High-precision determination of the pion–nucleon σ-term from Roy–Steiner equations

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Outline

Pion–nucleon scattering and the σ-term

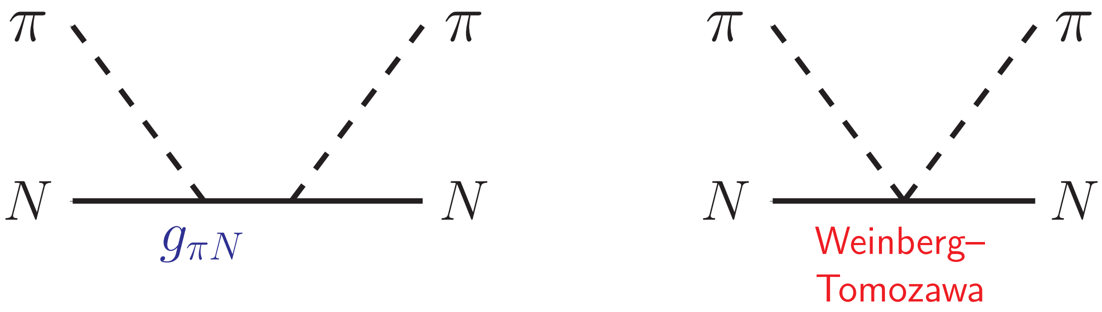
A new dispersive analysis: Roy–Steiner equations

- phase shifts — Phys. Rept. 625 (2016) 1
- σ-term and comparison to lattice results — PRL 115 (2015) 092301, PLB 760 (2016) 74
- comparison to low-energy πN scattering data — preliminary
- form factor spectral functions — arXiv:1609.06722 (with H.-W. Hammer)
- chiral low-energy constants — PRL 115 (2015) 192301

in collaboration with M. Hoferichter, J. Ruiz de Elvira, and U.-G. Meißner

Chiral pion–nucleon interaction

- simplest process for chiral pion interaction with nucleons



- leading-order $\mathcal{O}(p) = \mathcal{O}(M_\pi)$ predictions for πN:

  scattering lengths:

$$a^- = \frac{M_\pi m_N}{8\pi(m_N + M_\pi)F_\pi^2} + \mathcal{O}\big(M_\pi^3\big) \qquad a^+ = \mathcal{O}\big(M_\pi^2\big)$$

Weinberg 1966

  Goldberger–Treiman relation: $g_{\pi N} = \dfrac{g_A m_N}{F_\pi}$

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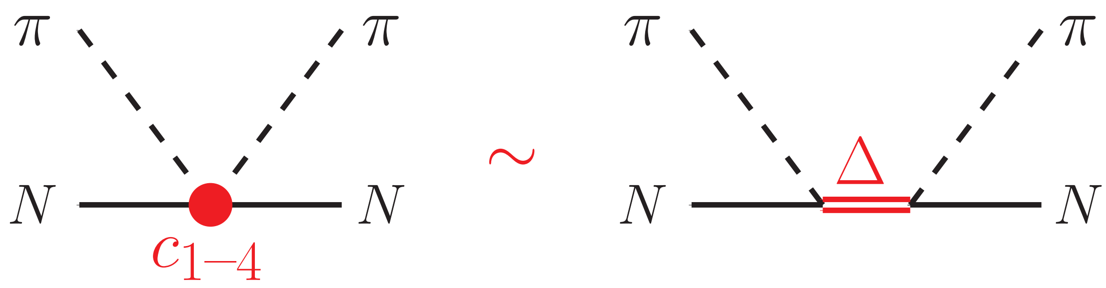
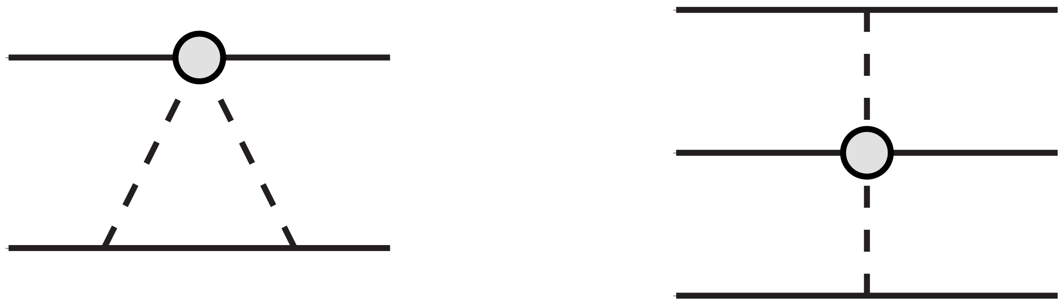


- next-to-leading order $\mathcal{O}(p^2)$: low-energy constants (LECs) c_{1-4} effectively incorporate effects of the $\Delta(1232)$ resonance: low mass $m_\Delta - m_N \approx 2M_\pi$ and strong couplings
- determination of c_i very important for nuclear physics: πN important for NN / determines longest-range $3N$ forces



The pion–nucleon σ-term

- scalar form factor of the nucleon:

$$\langle N(p')|\hat{m}(\bar{u}u+\bar{d}d)|N(p)\rangle = \sigma(t)\bar{u}(p')u(p) \qquad t=(p-p')^2$$

$$\sigma_{\pi N} \equiv \sigma(0) = \frac{\hat{m}}{2m_N}\langle N|\bar{u}u+\bar{d}d|N\rangle \qquad \hat{m}=\frac{m_u+m_d}{2}$$

- $\sigma_{\pi N}$ determines light quark contribution to nucleon mass: Feynman–Hellmann theorem

$$\sigma_{\pi N} = \hat{m}\frac{\partial m_N}{\partial \hat{m}} = -4c_1 M_\pi^2 + \mathcal{O}(M_\pi^3)$$

$\longrightarrow$ at leading order, related to the chiral coupling c_1

- $\sigma_{\pi N}$ determines scalar couplings wanted for direct-detection dark matter searches

e.g. Ellis et al. 2008

see also talks this morning

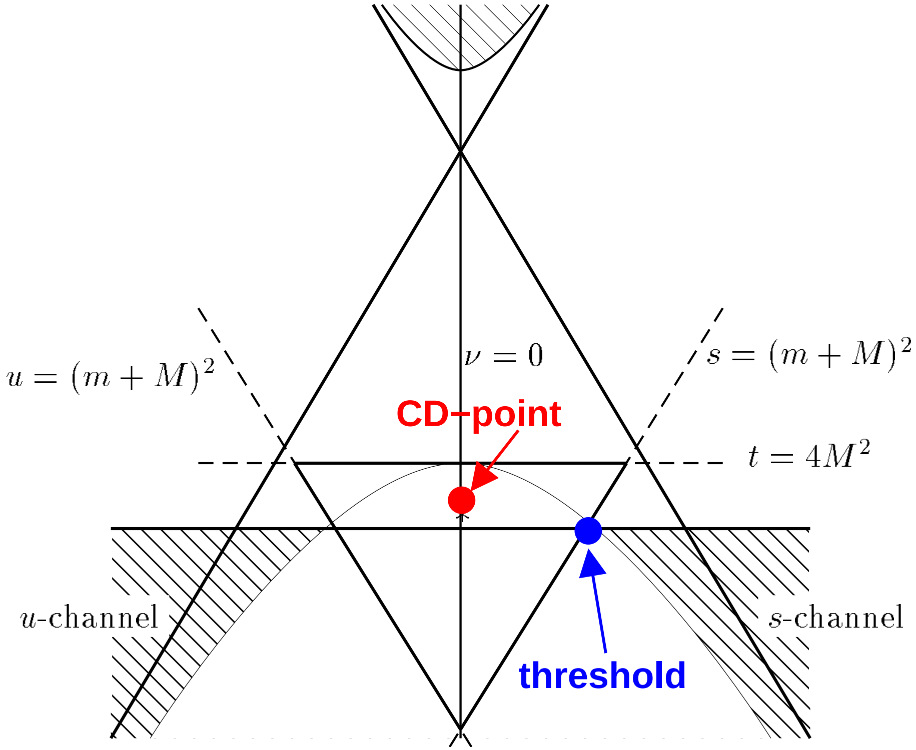
On the chiral extractions of $\sigma_{\pi N}$

The Cheng–Dashen theorem

- isoscalar amplitude at CD point related to scalar form factor

$$\underbrace{F_\pi^2 \bar{D}^+(s=u, t=2M_\pi^2)}_{F_\pi^2(d_{00}^+ + 2M_\pi^2 d_{01}^+) + \Delta_D} = \underbrace{\sigma(2M_\pi^2)}_{\sigma_{\pi N} + \Delta_\sigma} + \Delta_R$$

$|\Delta_R| \lesssim 2\,\text{MeV}$ Bernard et al. 1996



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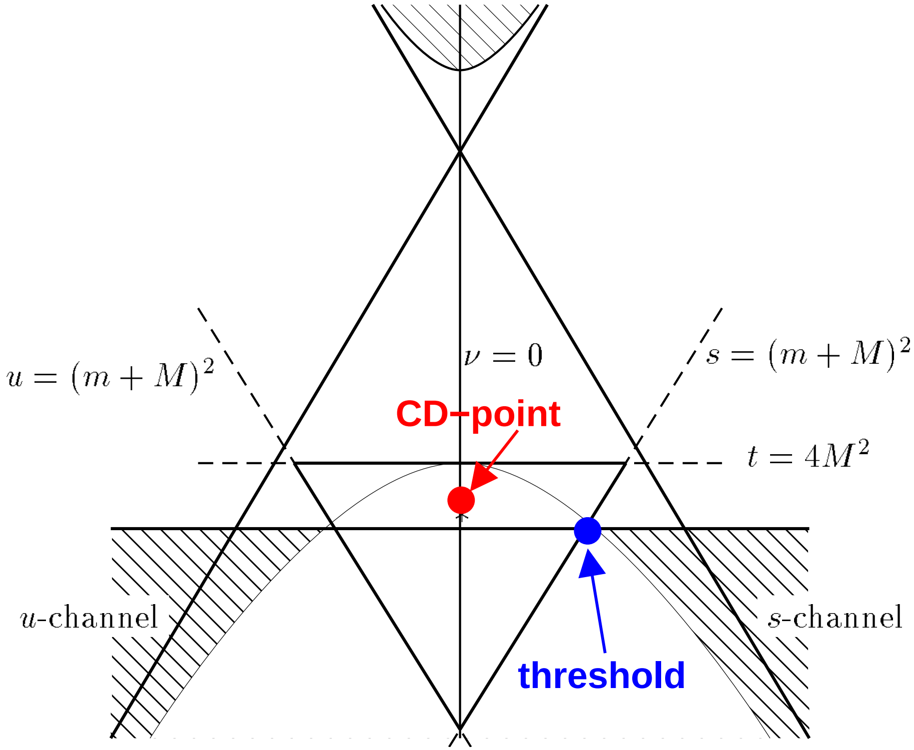

- ChPT fulfils all these relations perturbatively only

  is known to fail at one loop for Δ_D, Δ_σ: Gasser, Leutwyler, Sainio 1991

  curvature d_{02}^+ not reproduced at one loop Alarcón et al. 2013

- we're lucky: $\Delta_D - \Delta_\sigma = (-1.8 \pm 0.2)$ MeV cancels to large extent

$\longrightarrow$ one-loop ChPT does not describe pion–nucleon scattering accurately in the whole low-energy region

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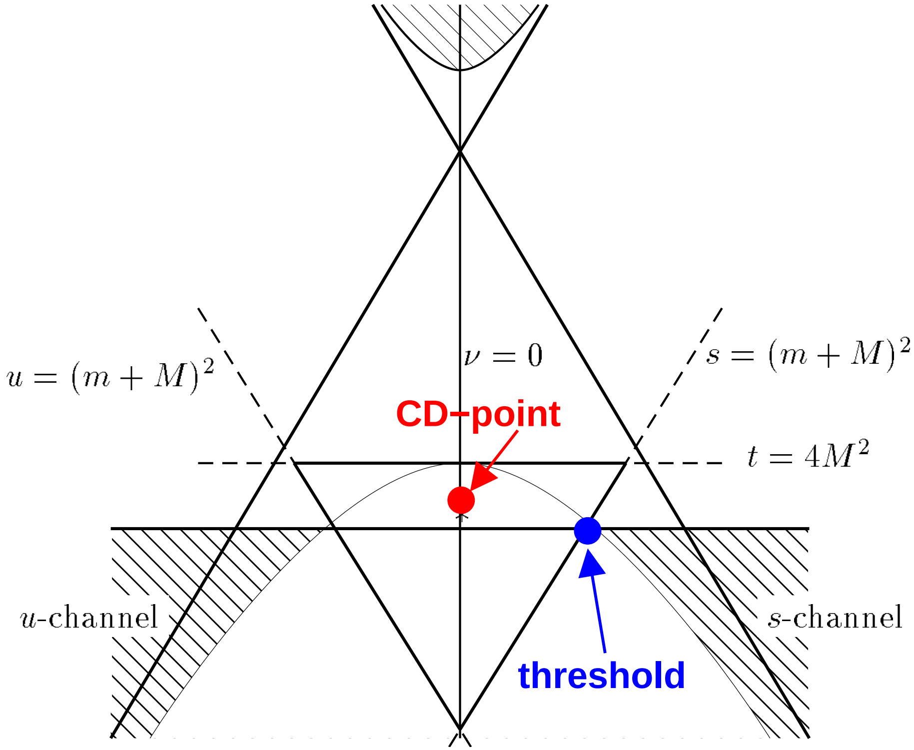

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$\longrightarrow$ update dispersive analysis, Roy–Steiner equations

Hoferichter, Ruiz de Elvira, BK, Meißner

The well-known paradigm: $\pi\pi$ Roy equations

Roy equations = coupled system of partial-wave dispersion relations + crossing symmetry + unitarity

- twice-subtracted fixed-t dispersion relation:

$$T(s,t) = c(t) + \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} ds' \Bigg\{ \underbrace{\frac{s^2}{s'^2(s'-s)}}_{s\text{-channel cut}} + \underbrace{\frac{u^2}{s'^2(s'-u)}}_{u\text{-channel cut}} \Bigg\} \mathrm{Im}T(s',t)$$

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- subtraction function $c(t)$ determined from crossing symmetry
- project onto partial waves $t_J^I(s)$ (angular momentum J, isospin I)
  expand $\mathrm{Im}T(s',t)$ in partial waves

$$t_J^I(s) = \text{polynomial}(a_0^0, a_0^2) + \sum_{I'=0}^{2}\sum_{J'=0}^{\infty}\int_{4M_\pi^2}^{\infty} ds' K_{JJ'}^{II'}(s,s')\,\mathrm{Im}t_{J'}^{I'}(s')$$

kernel functions $K_{JJ'}^{II'}(s,s')$ known analytically Roy 1971

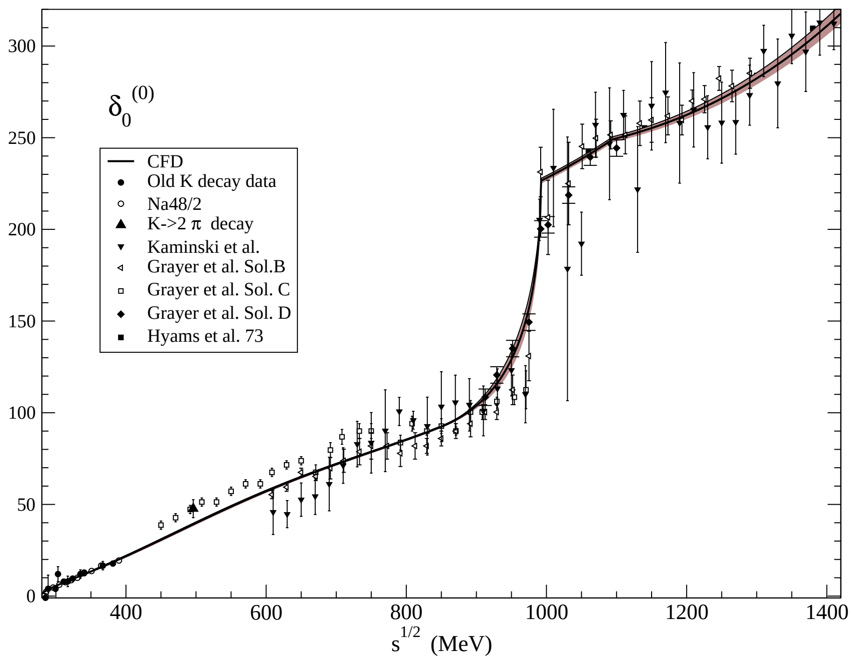
$\pi\pi$ Roy equations

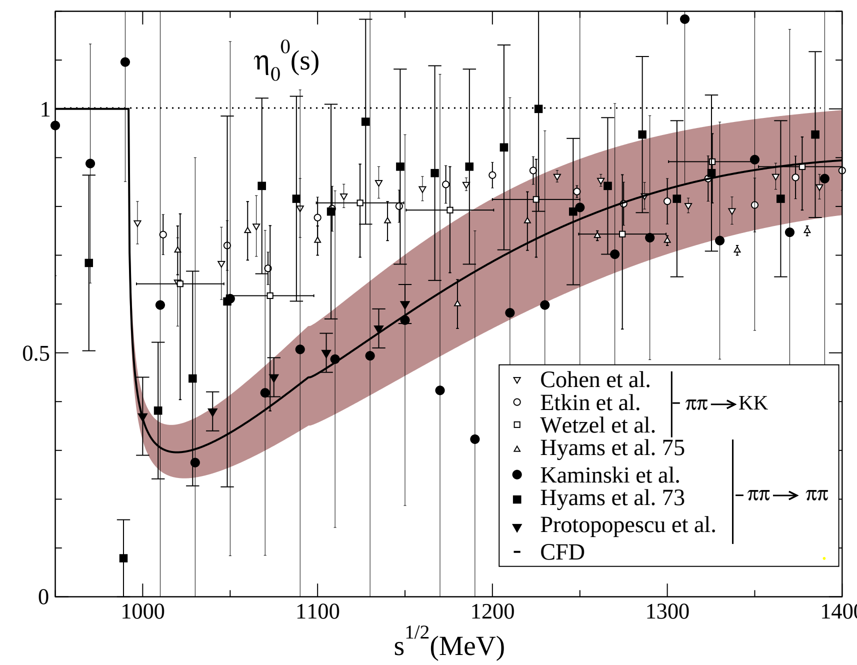
- elastic unitarity:

$$t_J^I(s) = \frac{e^{2i\delta_J^I(s)} - 1}{2i\sigma} \qquad \sigma = \sqrt{1 - \frac{4M_\pi^2}{s}}$$

$\longrightarrow$ coupled integral equations for phase shifts

- example: $\pi\pi$ $I = 0$ S-wave phase shift & inelasticity

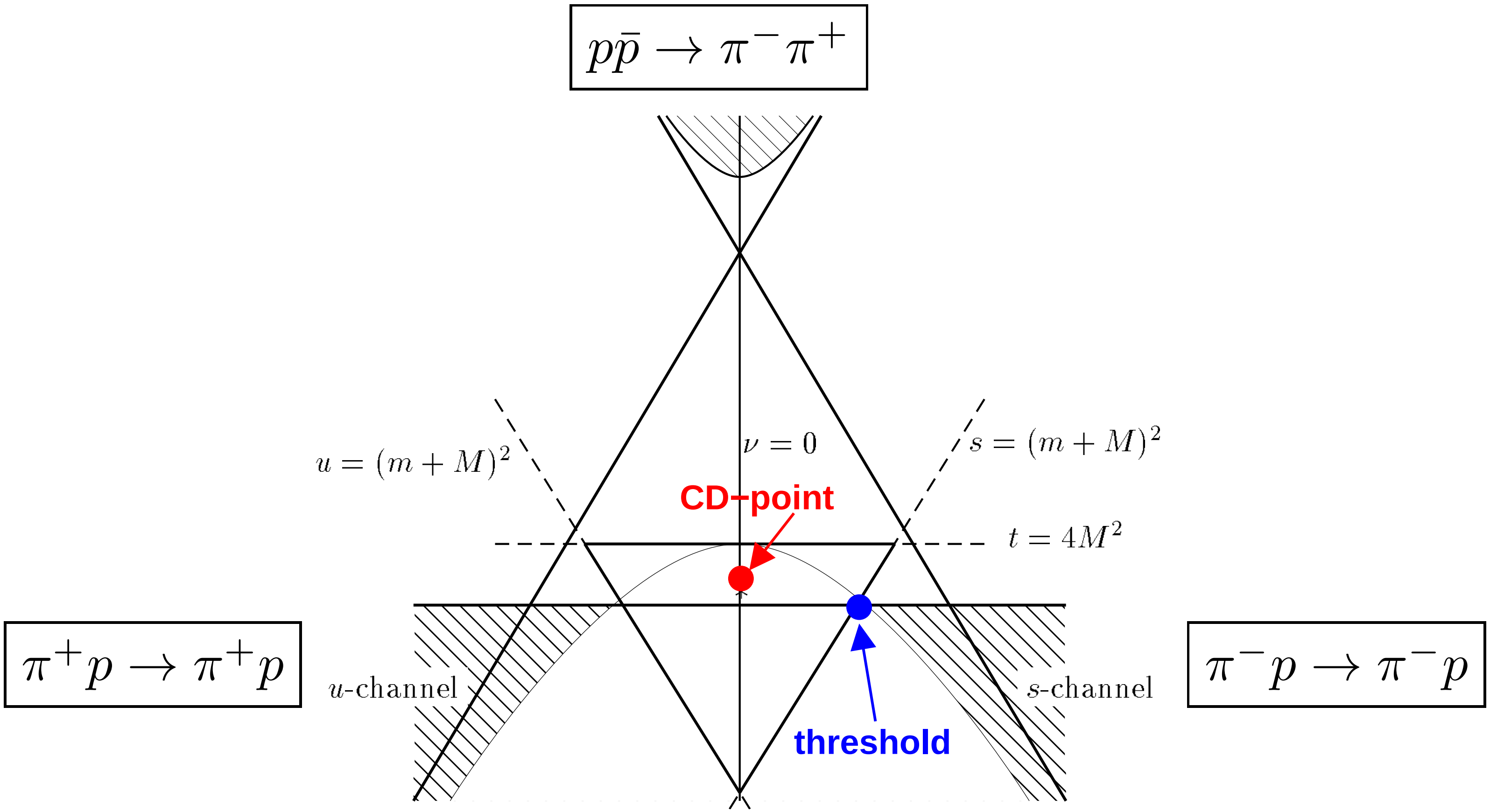




García-Martín et al. 2011

$\longrightarrow$ strong constraints on data from analyticity and unitarity!

Pion–nucleon scattering, crossing symmetry



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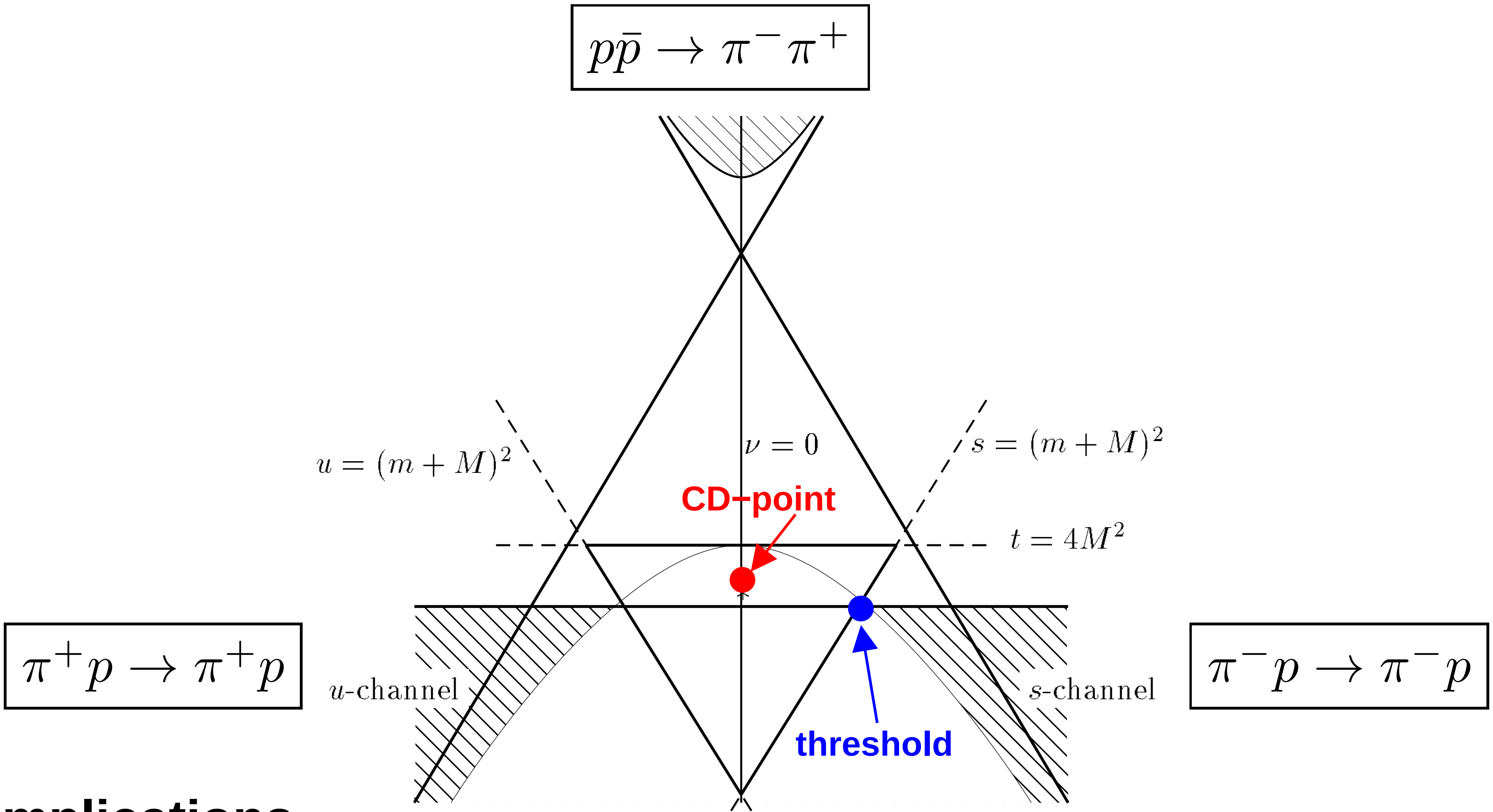

Complications

- crossing links two different processes, $\pi N \to \pi N$ and $\pi\pi \to \bar{N}N$
  $\longrightarrow$ use hyperbolic (instead of fixed-t) DR (Roy–Steiner)
- large pseudophysical region in the t-channel: $t = 4M_\pi^2 \longrightarrow 4m_N^2$,
  $\bar{K}K$ intermediate states ($f_0(980)$)

Roy–Steiner equations for pion–nucleon scattering

Limited range of validity:

$$\sqrt{s} \leq \sqrt{s_m} = 1.38\,\text{GeV} \qquad \sqrt{t} \leq \sqrt{t_m} = 2.00\,\text{GeV}$$

Input / constraints:

- S-, P-waves above matching point $s > s_m$ ($t > t_m$)
- inelasticities
- higher waves (D-, F-...)
- scattering lengths from hadronic atoms Baru et al. 2011

Output:

- S- and P-waves at low energies $s < s_m$, $t < t_m$
- subthreshold parameters
  - ▷ pion–nucleon σ-term
  - ▷ nucleon form factor spectral functions

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Important analysis steps:

- full analytic system Ditsche, Hoferichter, BK, Meißner 2012
- improved t-channel S-wave ($\pi\pi \leftrightarrow \bar{K}K \leftrightarrow \bar{N}N$) Hoferichter, Ditsche, BK, Meißner 2012
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Pionic atoms and pion–nucleon scattering lengths

Measurements of πH and πD PSI 1995-2010; see talk by D. Gotta

$\epsilon_{1s} = (7.120 \pm 0.012)$ eV $\quad \Gamma_{1s} = (0.823 \pm 0.019)$ eV $\quad \epsilon_{1s}^{D} = (2.356 \pm 0.031)$ eV

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Theory to match this accuracy requires

- isospin breaking in πN Hoferichter, BK, Meißner 2009
- three-body corrections in πD Weinberg 1992, ...
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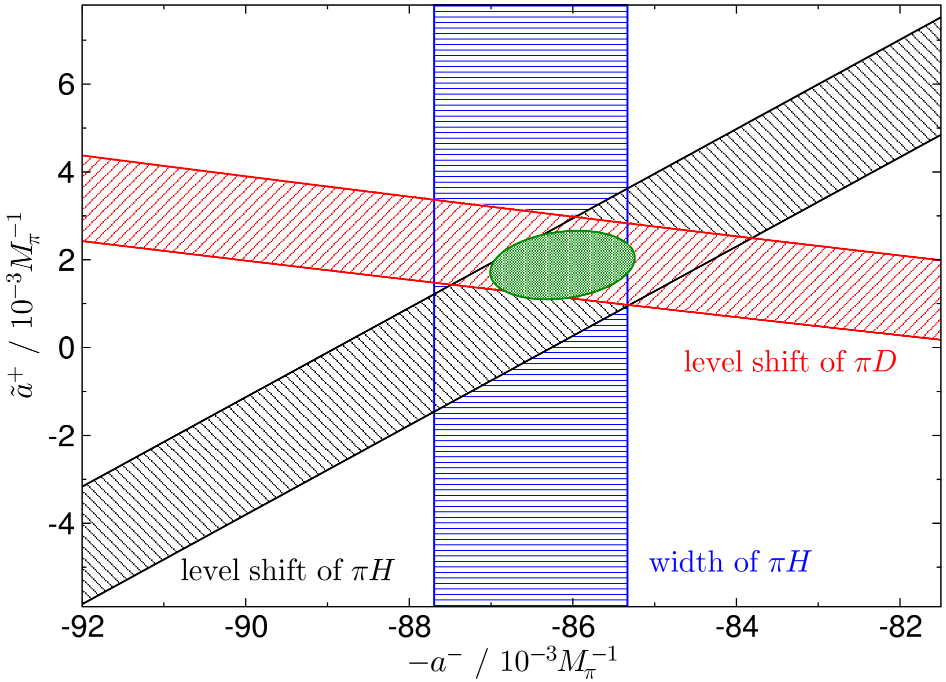

$$a_0^- = (86.1 \pm 0.9) \cdot 10^{-3} M_\pi^{-1}$$
$$a_0^+ = (7.6 \pm 3.1) \cdot 10^{-3} M_\pi^{-1}$$

but: $\frac{1}{2}(a_{\pi^- p} + a_{\pi^+ p}) = (-1.1 \pm 0.9) \cdot 10^{-3} M_\pi^{-1}$

$\longrightarrow$ large isospin-breaking effects in isoscalar sector

Baru et al. 2011

Solving the coupled system: paradigms, uncertainties

An update on Karlsruhe–Helsinki (KH) with modern input

- πN scattering lengths extracted from hadronic atoms
- Goldberger–Miyazawa–Oehme sum rule from those:

  $$g_{\pi N}^2/4\pi = 13.7 \pm 0.2$$ Baru et al. 2011

  in perfect agreement with NN extractions Navarro Pérez et al. 2016

  compare: $$g_{\pi N}^2/4\pi = 14.28$$ Höhler 1983

  $\longrightarrow$ check: always reproduce KH results with KH input
- modern s-channel partial waves from SAID above s_m

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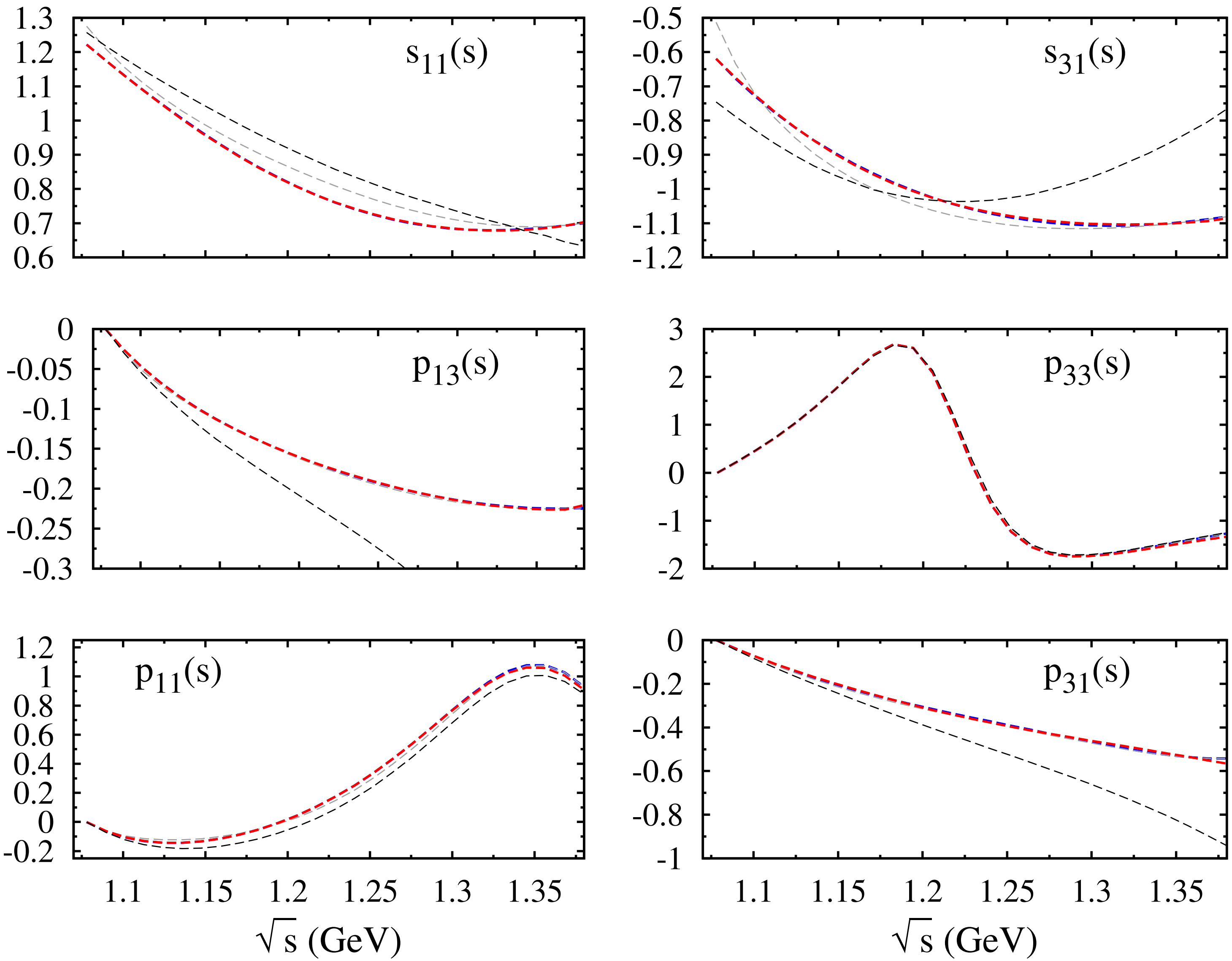
Dominant uncertainties

- near threshold: S-wave scattering lengths
- intermediate energies: significant correlations between 10 subtraction constants = subthreshold parameters ("flat minima")
- "large" energies: matching point uncertainties
- rather well under control: high-energy input, higher partial waves

Results: *s*-channel solution

LHS+RHS of Roy–Steiner eqs. *before* / LHS+RHS *after* fit/iteration



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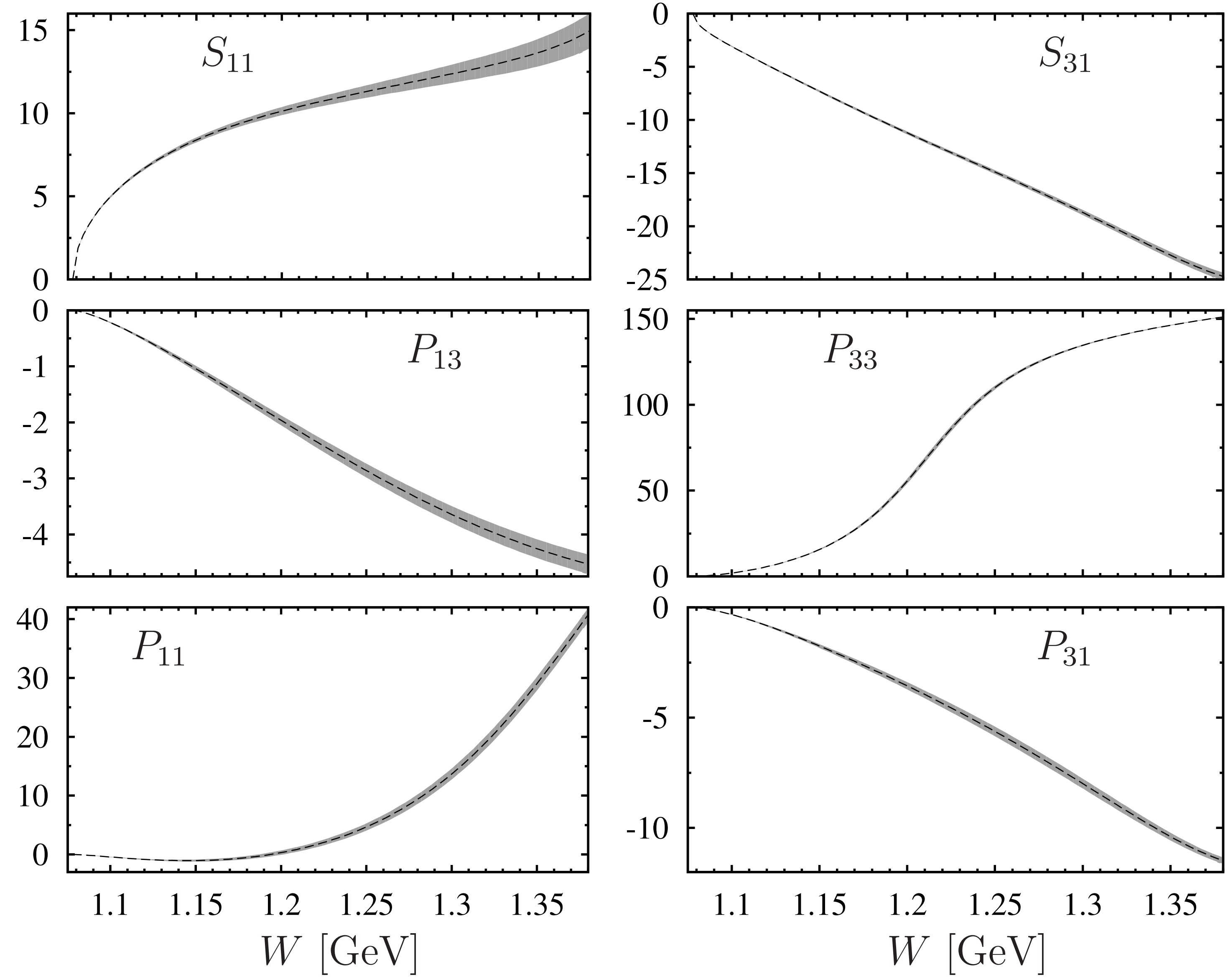
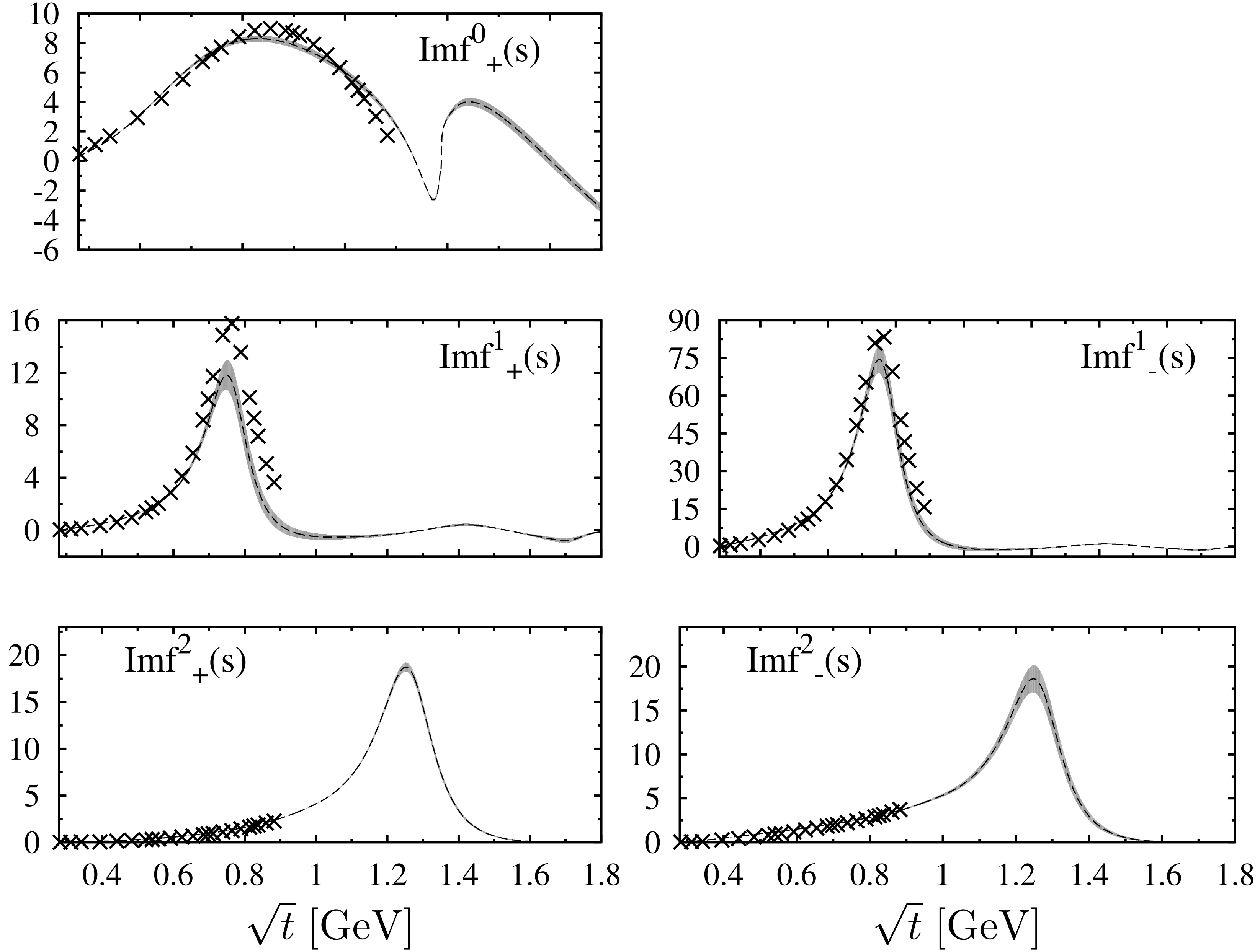


Hoferichter, Ruiz de Elvira, BK, Meißner 2015

Results: *t*-channel S-, P-, D-waves (compared to KH)



Results for the σ-term

$$\boxed{\sigma_{\pi N} = F_\pi^2(d_{00}^+ + 2M_\pi^2 d_{01}^+) + \Delta_D - \Delta_\sigma - \Delta_R}$$

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- subthreshold parameters output of the Roy–Steiner equations

$$d_{00}^+ = -1.36(3) M_\pi^{-1} \qquad \text{[KH:} \quad -1.46(10) M_\pi^{-1}\text{]}$$

$$d_{01}^+ = \quad 1.16(2) M_\pi^{-3} \qquad \text{[KH:} \quad 1.14(2) M_\pi^{-3}\text{]}$$

- $\Delta_D - \Delta_\sigma = (-1.8 \pm 0.2)$ MeV — Hoferichter et al. 2012
- $|\Delta_R| \lesssim 2$ MeV — Bernard, Kaiser, Meißner 1996
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- full result:

$$\sigma_{\pi N} = (59.1 \pm 1.9_{\text{RS}} \pm 3.0_{\text{LET}})\,\text{MeV} = (59.1 \pm 3.5)\,\text{MeV}$$

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Hoferichter, Ruiz de Elvira, BK, Meißner 2015

- KH input $\longrightarrow$ $\sigma_{\pi N} \approx 46$ MeV — Gasser, Leutwyler, Sainio 1991
- compare also $\sigma_{\pi N} \approx (64 \pm 8)$ MeV — Pavan et al. 2002

Comparison to lattice results – a puzzle (1)

- 4 new lattice calculations of $\sigma_{\pi N}$ at physical M_π since
  Hoferichter, Ruiz de Elvira, BK, Meißner 2015

$\sigma_{\pi N}$ [MeV]	collaboration	tension to RS
$38(3)(3)$	BMW 2015	3.8σ
$44.4(3.2)(4.5)$	χQCD 2015	2.2σ
$37.2(2.6)\left({}^{+1.0}_{-0.6}\right)$	ETMC 2016	4.9σ
$35.0(6.1)$	RQCD 2016	3.4σ

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$35.0(6.1)$	RQCD 2016	3.4σ

- robust correlation between $\sigma_{\pi N}$ and scattering lengths:

$$\sigma_{\pi N} = (59.1 \pm 3.1)\,\text{MeV} + \sum_I c_I \big(a_0^I - \bar{a}_0^I\big),$$

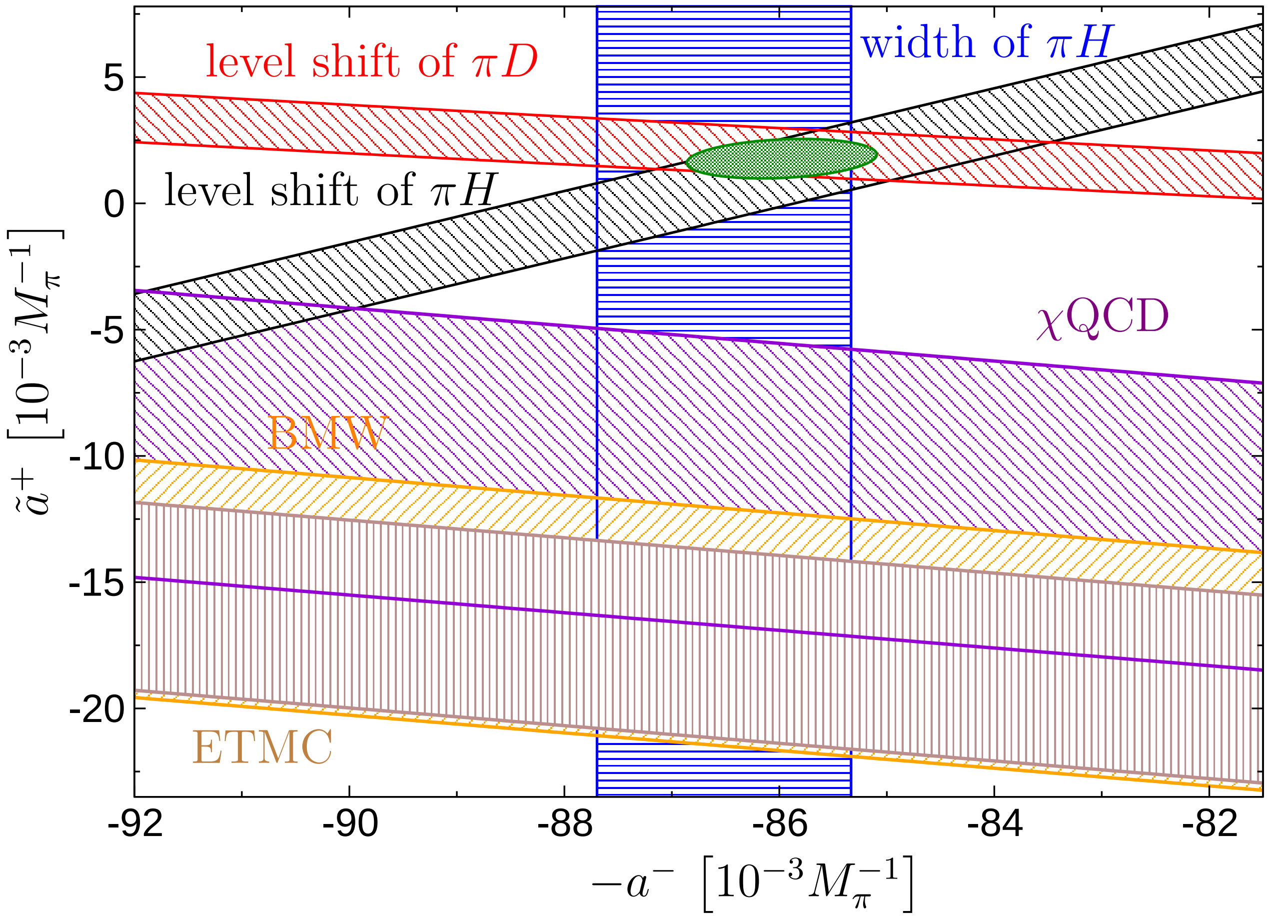
$$c_{1/2} = 0.242\,\text{MeV} \times 10^3 M_\pi \qquad c_{3/2} = 0.874\,\text{MeV} \times 10^3 M_\pi$$

$$\bar{a}_0^{1/2} = (169.8 \pm 2.0) \times 10^{-3}\, M_\pi^{-1} \quad \bar{a}_0^{3/2} = (-86.3 \pm 1.8) \times 10^{-3}\, M_\pi^{-1}$$

$\longrightarrow$ expansion around reference values from πH and πD

Comparison to lattice results – a puzzle (2)

- lattice $\sigma_{\pi N}$ as additional constraint in scattering lengths plane



$\longrightarrow$ lattice $\sigma_{\pi N}$ clearly at odds with hadronic atoms results

$\longrightarrow$ suggestion: determine πN scattering lengths on the lattice

Hoferichter, Ruiz de Elvira, BK, Meißner 2016

Comparison with experimental cross section data base

	HA/RS	KH80	
$a_{0+}^{1/2}$ $[10^{-3} M_\pi^{-1}]$	169.8 ± 2.0	173 ± 3	
$a_{0+}^{3/2}$ $[10^{-3} M_\pi^{-1}]$	-86.3 ± 1.8	-101 ± 4	$\longrightarrow \pi^+ p \to \pi^+ p$

- generate RS solutions for different scattering lengths
- uncertainties dominated by scatt. lengths below $T_\pi = 50\,$MeV

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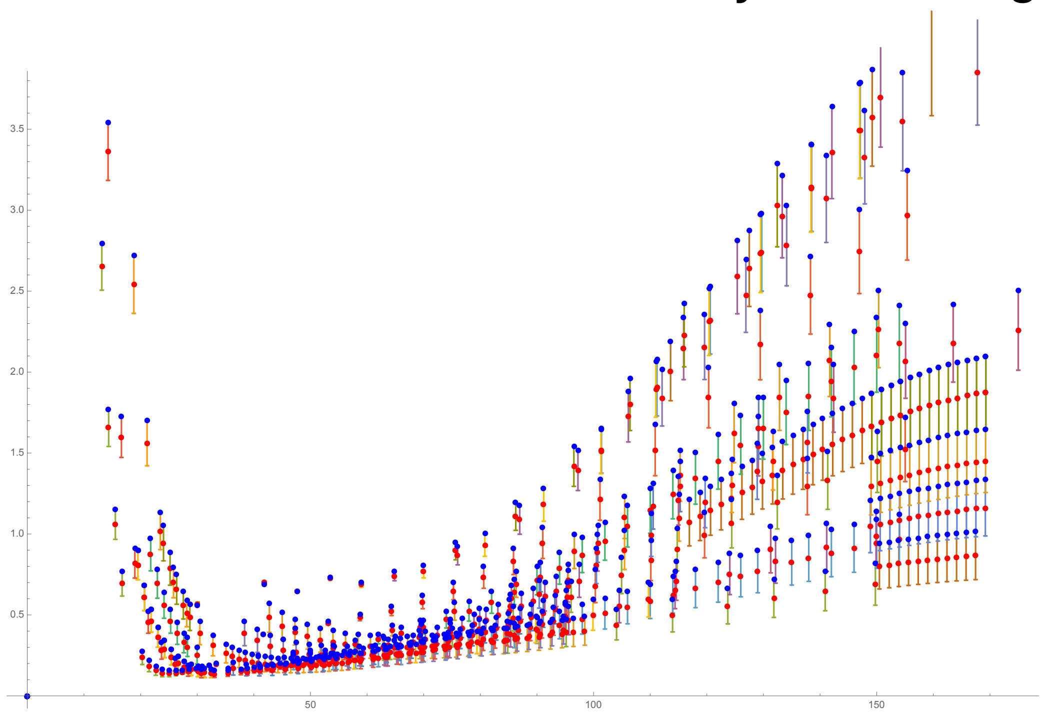


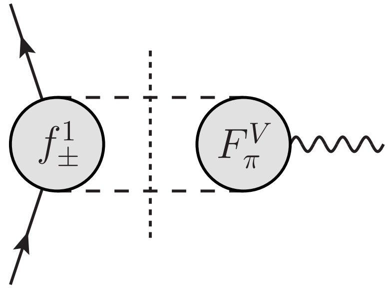
$\chi^2_{\text{HA}}/\text{d.o.f.} \approx 0.8$

$\chi^2_{\text{KH80}}/\text{d.o.f.} \approx 4.7$

$\longrightarrow$ scatt. lengths from had. atoms compatible with low-energy πN scatt. data

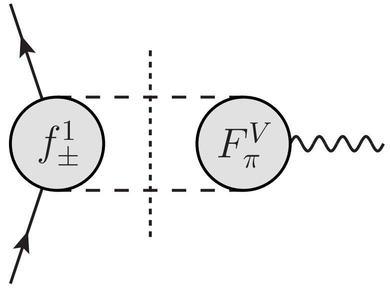
preliminary!

Nucleon form factor spectral functions



- $\pi\pi \to \bar{N}N$ partial waves + pion form factor
  $\longrightarrow \pi\pi$ contrib. to isovector spectral function
- consistent $\pi\pi$ phase shifts in $f^\pm_1$ and F^V_π
- modern pion form factor data BaBar 2009, KLOE 2012, BESIII 2015
- isospin breaking: $m_p - m_n$ in nucleon pole terms — effects in subthreshold parameters to $\mathcal{O}(pe^2)$ — consistent ρ–ω mixing

Nucleon form factor spectral functions



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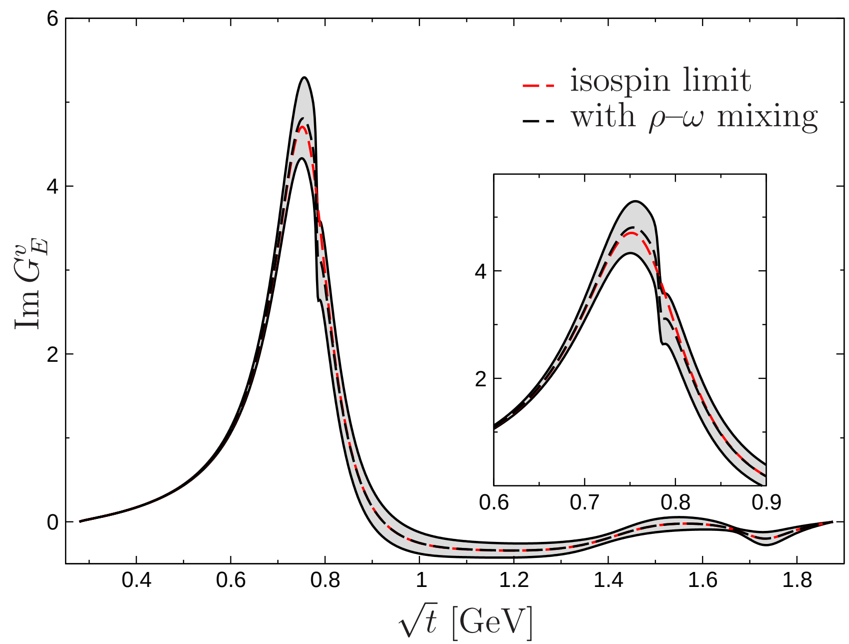
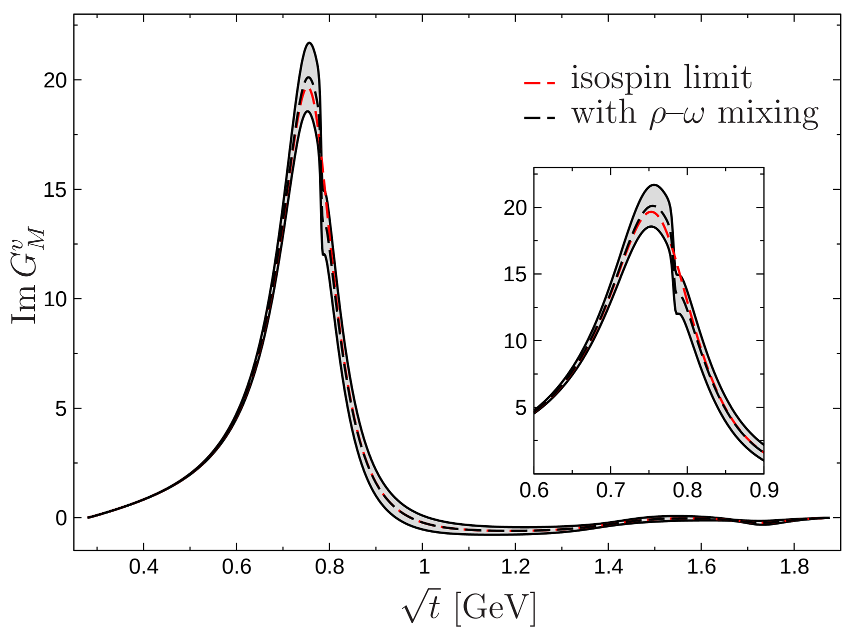
- isospin breaking: $m_p - m_n$ in nucleon pole terms — effects in subthreshold parameters to $\mathcal{O}(pe^2)$ — consistent ρ–ω mixing





Hoferichter, BK, Ruiz de Elvira, Hammer, Meißner 2016

$\pi\pi$ continuum and proton radius puzzle

- sum rules for isovector radii: $$\langle r^2_{E/M}\rangle^v = \frac{6}{\pi}\int_{4M_\pi^2}^{\Lambda} dt' \frac{\mathrm{Im}\, G^v_{E/M}(t')}{t'^2}$$

	$\Lambda = 1\,\text{GeV}$	$\Lambda = 2m_N$
$\langle r^2_E\rangle^v$ [fm^2]	0.418(32)	0.405(36)
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- with $\langle r^2_E\rangle^n = -0.1161(22)\,\text{fm}^2$ (n scattering on heavy atoms):
  proton radius puzzle $\hat{=}$ isovector radius puzzle
  $\langle r^2_E\rangle^v = 0.412\,\text{fm}^2$ (μH) vs. $\langle r^2_E\rangle^v = 0.442\,\text{fm}^2$ (CODATA)
  $\longrightarrow$ mild preference for small proton charge radius

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Chiral low-energy constants

- chiral expansion expected to work best at subthreshold point:
  - ▷ maximal distance from threshold singularities
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	LO	NLO	NNLO
c_1 [GeV^{-1}]	-0.74 ± 0.02	-1.07 ± 0.02	-1.11 ± 0.03
c_2 [GeV^{-1}]	1.81 ± 0.03	3.20 ± 0.03	3.13 ± 0.03
c_3 [GeV^{-1}]	-3.61 ± 0.05	-5.32 ± 0.05	-5.61 ± 0.06
c_4 [GeV^{-1}]	2.17 ± 0.03	3.56 ± 0.03	4.26 ± 0.04

$\longrightarrow$ subthreshold errors tiny, chiral expansion dominates uncertainty

$\longrightarrow$ higher-order LECs much more problematic — see talk by D. Siemens

Summary

Pion–nucleon Roy–Steiner equations

- allow to determine low-energy πN scattering with precision
  - ▷ obeying analyticity, unitarity, crossing symmetry
  - ▷ new input on scattering lengths from hadronic atoms
- provide πN phase shifts with systematic uncertainties
- phenomenological determination of sigma term:

  $$\sigma_{\pi N} = 59.1 \pm 3.5\,\text{MeV}$$

  currently at odds with lattice QCD results
- consistency check: Karlsruhe–Helsinki input leads to Karlsruhe–Helsinki results
- t-channel $\longrightarrow$ nucleon form factor spectral functions
  sum rules for isovector radii $\longrightarrow$ proton radius puzzle
- chiral low-energy constants obtained algebraically from subtreshold coefficients $\longrightarrow$ to be used in chiral NN potentials

Spares

Roy–Steiner equations: subtractions

	$m_{f_{0+}^{1/2}}$	$m_{f_{0+}^{3/2}}$	$m_{f_{1+}^{1/2}}$	$m_{f_{1+}^{3/2}}$	$m_{f_{1-}^{1/2}}$	$m_{f_{1-}^{3/2}}$	m
$W_+ \leq W_\mathrm{m} \leq 1.20\,\mathrm{GeV}$	0	-1	-1	0	-1	-1	-4
$1.20\,\mathrm{GeV} \leq W_\mathrm{m} \leq 1.23\,\mathrm{GeV}$	0	-1	-1	0	0	-1	-3
$1.23\,\mathrm{GeV} \leq W_\mathrm{m} \leq 1.52\,\mathrm{GeV}$	0	-1	-1	1	0	-1	-2
$1.52\,\mathrm{GeV} \leq W_\mathrm{m} \leq 1.69\,\mathrm{GeV}$	0	-1	-1	1	1	-1	-1
$1.69\,\mathrm{GeV} \leq W_\mathrm{m} \leq 1.80\,\mathrm{GeV}$	1	-1	-1	1	1	-1	0

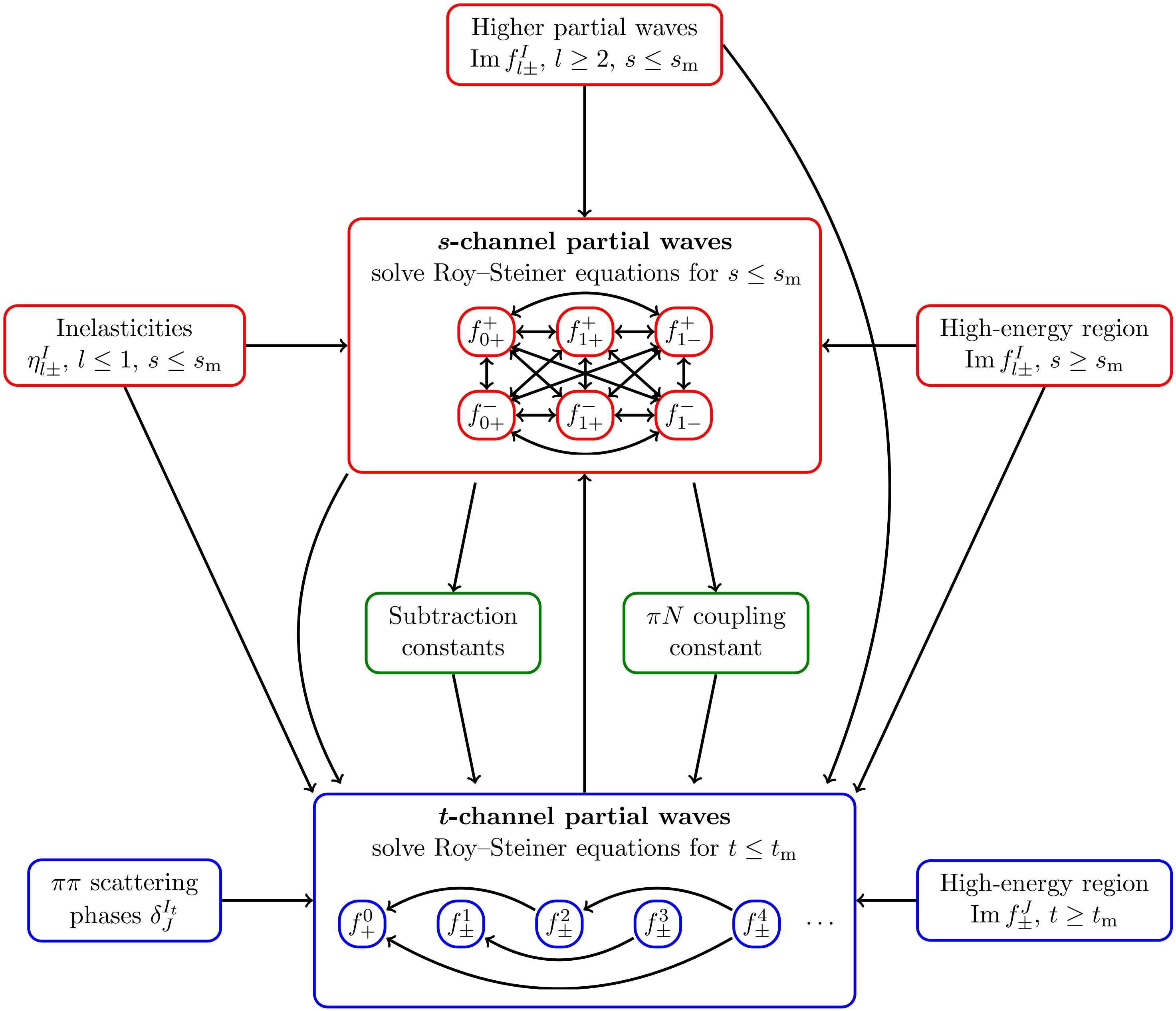
- $m = -2$ at $W_\mathrm{m} = 1.38\,\mathrm{GeV}$

  + 6 no-cusp conditions for 2 S- and 4 P-waves

  $\longrightarrow$ 8 free parameters (subtraction constants = subthreshold parameters) to guarantee unique solution
- 2 S-wave scattering lengths as additional constraints

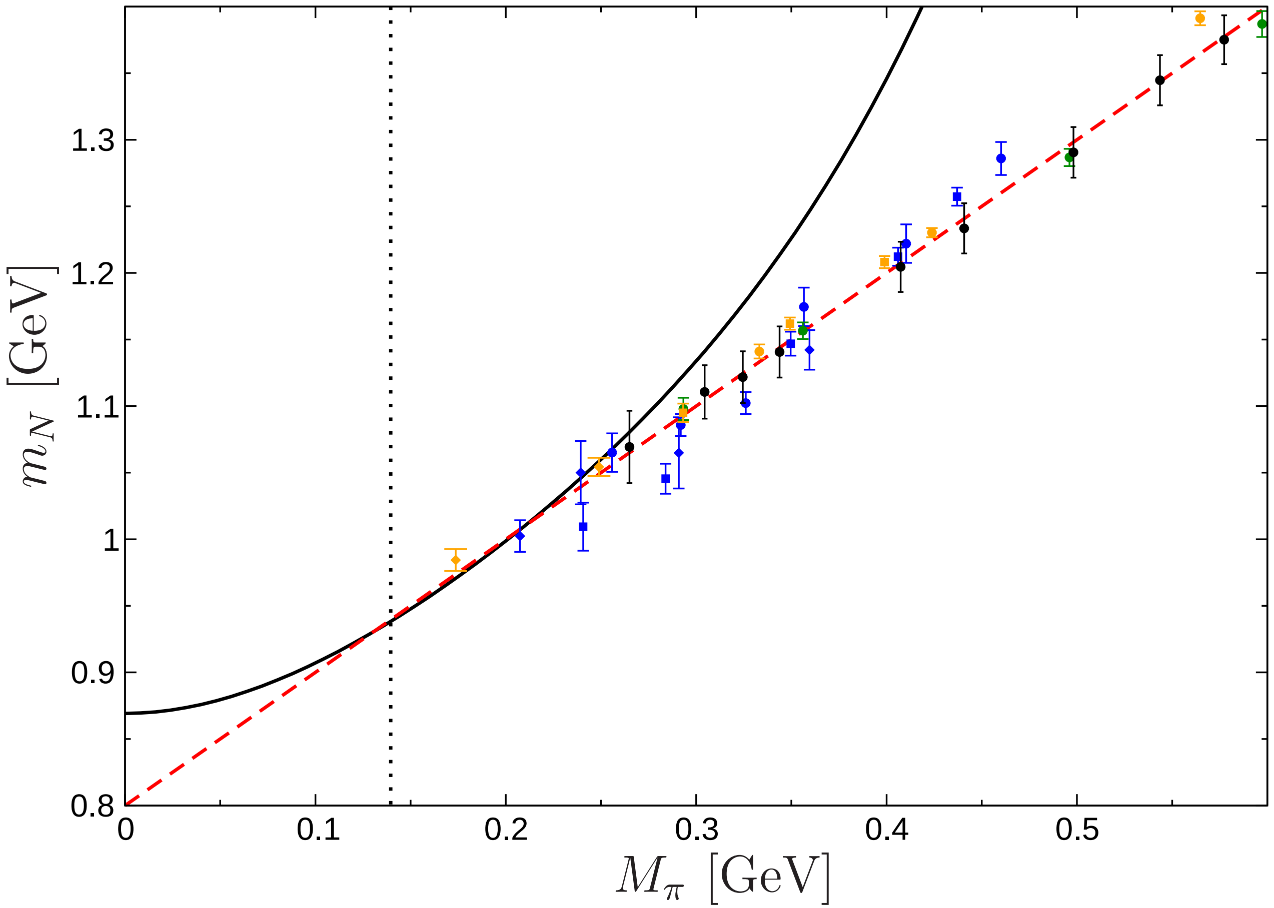
  $\longrightarrow$ need 10 free parameters
- correspond to (partially) 3 subtractions

Roy–Steiner equations: information flowchart



The "ruler plot" vs. ChPT

- pion mass dependence of m_N, using
  - ▷ c_1 from subthreshold matching to Roy–Steiner solution
  - ▷ combination of e_i from $\sigma_{\pi N}$



thanks to A. Walker-Loud for providing the lattice data

Nucleon strangeness

- relate $\sigma_{\pi N}$ to strangeness content of the nucleon:

$$\sigma_{\pi N} = \frac{\hat{m}}{2m_N} \frac{\langle N|\bar{u}u + \bar{d}d - 2\bar{s}s|N\rangle}{1-y}\,, \qquad y = \frac{2\langle N|\bar{s}s|N\rangle}{\langle N|\bar{u}u + \bar{d}d|N\rangle}$$

$(m_s - \hat{m})(\bar{u}u + \bar{d}d - 2\bar{s}s) \subset \mathcal{L}_{\text{QCD}}$ produces SU(3) mass splittings:

$$\sigma_{\pi N} = \frac{\sigma_0}{1-y}\,, \qquad \sigma_0 = \frac{\hat{m}}{m_s - \hat{m}}\big(m_\Xi + m_\Sigma - 2m_N\big) \simeq 26 \text{ MeV}$$

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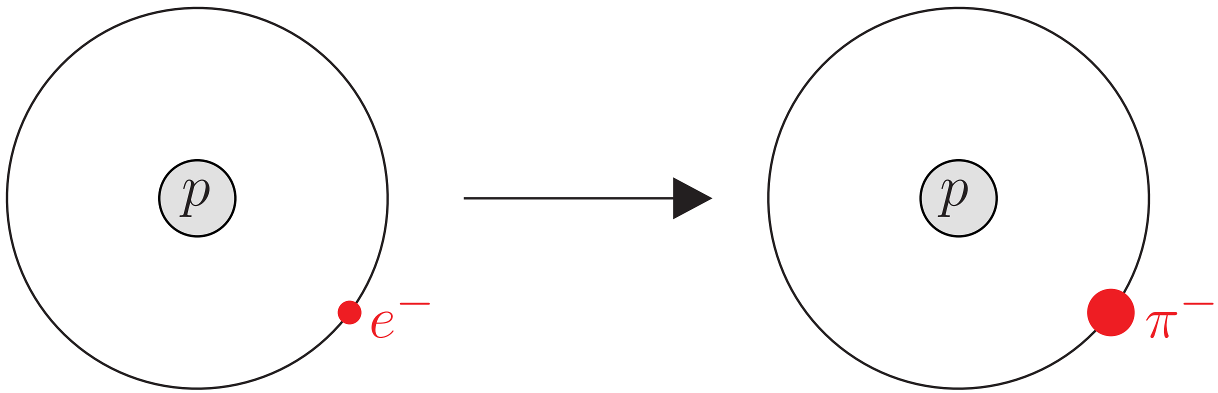
- conclusion:
  - ▷ $\sigma_{\pi N} = (59.1 \pm 3.5)$ MeV not incompatible with small y
  - ▷ chiral convergence of σ_0 (hence $\langle N|\bar{s}s|N\rangle$) very doubtful

Pionic atoms and pion–nucleon scattering lengths

cf. Gasser, Lyubovitskij, Rusetsky 2008

- pionic hydrogen πH, pionic deuterium πD: atoms with $e^- \to \pi^-$
  calculate energy levels as for hydrogen in quantum mechanics!



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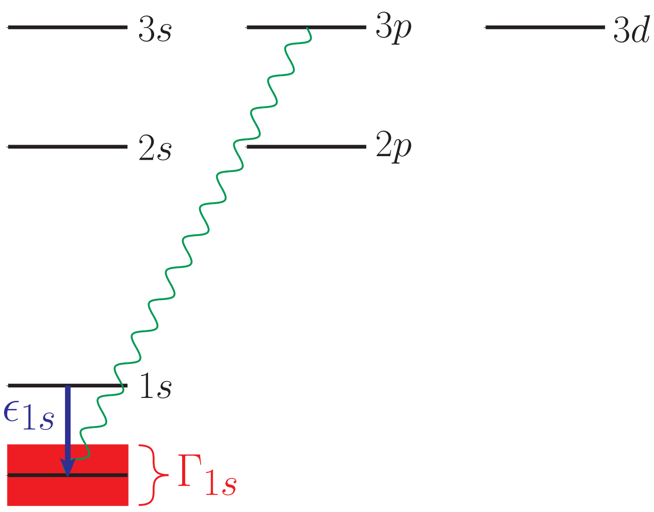
- energy levels perturbed by strong interactions:
  - ▷ ground state instable, decays:
    $\pi^- p \to \pi^0 n \quad \longrightarrow$ width Γ_{1s}
  - ▷ ground state energy shift ϵ_{1s}



- linked to πN scattering at threshold:

$$\epsilon_{1s} \propto T(\pi^- p \to \pi^- p) \propto a_0^+ + a_0^-$$
$$\Gamma_{1s} \propto |T(\pi^- p \to \pi^0 n)|^2 \propto |a_0^-|^2$$

Deser, Goldberger, Baumann, Thirring 1954

- πD: add. information from energy shift (diff. isospin combination)