)$
O_{uu}	$(\bar{u}\gamma_\mu u)(\bar{u}\gamma_\mu u)$
O_{dd}	$(\bar{d}\gamma_\mu d)(\bar{d}\gamma_\mu d)$
O_{eu}	$(\bar{e}\gamma_\mu e)(\bar{u}\gamma_\mu u)$
O_{ed}	$(\bar{e}\gamma_\mu e)(\bar{d}\gamma_\mu d)$
O_{ud}	$(\bar{u}\gamma_\mu u)(\bar{d}\gamma_\mu d)$
O'_{ud}	$(\bar{u}\gamma_\mu T^a u)(\bar{d}\gamma_\mu T^a d)$

$(\bar{L}L)(\bar{R}R)$

$O_{\ell e}$	$(\bar{\ell}\gamma_\mu \ell)(\bar{e}\gamma_\mu e)$
$O_{\ell u}$	$(\bar{\ell}\gamma_\mu \ell)(\bar{u}\gamma_\mu u)$
$O_{\ell d}$	$(\bar{\ell}\gamma_\mu \ell)(\bar{d}\gamma_\mu d)$
O_{qe}	$(\bar{q}\gamma_\mu q)(\bar{e}\gamma_\mu e)$
O_{qu}	$(\bar{q}\gamma_\mu q)(\bar{u}\gamma_\mu u)$
O'_{qu}	$(\bar{q}\gamma_\mu T^a q)(\bar{u}\gamma_\mu T^a u)$
O_{qd}	$(\bar{q}\gamma_\mu q)(\bar{d}\gamma_\mu d)$
O'_{qd}	$(\bar{q}\gamma_\mu T^a q)(\bar{d}\gamma_\mu T^a d)$

Grządkowski et al.
1008.4884

59 different kinds of operators, of which 17 are complex

2499 distinct operators, including flavor structure and CP conjugates

Alonso et al 1312.2014

This slide is taken from a talk by Adam Falkowski given at the GGI in Florence in September, 2015

Extended Higgs sectors: 2HDM and beyond

It is possible that new BSM physics is lurking below the TeV scale, and thus cannot be integrated out and incorporated into an effective field theory.

Example: a theory with an enlarged Higgs sector. In theories of this type, new scalar states exist that can mix with the SM Higgs boson. Thus, the observed Higgs boson can contain an admixture of new scalar states leading to deviations in the observed couplings of the Higgs boson to SM particles.

Motivations for an extended Higgs sector

- The fermion sector of the SM is not of minimal form ("Who ordered that?"). So, why should the scalar sector be minimal?
- Adding new scalar states can alleviate the metastability of the vacuum, allowing the SM to be extended all the way up to the Planck scale.
- Models of BSM physics often require additional scalar Higgs states. The MSSM is a famous example of this.

For an arbitrary Higgs sector, the tree-level ρ-parameter is given by

$$\rho_0 \equiv \frac{m_W^2}{m_Z^2 \cos^2\theta_W} = 1 \qquad \Longleftrightarrow \qquad (2T+1)^2 - 3Y^2 = 1\,,$$

independently of the Higgs vacuum expection values (vevs), where T and Y specify the weak-isospin and the hypercharge of the Higgs representation to which it belongs. Y is normalized such that the electric charge of the scalar field is $Q = T_3 + Y/2$. The simplest solutions are Higgs singlets $(T,Y) = (0,0)$ and hypercharge-one complex Higgs doublets $(T,Y) = (\frac{1}{2},1)$.

Thus, we shall consider non-minimal Higgs sectors consisting of multiple Higgs doublets (and perhaps Higgs singlets), but no higher Higgs representations, to avoid the fine-tuning of Higgs vevs.

Higgs boson phenomena beyond the SM

The two-Higgs-doublet model (2HDM) consists of two hypercharge-one scalar doublets. Of the eight initial degrees of freedom, three correspond to the Goldstone bosons and five are physical: a charged Higgs pair, $H^{\pm}$ and three neutral scalars.

In contrast to the SM, whereas the Higgs-sector is CP-conserving, the 2HDM allows for Higgs-mediated CP-violation. If CP is conserved, the Higgs spectrum contains two CP-even scalars, h^0 and H^0 and a CP-odd scalar A^0. Thus, new features of the extended Higgs sector include:

- Charged Higgs bosons
- A CP-odd Higgs boson (if CP is conserved in the Higgs sector)
- Higgs-mediated CP-violation (and neutral Higgs states of indefinite CP)

More exotic Higgs sectors allow for doubly-charged Higgs bosons, etc.

The general 2HDM

Consider the most general 2HDM potential,

$$\begin{aligned}\mathcal{V} &= m_{11}^2\Phi_1^\dagger\Phi_1 + m_{22}^2\Phi_2^\dagger\Phi_2 - [m_{12}^2\Phi_1^\dagger\Phi_2 + \text{h.c.}] + \tfrac{1}{2}\lambda_1(\Phi_1^\dagger\Phi_1)^2 \\ &\quad +\tfrac{1}{2}\lambda_2(\Phi_2^\dagger\Phi_2)^2 + \lambda_3(\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2) + \lambda_4(\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1) \\ &\quad + \left\{\tfrac{1}{2}\lambda_5(\Phi_1^\dagger\Phi_2)^2 + \left[\lambda_6(\Phi_1^\dagger\Phi_1) + \lambda_7(\Phi_2^\dagger\Phi_2)\right]\Phi_1^\dagger\Phi_2 + \text{h.c.}\right\}\,.\end{aligned}$$

After minimizing the scalar potential, assume that $\langle\Phi_i^0\rangle = v_i/\sqrt{2}$, such that $v^2 \equiv |v_1|^2 + |v_2|^2 = (246\ \text{GeV})^2$, and $\tan\beta \equiv |v_2|/|v_1|$. Define new linear combinations of the Higgs doublet fields (the so-called *Higgs basis*):

$$H_1 = \begin{pmatrix} H_1^+ \\ H_1^0 \end{pmatrix} \equiv \frac{v_1^*\Phi_1 + v_2^*\Phi_2}{v}\,, \qquad H_2 = \begin{pmatrix} H_2^+ \\ H_2^0 \end{pmatrix} \equiv \frac{-v_2\Phi_1 + v_1\Phi_2}{v}\,,$$

such that $\langle H_1^0\rangle = v/\sqrt{2}$ and $\langle H_2^0\rangle = 0$. The Higgs basis is uniquely defined up to an overall rephasing, $H_2 \to e^{i\chi}H_2$.

In the Higgs basis, the scalar potential is given by:

$$
\begin{aligned}
\mathcal{V} = Y_1 H_1^\dagger H_1 + Y_2 H_2^\dagger H_2 + [Y_3 H_1^\dagger H_2 + \mathrm{h.c.}] + \tfrac{1}{2} Z_1 (H_1^\dagger H_1)^2 \\
+ \tfrac{1}{2} Z_2 (H_2^\dagger H_2)^2 + Z_3 (H_1^\dagger H_1)(H_2^\dagger H_2) + Z_4 (H_1^\dagger H_2)(H_2^\dagger H_1) \\
+ \left\{ \tfrac{1}{2} Z_5 (H_1^\dagger H_2)^2 + \left[Z_6 (H_1^\dagger H_1) + Z_7 (H_2^\dagger H_2) \right] H_1^\dagger H_2 + \mathrm{h.c.} \right\} ,
\end{aligned}
$$

where Y_1, Y_2 and $Z_1, \ldots, Z_4$ are real and uniquely defined, whereas Y_3, Z_5, Z_6 and Z_7 are complex and transform under the rephasing of H_2,

$$
[Y_3, Z_6, Z_7] \to e^{-i\chi} [Y_3, Z_6, Z_7] \quad \text{and} \quad Z_5 \to e^{-2i\chi} Z_5 \,.
$$

Physical observables must be independent of χ.

After minimizing the scalar potential,

$$
Y_1 = -\tfrac{1}{2} Z_1 v^2 \,, \qquad\qquad Y_3 = -\tfrac{1}{2} Z_6 v^2 \,.
$$

This leaves 11 free parameters: 1 vev, 8 real parameters, Y_2, $Z_{1,2,3,4}$, $|Z_{5,6,7}|$, and two relative phases.

The three physical neutral Higgs boson mass-eigenstates are determined by diagonalizing a 3×3 real symmetric squared-mass matrix that is defined in the Higgs basis.[§] The diagonalizing matrix is a 3×3 real orthogonal matrix that depends on three angles: θ_{12}, θ_{13} and θ_{23}. Under the rephasing of H_2,

$$\theta_{12}\,,\ \theta_{13} \text{ are invariant, and} \quad \theta_{23} \to \theta_{23} - \chi\,.$$

It is convenient to define the $q_{k\ell}$ which are defined in terms of the invariant angles θ_{12} and θ_{13}, where $c_{ij} \equiv \cos\theta_{ij}$ and $s_{ij} \equiv \sin\theta_{ij}$.

k	q_{k1}	q_{k2}
0	i	0
1	$c_{12}c_{13}$	$-s_{12} - ic_{12}s_{13}$
2	$s_{12}c_{13}$	$c_{12} - is_{12}s_{13}$
3	s_{13}	ic_{13}

[§]For details, see H.E. Haber and D. O'Neil, "Basis-independent methods for the two-Higgs-doublet model. II: The significance of $\tan\beta$," *Phys. Rev.* **D74**, 015018 (2006) [hep-ph/0602242].

The Higgs mass eigenstates consist of charged Higgs states $H^{\pm} = e^{\pm i\theta_{23}} H_2^{\pm}$ and the neutral Higgs states $(h_{1,2,3})$:

$$h_k = \frac{1}{\sqrt{2}} \left\{ q_{k1}^* \left(H_1^0 - \frac{v}{\sqrt{2}} \right) + q_{k2}^* H_2^0 e^{i\theta_{23}} + \text{h.c.} \right\} .$$

The Higgs-quark interactions in the Higgs basis is given by:

$$\begin{aligned} -\mathscr{L}_{\mathrm{Y}} = {} & \overline{U}_L (\kappa^U H_1^{0\,\dagger} + \rho^U H_2^{0\,\dagger}) U_R - \overline{D}_L K^\dagger (\kappa^U H_1^- + \rho^U H_2^-) U_R \\ & + \overline{U}_L K (\kappa^{D\,\dagger} H_1^+ + \rho^{D\,\dagger} H_2^+) D_R + \overline{D}_L (\kappa^{D\,\dagger} H_1^0 + \rho^{D\,\dagger} H_2^0) D_R + \text{h.c.}\,, \end{aligned}$$

where K is the CKM matrix. After defining quark mass-eigenstate fields as before, we identify the $\kappa^Q = \sqrt{2} M_Q / v$ $(Q = U, D)$, where the M_Q are the (real non-negative) diagonal quark mass matrices .

In contrast, the matrices ρ^Q $(Q = U, D)$ are independent complex 3×3 matrices. Under the rephasing of H_2, the κ^Q are invariant and $\rho^Q \to e^{i\chi} \rho^Q$.

The Yukawa couplings of the mass-eigenstate Higgs bosons to the quarks are

$$-\mathscr{L}_Y = \frac{1}{v}\overline{D}\sum_k \left\{ M_D(q_{k1}P_R + q_{k1}^* P_L) + \frac{v}{\sqrt{2}}\left[q_{k2}\,[e^{i\theta_{23}}\rho^D]^\dagger P_R + q_{k2}^*\, e^{i\theta_{23}}\rho^D P_L\right]\right\} D h_k$$

$$+\frac{1}{v}\overline{U}\sum_k \left\{ M_U(q_{k1}P_L + q_{k1}^* P_R) + \frac{v}{\sqrt{2}}\left[q_{k2}^*\, e^{i\theta_{23}}\rho^U P_R + q_{k2}\,[e^{i\theta_{23}}\rho^U]^\dagger P_L\right]\right\} U h_k$$

$$+\left\{\overline{U}\left[K[e^{i\theta_{23}}\rho^D]^\dagger P_R - [e^{i\theta_{23}}\rho^U]^\dagger K P_L\right] D H^+ + \frac{\sqrt{2}}{v}\overline{U}\,[K M_D P_R - M_U K P_L]\, D G^+ + \text{h.c.}\right\}$$

where $P_{L,R} = \frac{1}{2}(1 \mp \gamma_5)$.

Remarks:

- The factors of $e^{i\theta_{23}}$ guarantee that the interactions above are invariant under the rephasing of H_2. Note that no $\tan\beta$ parameter appears!

- The general 2HDM exhibits in general flavor changing neutral currents (FCNCs) and new sources of CP-violation via the ρ^Q. The FCNCs are especially problematic given that such effects must be extremely suppressed to avoid conflict with experimental data.

Type I and II Higgs-quark Yukawa couplings in the 2HDM

In a generic basis, the 2HDM Higgs-quark Yukawa Lagrangian is:

$$-\mathscr{L}_{\rm Y} = \overline{U}_L \Phi_a^{0\,*} h_a^U U_R - \overline{D}_L K^\dagger \Phi_a^- h_a^U U_R + \overline{U}_L K \Phi_a^+ h_a^{D\,\dagger} D_R + \overline{D}_L \Phi_a^0 h_a^{D\,\dagger} D_R + {\rm h.c.}\,,$$

where K is the CKM mixing matrix, and there is an implicit sum over $a = 1, 2$. The $h^{U,D}$ are 3×3 Yukawa coupling matrices.

As previously noted, tree-level Higgs-mediated flavor-changing neutral currents (FCNCs) and CP-violating neutral Higgs-quark couplings are both present in the most general model. Both can be avoided by imposing a discrete symmetry to restrict the structure of the Higgs-quark Yukawa Lagrangian.

Two types of Higgs-quark Yukawa couplings can be implemented by a discrete symmetry as shown in the table below.

	Φ_1	Φ_2	U_R	D_R	Q_L,
Type I	$+$	$-$	$-$	$-$	$+$
Type II (MSSM like)	$+$	$-$	$-$	$+$	$+$

Different choices for the discrete symmetry yield:[¶]

- Type-I Yukawa couplings: $h_1^U = h_1^D = 0$,
- Type-II Yukawa couplings: $h_1^U = h_2^D = 0$,

The imposition of the discrete symmetry also restricts the form of the Higgs scalar potential by setting $m_{12}^2 = \lambda_6 = \lambda_7 = 0$. In this case, one can always rephase Φ_1 such that λ_5 is real, in which case the scalar potential is CP-conserving. Moreover, assuming that a U(1)$_{\rm EM}$-conserving potential minimum exists, the corresponding vacuum is CP-conserving, corresponding to real vacuum expectation values, $\langle\Phi_i^0\rangle \equiv v_i/\sqrt{2}$. Thus, the parameter

$$\tan\beta \equiv \frac{v_2}{v_1}\,,$$

is now meaningful since it refers to vacuum expectation values with respect to the basis of scalar fields where the discrete symmetry has been imposed.

Remark: One can generalize slightly by allowing for the $\mathbb{Z}_2$ to be broken softly by taking $m_{12}^2 \neq 0$. In this case, an additional source of CP-violation will be present if $\mathrm{Im}(\lambda_5^*[m_{12}^2]^2) \neq 0$. Nevertheless, Higgs-mediated FCNC effects remain suppressed.

[¶] The Higgs couplings to leptons can also follow either the Type-I or II coupling pattern, which yields a total of four possible models of Higgs-fermion interactions.

The alignment limit in the Type-I and Type II 2HDM

For simplicity, we assume that the scalar potential and vacuum are CP-conserving. In this case, one can rephase the Higgs basis field H_2 such that all scalar potential parameters in the Higgs basis are real.

Working out the squared-mass matrix of the three neutral scalars, one identifies the CP-odd Higgs boson $A = \sqrt{2}\,\mathrm{Im}\ H_2^0$ with squared mass, $m_A^2 = Y_2 + \frac{1}{2}(Z_3 + Z_4 - Z_5)v^2$. The CP-even Higgs mass eigenstates are obtained by diagonalizing the squared-mass matrix, expressed with respect to interaction eigenstates, $\{\sqrt{2}\,\mathrm{Re}\ H_1^0 - v\,, \sqrt{2}\,\mathrm{Re}\ H_2^0\}$,

$$\mathcal{M}_H^2 = \begin{pmatrix} Z_1 v^2 & Z_6 v^2 \\ Z_6 v^2 & m_A^2 + Z_5 v^2 \end{pmatrix} ,$$

after eliminating Y_2 in favor of m_A^2.

To diagonalize $\mathcal{M}_H^2$, we define the CP-even mass eigenstates, h and H (with $m_h \leq m_H$) by

$$\begin{pmatrix} H \\ h \end{pmatrix} = \begin{pmatrix} c_{\beta-\alpha} & -s_{\beta-\alpha} \\ s_{\beta-\alpha} & c_{\beta-\alpha} \end{pmatrix} \begin{pmatrix} \sqrt{2}\,\mathrm{Re}\,H_1^0 - v \\ \sqrt{2}\,\mathrm{Re}\,H_2^0 \end{pmatrix},$$

where $c_{\beta-\alpha} \equiv \cos(\beta-\alpha)$ and $s_{\beta-\alpha} \equiv \sin(\beta-\alpha)$ are defined in terms of the mixing angle α that diagonalizes the CP-even Higgs squared-mass matrix when expressed in the original basis of scalar fields, $\{\sqrt{2}\,\mathrm{Re}\,\Phi_1^0 - v_1\,,\ \sqrt{2}\,\mathrm{Re}\,\Phi_2^0 - v_2\}$.

Recall that in the Higgs basis, the direction of the vev in field space lies along the field H_1^0. If $\sqrt{2}\,\mathrm{Re}\,H_1^0 - v$ were a mass-eigenstate, its properties would coincide with the SM Higgs boson. The *alignment limit* refers to the limit in which one of the neutral Higgs mass eigenstates is exactly aligned in the direction of the vev.

Current LHC data imply that the observed Higgs boson is SM-like. Thus, we anticipate that if the 2HDM is realized in nature, its parameters must be consistent with an approximate alignment limit.

The alignment limit in the general 2HDM

In the general 2HDM, the scalar potential is generically CP-violating. In this case, the neutral Higgs mass-eigenstates are linear combinations of $\{\sqrt{2}\,\mathrm{Re}\,H_1^0-v\,,\,\sqrt{2}\,\mathrm{Re}\,H_2^0\,,\,\sqrt{2}\,\mathrm{Im}\,H_2^0\}$, which are determined by diagonalizing the 3×3 real symmetric squared-mass matrix

$$\mathcal{M}^2=v^2\begin{pmatrix} Z_1 & \mathrm{Re}(Z_6) & -\mathrm{Im}(Z_6) \\ \mathrm{Re}(Z_6) & \frac{1}{2}Z_{345}+Y_2/v^2 & -\frac{1}{2}\mathrm{Im}(Z_5) \\ -\mathrm{Im}(Z_6) & -\frac{1}{2}\mathrm{Im}(Z_5) & \frac{1}{2}Z_{345}-\mathrm{Re}(Z_5)+Y_2/v^2\end{pmatrix},$$

where $Z_{345}\equiv Z_3+Z_4+\mathrm{Re}(Z_5)$. The diagonalizing matrix is a 3×3 real orthogonal matrix that depends on three angles, θ_{12}, θ_{13} and θ_{23}, previously introduced. The corresponding neutral Higgs mass eigenstates will be denoted by h_1, h_2 and h_3 with masses m_1, m_2 and m_3, respectively.

The alignment limit again corresponds to two cases:

1. $Y_2\gg v^2$, corresponding to the decoupling limit.
2. $|Z_6|\ll 1$, corresponding to the alignment limit without decoupling.

The alignment limit can be realized in two different ways. Examining the structure of

$$\mathcal{M}_H^2 = \begin{pmatrix} Z_1 v^2 & Z_6 v^2 \\ Z_6 v^2 & m_A^2 + Z_5 v^2 \end{pmatrix},$$

we see that $\sqrt{2}\,\mathrm{Re}\,H_1^0 - v$ is an approximate mass-eigenstate if either

1. $m_A^2 \gg (Z_1 - Z_5)v^2$. This is the so-called *decoupling limit*. In this case, h is SM-like and $m_A \sim m_H \sim m_{H^\pm} \gg m_h$. In this case, $|c_{\beta-\alpha}| \ll 1$.

2. $|Z_6| \ll 1$. In this case either $|c_{\beta-\alpha}| \ll 1$ (if h is SM-like) or $|s_{\beta-\alpha}| \ll 1$ (if H is SM-like).

It is straightforward to derive

$$s_{\beta-\alpha} c_{\beta-\alpha} = -\frac{Z_6 v^2}{m_H^2 - m_h^2},$$

which exhibits the features of the alignment limit described above.

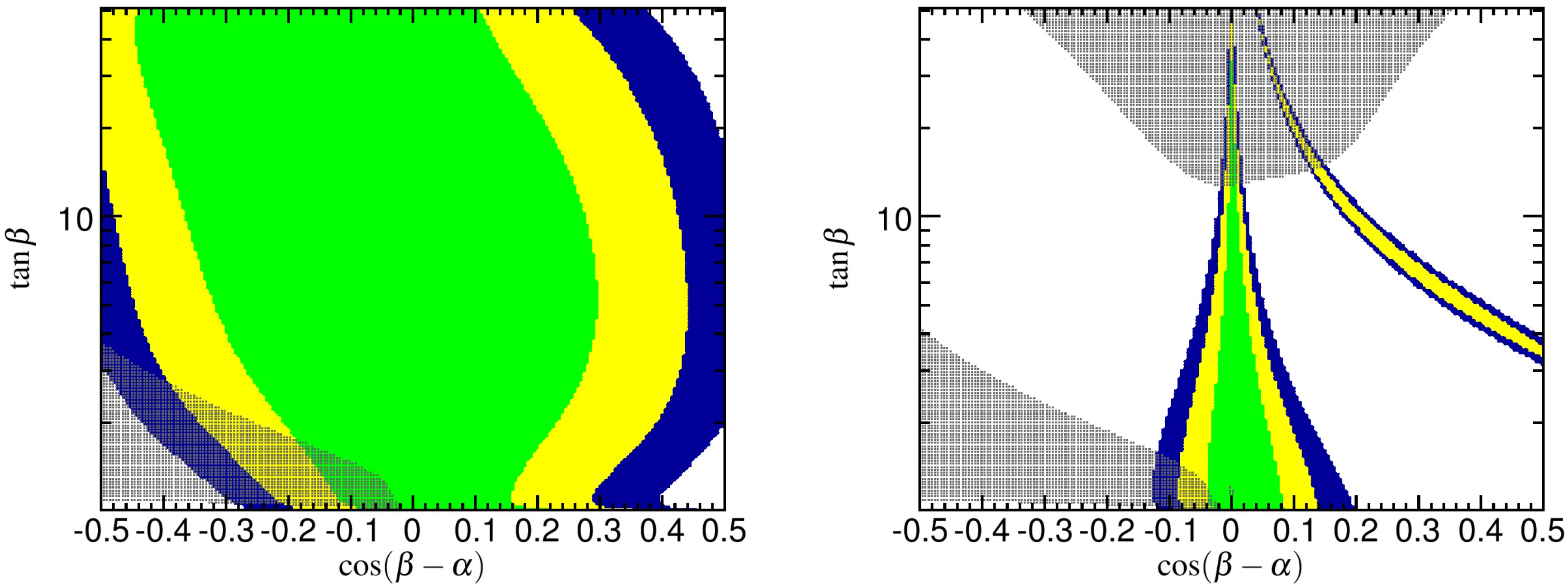
Higgs interaction	2HDM coupling	alignment limit
hVV	$s_{\beta-\alpha}$	$1-\frac{1}{2}c^2_{\beta-\alpha}$
hhh	*	$1+2(Z_6/Z_1)c_{\beta-\alpha}$
hH^+H^-	*	$\frac{1}{3}\left[(Z_3/Z_1)+(Z_7/Z_1)c_{\beta-\alpha}\right]$
$hhhh$	*	$1+3(Z_6/Z_1)c_{\beta-\alpha}$
$h\overline{D}D$	$s_{\beta-\alpha}\mathbb{1}+c_{\beta-\alpha}\rho^D_R$	$\mathbb{1}+c_{\beta-\alpha}\rho^D_R$
$h\overline{U}U$	$s_{\beta-\alpha}\mathbb{1}+c_{\beta-\alpha}\rho^U_R$	$\mathbb{1}+c_{\beta-\alpha}\rho^U_R$

Type I and II 2HDM couplings of the SM-like Higgs boson h normalized to those of the SM Higgs boson, in the alignment limit. The hH^+H^- coupling given below is normalized to the SM hhh coupling. The scalar Higgs potential is taken to be CP-conserving. For the fermion couplings, D is a column vector of three down-type fermion fields (either down-type quarks or charged leptons) and U is a column vector of three up-type quark fields. In the third column, the first non-trivial correction to alignment is exhibited. Finally, complete expressions for the entries marked with a * can be found in H.E. Haber and D. O'Neil, Phys. Rev. D **74**, 015018 (2006) [Erratum: ibid. D **74** (2006) 059905].

$$\text{Type I}: \qquad \rho^D_R=\rho^U_R=\mathbb{1}\cot\beta\,,$$

$$\text{Type II}: \qquad \rho^D_R=-\mathbb{1}\tan\beta\,, \qquad \rho^U_R=\mathbb{1}\cot\beta\,.$$

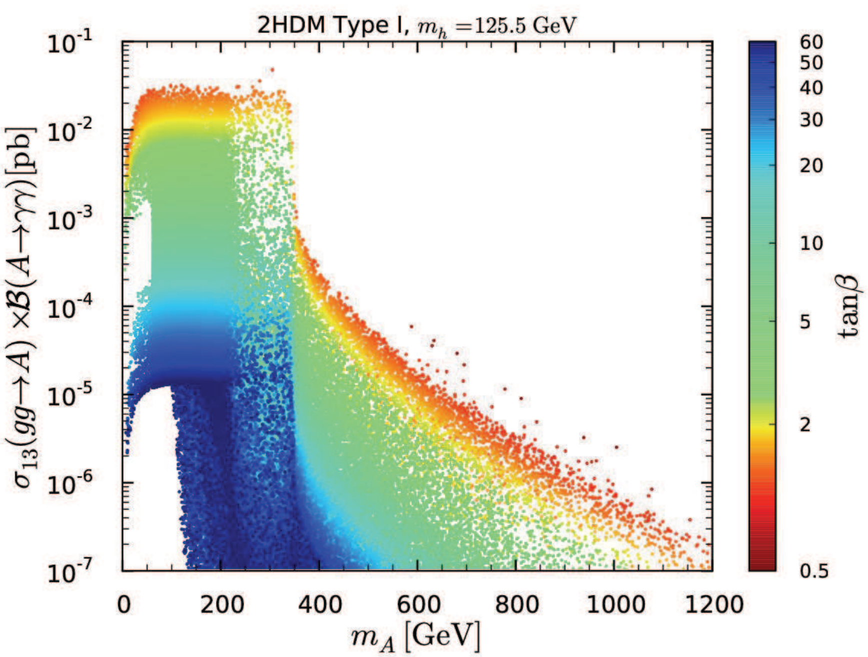
Constraints on Type-I and II 2HDMs from Higgs data

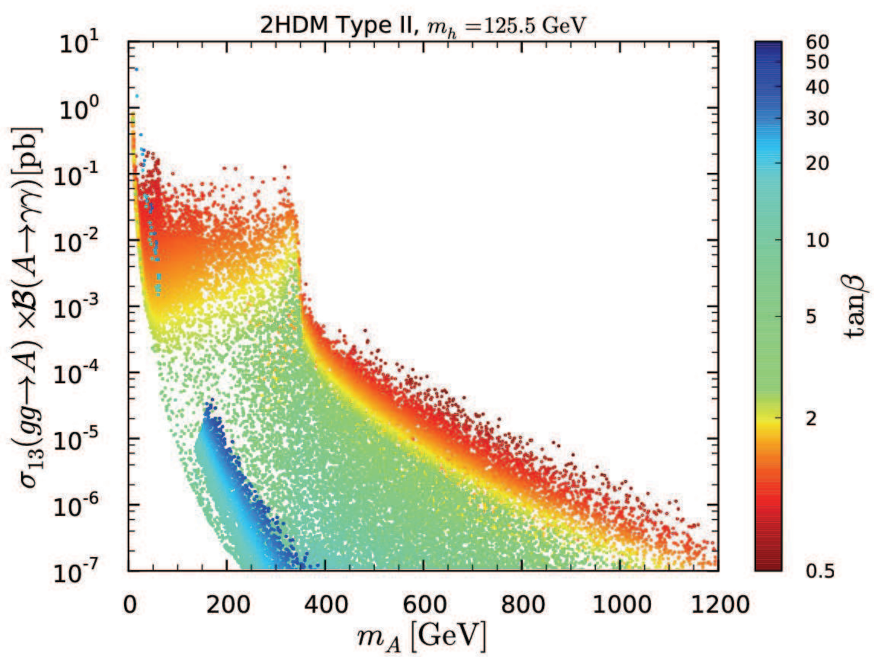


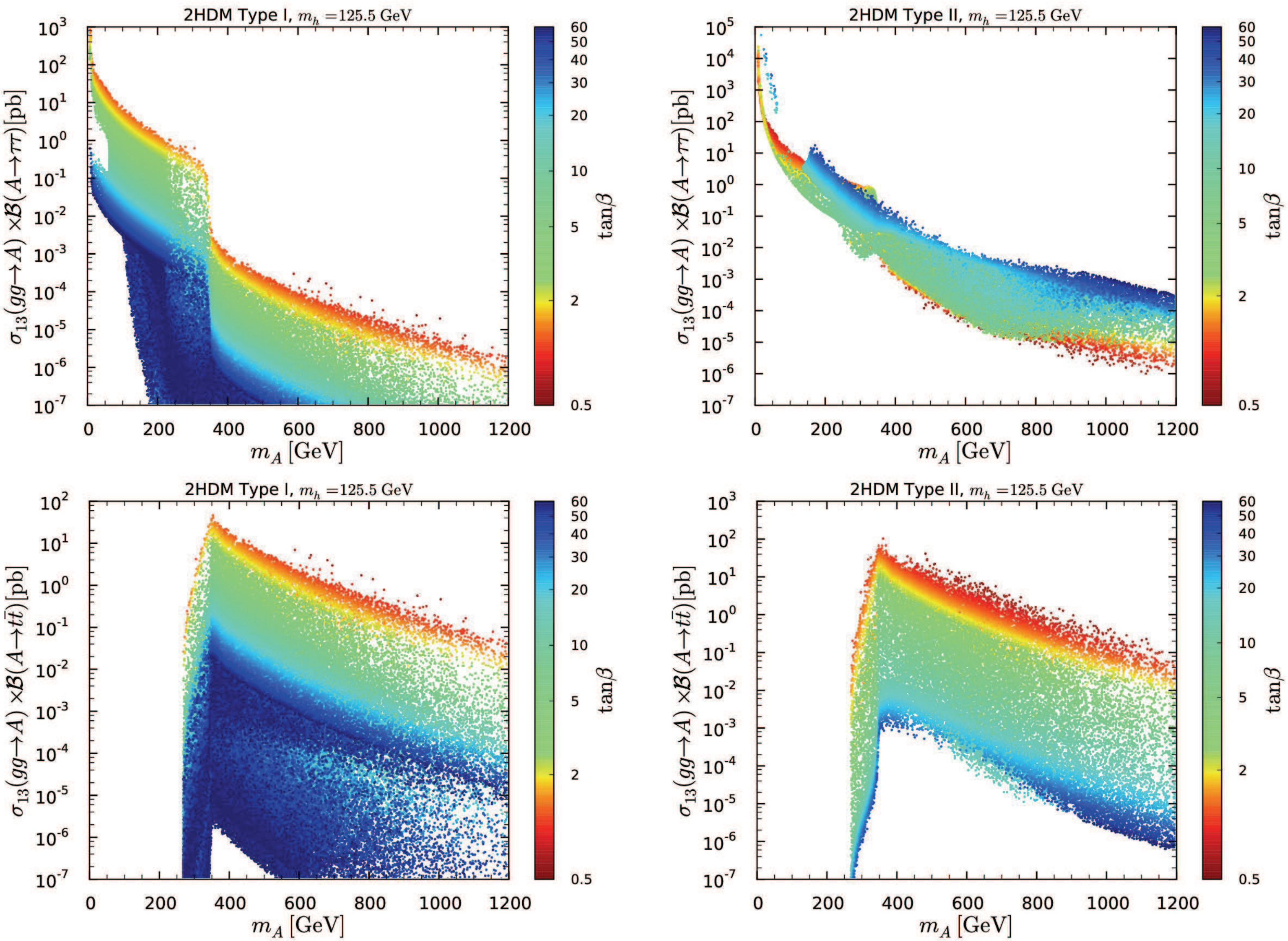
Direct constraints from LHC Higgs searches for Type-I (left) and Type-II (right) 2HDM with $m_H = 300\mathrm{GeV}$ with $m_h = 125\mathrm{GeV}$, $Z_4 = Z_5 = -2$ and $Z_7 = 0$. Colors indicate compatibility with the observed Higgs signal at $1\,\sigma$ (green), $2\,\sigma$ (yellow) and $3\,\sigma$ (blue). Exclusion bounds at 95% C.L. from the non-observation of the additional Higgs states overlaid in gray. From H.E. Haber and O. Stål, Eur. Phys. J. C **75**, 491 (2015) [Erratum: ibid., **76**, 312 (2016)].

Projections for future LHC running

Since present data suggests a SM-like Higgs boson, one should take this into account in devising searches for the heavier Higgs states of the 2HDM. For example, in J. Bernon, J.F. Gunion, H.E. Haber, Y. Jiang and S. Kraml, Phys. Rev. D **92**, 075004 (2015), we assumed that $|\cos(\beta-\alpha)| \leq 0.14$ and scanned the Type-I and II 2HDM parameter spaces. Sample results are shown below for the search for A in gluon-gluon fusion.







Cross sections times branching ratio in Type I (left panels) and in Type II (right panels) for $gg \to A \to Y$ at the 13 TeV LHC as functions of m_A for $Y = \gamma\gamma$ (previous page panels), $Y = \tau\tau$ (upper panels) and $Y = t\bar{t}$ (lower panels) with $\tan\beta$ color code.

A symmetry origin for alignment without decoupling

Alignment without decoupling appears to required an unexplained tuning to achieve $|Z_6| \ll 1$. However, there exists a symmetry that can impose the exact alignment limit of $Z_6 = 0$—a discrete $\mathbb{Z}_2$ symmetry where the Higgs basis field H_1 is unchanged but $H_2 \to -H_2$. If we impose this symmetry on the scalar potential, then it follows that

$$Y_3 = Z_6 = Z_7 = 0\,.$$

Note that the minimum condition $Y_3 = -\frac{1}{2}Z_6 v^2$ requires that $Y_3 = 0$ if $Z_6 = 0$, so this $\mathbb{Z}_2$ symmetry *cannot* be softly broken.

Having imposed the above $\mathbb{Z}_2$ symmetry in the bosonic sector of the theory, we can extend it to the Yukawa interactions. If we demand that all fermions are even under the $\mathbb{Z}_2$ symmetry, then the H_1 couplings to fermions are those of the SM Higgs boson and the Yukawa couplings of H_2 to the fermions are absent.

This is the inert doublet model (IDM). The lightest scalar of the H_2 doublet is therefore absolutely stable and a possible candidate for dark matter. That is, the inert 2HDM is a Type-I 2HDM in which there exists an unbroken $\mathbb{Z}_2$ symmetry in the Higgs basis.

Further details on the IDM

By imposing the discrete $\mathbb{Z}_2$ symmetry, the scalar potential is CP-conserving. The SM Higgs state is $h = \sqrt{2}\,\mathrm{Re}\,H_1^0 - v$. The inert doublet is

$$H_2 = \begin{pmatrix} H^+ \\ (H + iA)/\sqrt{2} \end{pmatrix},$$

where the mass eigenstates consist of two neutral scalars, H, A and a charged Higgs pair. One is free to rephase H_2 so that the model depends on Y_2, Z_1, Z_2, Z_3, Z_4 and $|Z_5|$. Vacuum stability implies that $Z_{1,2} > 0$, $Z_3 > -(Z_1 Z_2)^{1/2}$ and $Z_3 + Z_4 \pm |Z_5| > -(Z_1 Z_2)^{1/2}$.

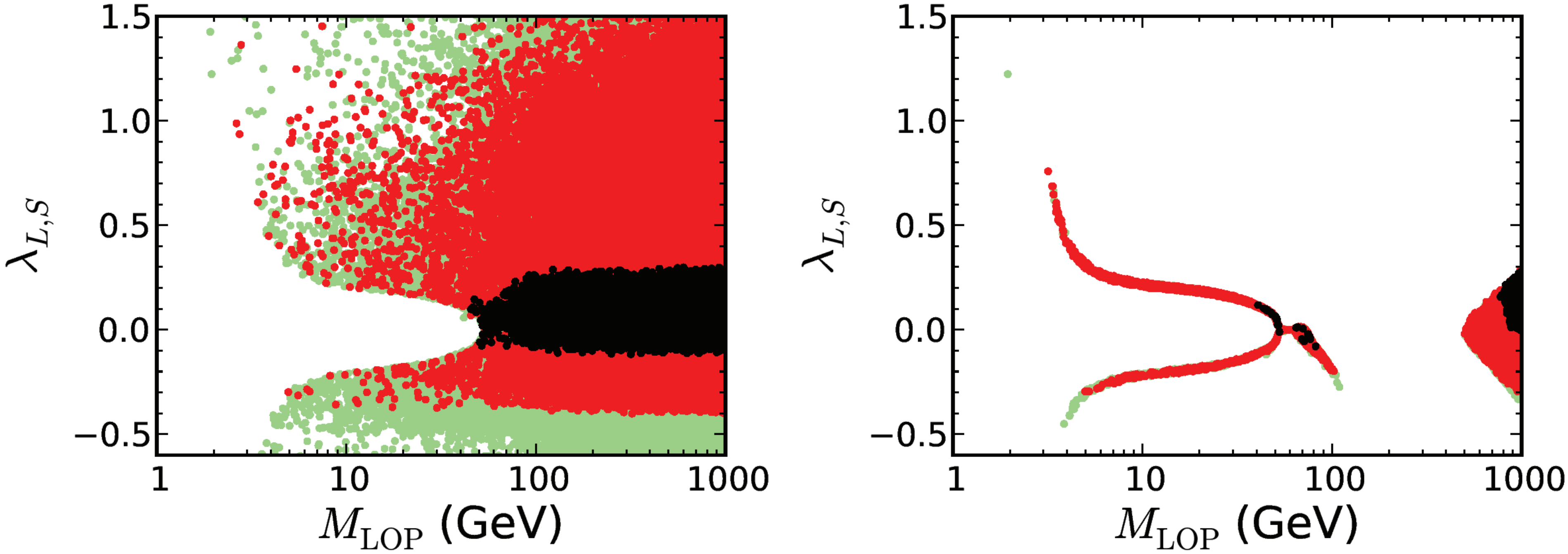
The physical Higgs masses are

$$m_h^2 = Z_1 v^2\,, \qquad m_{H^\pm}^2 = Y_2 + \tfrac{1}{2} Z_3 v^2\,,$$
$$m_{H,A}^2 = m_{H^\pm}^2 + \tfrac{1}{2}(Z_4 \pm |Z_5|) v^2\,.$$

H and A have opposite CP-quantum numbers, but there is no interaction that can determine separate CP quantum number for these states. The lighter of these two states will henceforth be denoted as H_L.

The lightest $\mathbb{Z}_2$–odd particle (LOP) is a candidate for dark matter. Assuming that $Z_4 < |Z_5|$ (in which case H_L is lighter than $H^\pm$), we identify the squared-mass of the LOP as $M_{\rm LOP}^2 = Y_2 + \frac{1}{2}(Z_3 + Z_4 - |Z_5|)v^2$.

Including the exclusion limits from the current dark matter direct detection experiments, a cosmologically relevant LOP is ruled out by Goudelis, Herrmann and Stål for all LOP masses below 500 GeV except for a narrow window around $\frac{1}{2} m_h$ (roughly 50 GeV $\lesssim M_{\rm LOP} \lesssim 80$ GeV).



The viable IDM parameter space projected on the (M_{LOP} , $\lambda_{L,S}$) plane imposing only the upper limit (left) and the upper and lower limits (right) of the WMAP range, $0.1018 \leq M_{\text{LOP}} h^2 \leq 0.1234$. The green points correspond to all valid points in the scan, while the red and black regions show the points which remain valid when the model satisfies stability and perturbativity up to a scale $\Lambda = 10^4$ GeV and the GUT scale $\Lambda = 10^{16}$ GeV, respectively. Above, $\lambda_{L,S} \equiv \frac{1}{2}(Z_3 + Z_4 \mp |Z_5|)$; when multiplied by v the latter corresponds to the hH_LH_L coupling. Taken from A. Goudelis, B. Herrmann and O. Stål, JHEP **1309** (2013) 106.

The alignment limit of the general 2HDM in equations

To obtain the conditions in which h_1 is the SM-like Higgs boson, noting that:

$$\frac{g_{h_1VV}}{g_{h_{\rm SM}VV}} = c_{12}c_{13}\,, \qquad \text{where } V = W \text{ or } Z\,,$$

where $h_{\rm SM}$ is the SM Higgs boson, we demand that

$$s_{12}\,,\, s_{13} \ll 1\,.$$

Here, $s_{12} \equiv \sin\theta_{12}$, $c_{12} \equiv \cos\theta_{12}$, etc. The following (exact) relations are noteworthy:

$$\begin{aligned}
Z_1 v^2 &= m_1^2 c_{12}^2 c_{13}^2 + m_2^2 s_{12}^2 c_{13}^2 + m_3^2 s_{13}^2\,,\\
\mathrm{Re}(Z_6\, e^{-i\theta_{23}})\, v^2 &= c_{13} s_{12} c_{12}(m_2^2 - m_1^2)\,,\\
\mathrm{Im}(Z_6\, e^{-i\theta_{23}})\, v^2 &= s_{13} c_{13}(c_{12}^2 m_1^2 + s_{12}^2 m_2^2 - m_3^2)\,,\\
\mathrm{Re}(Z_5\, e^{-2i\theta_{23}})\, v^2 &= m_1^2(s_{12}^2 - c_{12}^2 s_{13}^2) + m_2^2(c_{12}^2 - s_{12}^2 s_{13}^2) - m_3^2 c_{13}^2\,,\\
\mathrm{Im}(Z_5\, e^{-2i\theta_{23}})\, v^2 &= 2 s_{12} c_{12} s_{13}(m_2^2 - m_1^2)\,.
\end{aligned}$$

Assuming no mass degeneracies in the neutral scalar sector, it then follows that in the alignment limit,

$$\boxed{\begin{aligned} s_{12} \equiv \sin\theta_{12} &\simeq \frac{\mathrm{Re}(Z_6 e^{-i\theta_{23}})v^2}{m_2^2 - m_1^2} \ll 1\,, \\ s_{13} \equiv \sin\theta_{13} &\simeq -\frac{\mathrm{Im}(Z_6 e^{-i\theta_{23}})v^2}{m_3^2 - m_1^2} \ll 1\,, \end{aligned}}$$

One additional small quantity characterizes the alignment limit,

$$\mathrm{Im}(Z_5 e^{-2i\theta_{23}}) \simeq \frac{(m_2^2 - m_1^2)s_{12}s_{13}}{v^2} \simeq -\frac{2\,\mathrm{Im}(Z_6^2 e^{-2i\theta_{23}})v^2}{m_3^2 - m_1^2} \ll 1\,.$$

Finally, the following mass relations in the alignment limit are noteworthy,

$$\begin{aligned} m_1^2 &\simeq Z_1 v^2\,, \\ m_2^2 - m_3^2 &\simeq \mathrm{Re}(Z_5 e^{-2i\theta_{23}})v^2\,. \end{aligned}$$