S-matrix solution of the Lippmann-Schwinger equation for regular and singular potentials

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- [II] Long version, in preparation

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Overview

1. Lippmann-Schwinger equation
2. New exact equation in NR scattering theory
3. LS equation in the complex plane
4. N/D method with non-perturbative $\Delta(A)$
5. Regular interactions
6. Singular Interactions
7. $T(A)$ in the complex plane
8. Conclusions

Lippmann-Schwinger equation (LS)

Potential two-body scattering. Given a potential V

Scattering T-matrix $T(z)$, Im$(z) \neq 0$

$$\begin{aligned} T(z) =& V - V R_0(z) T(z) \\ R_0(z) =& [H_0 - z]^{-1} \\ H_0 =& -\frac{1}{2\mu}\nabla^2 \\ H =& H_0 + V \end{aligned}$$

- LS in partial waves

$$T_\ell(p', p, z) = V_\ell(p', p) + \frac{\mu}{\pi^2}\int_0^\infty dq q^2 \frac{V_\ell(p', q) T_\ell(q, p, z)}{q^2 - 2\mu z}$$

Criterion for Singular Potentials

$$V(r) \xrightarrow[r \to 0]{} \alpha r^{-\gamma}$$

$$\bar{\alpha} = \alpha + \ell(\ell+1)$$

Potential	Ordinary	Singular
γ	< 2	> 2
$\gamma = 2$	$\bar{\alpha} > 0$	$\bar{\alpha} \leq 0$

Ordinary/Regular Potentials:

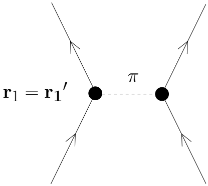
Standard quantum mechanical treatment
Boundary condition: $u(0) = 0$
No extra free parameters

The One-Pion-Exchange (OPE) potential for the singlet NN interaction $(r > 0)$:

Yukawa potential

$$V(r) = -\,\tau_1 \cdot \tau_2 \left(\frac{g_A m_\pi}{2 f_\pi} \right)^2 \frac{e^{-m_\pi r}}{4\pi r}$$

Exchange of a pion between two nucleons



In many instances one has singular potentials

- **Molecular physics: Van der Waals Force**

$$V(r) = -\frac{3\alpha_A \alpha_B I_A I_B / 2(I_A + I_B)}{\mathbf{r}^6} + \sum_{n=4} \frac{\lambda_n}{r^{2n}}$$

Nuclear physics

Triplet component of one-Pion Exchange (OPE)

$$T(r) = \frac{e^{-m_\pi r}}{r}\left[1 + \frac{3}{m_\pi r} + \frac{3}{(m_\pi r)^2}\right] \xrightarrow[r \to 0]{} r^{-3}$$

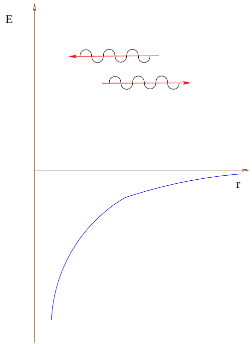
Higher orders in CHPT add extra powers of $1/r^n$

Particle physics:

Scattering of heavy-quarkonium states, color dipoles, QCD analog to Van der Waals forces
pNRQCD, pNRQED
NR Quark Models, etc

- Full range $r \in]0, \infty[$: Case, PR60,797(1950)

$$-u''(r) + \left(2\mu V(r) + \frac{\ell(\ell+1)}{r^2} - k^2\right) u(r) = 0$$



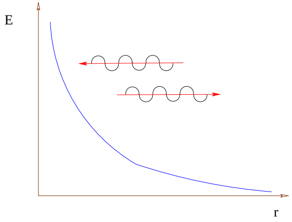
- Singular Attractive Potential

Two linearly independent wave functions
One has to fix a relative phase, $\varphi(p)$ **How to do it?.**

Which are the appropriate boundary conditions?
Historical Mess

Theoretical control within LS only for $\varphi(p) = const.$
Case, PR60,797('50); Arriola, Pavón, PRC72,054002('05)

The attractive singular potential does not determine uniquely the scattering problem Plesset, PR41,278(1932), Case, PR60,797(1950)



- Singular Repulsive Potential

There is only one finite (vanishing) reduced wave function at $r = 0$

The solution is fixed

Pavón Valderrama, Ruiz Arriola, Ann.Phys.323,1037(2008)

New exact equation in NR scattering theory

Yukawa potential,

$$V(\mathbf{q}) = \frac{2g}{\mathbf{q}^2 + m_\pi^2}$$

Singularity for $\mathbf{q}^2 = -m_\pi^2$

$^1S_0$ potential: $(^{2S+1}L_J)$ $\qquad g = (g_A m_\pi/\sqrt{8} f_\pi)^2$

$$V(p) = \frac{g}{2p^2}\log(4p^2/m_\pi^2 + 1)$$

$$\Delta_{1\pi}(p^2) = \frac{V(p^2 + i0^+) - V(p^2 - i0^+)}{2i} = \mathrm{Im}V(p^2 + i0^+) = \frac{g\pi}{2p^2}$$

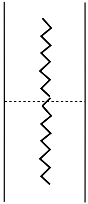
Left-hand cut (LHC) discontinuity **for On-shell scattering**

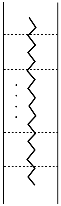
$$p^2 < -m_\pi^2/4 = L$$

Born approximation

Full LHC Discontinuity , $p^2 = -k^2 < L$

$$2i\Delta(p^2) = T(p^2 + i0^+) - T(p^2 - i0^+)$$
$$\Delta(p^2) = \mathrm{Im}T(p^2 + i0^+)$$





The LS generates contributions with any number of pions to $\Delta(p^2)$, $p^2 < L$

$\Delta_{n\pi}(p^2)$ for $p^2 < -n^2 m_\pi^2/4$

Notation: $A = p^2$; How to calculate $\Delta(A)$?:

G.E. Brown, A.D. Jackson "The Nucleon-Nucleon interaction", North-Holland, 1976. Page 86: *"In practice, of course, we do not know the exact form of $\Delta(p^2)$ for a given potential ..."*

$$p = ik \pm \varepsilon \ , \ \varepsilon = 0^+ \ , \ p^2 = -k^2 < L$$

$$T(ik \pm \varepsilon, ik \pm \varepsilon) = V(ik \pm \varepsilon, ik \pm \varepsilon) + \frac{\mu}{2\pi^2} \int_0^\infty dq q^2 \frac{V(ik \pm \varepsilon, q) T(q, ik \pm \varepsilon)}{q^2 + k^2}$$

The last integral, so calculated, IS PURELY REAL!!

You can try to calculate numerically just the once iterated OPE

$$\frac{\mu}{2\pi^2} \int_0^\infty dq q^2 \frac{V(ik \pm \varepsilon, q) V(q, ik \pm \varepsilon)}{q^2 + k^2} \in \mathbb{R}$$

- **GENERAL method:**

Analytic extrapolation of the LS from its integral expression

$$\Delta(A) = \frac{1}{2}\mathfrak{f}(-k)\ , \ \ A = -k^2$$

$$\mathfrak{f}(\nu) = \Delta v(\nu,k) + \frac{\theta(k-2m_\pi-\nu)m}{2\pi^2}\int_{m_\pi+\nu}^{k-m_\pi}\frac{d\nu_1\nu_1^2}{k^2-\nu_1^2}\Delta v(\nu,\nu_1)\mathfrak{f}(\nu_1)$$

$$\text{IE}:\ -k+m_\pi<\nu<k-m_\pi$$

- The limits in the IE ARE FINITE
- The denominator never vanishes , $|\nu_1| \le k - m_\pi$ in the IE
- **NO FREE PARAMETERS**

Reason: Contact interactions (monomials) do not contribute to the discontinuity of $T(A)$
Short-distance physics is not resolved→ Contact interactions

$$\Delta(A) = \frac{\mathfrak{f}(-k)}{2} \; , \; k = \sqrt{-A}$$
$$\mathfrak{f}(\nu) = \Delta v(\nu,k) + \frac{\theta(k-2m_\pi-\nu)m}{2\pi^2}\int_{m_\pi+\nu}^{k-m_\pi}\frac{d\nu_1\nu_1^2}{k^2-\nu_1^2}\Delta v(\nu,\nu_1)\mathfrak{f}(\nu_1)$$

$$\text{IE}: \; -k+m_\pi < \nu < k-m_\pi$$

It can be applied to:

- Any local potential (spectral decomposition:)

$$V(\mathbf{p}',\mathbf{p}) = \frac{1}{\pi}\int_{\mu_0^2}^{\infty} d\mu^2 \frac{\eta(\mu^2)}{\mathbf{q}^2+\mu^2} \; , \; \mathbf{q} = \mathbf{p}'-\mathbf{p}$$

- Higher partial waves, $\ell \geq 0$
- Coupled Channels
- Nonlocal potentials due to relativistic corrections $\to \eta(\mu^2)$

LS equation in the complex plane

Analytical properties of the potential

- Local potential, spectral decomposition:

$$V(\mathbf{q}^2) = \frac{1}{\pi}\int_{\mu_0^2}^{\infty} d\mu^2 \frac{\eta(\mu^2)}{\mathbf{q}^2+\mu^2}\ ,\ \ \mathbf{q}=\mathbf{p}_2-\mathbf{p}_1$$

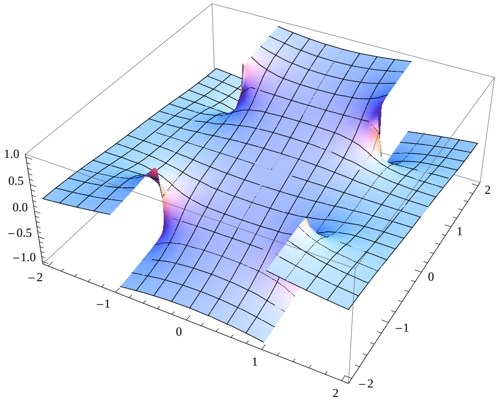
- S-wave projection:

$$v(p_1,p_2) = \frac{1}{2\pi}\int_{-1}^{+1} dt \int_{\mu_0^2}^{\infty} d\mu^2 \frac{\eta(\mu^2)}{p_1^2+p_2^2-2p_1p_2t+\mu^2}$$
$$= \frac{1}{4\pi p_1 p_2}\int_{\mu_0^2}^{\infty} d\mu^2 \eta(\mu^2)$$
$$\times\left\{\log\left[\mu^2+(p_1+p_2)^2\right]-\log\left[\mu^2+(p_1-p_2)^2\right]\right\}$$

Vertical cuts:

$$p_2 = \pm(p_1 \pm i\sqrt{m_\pi^2 + x^2})\ x \in \mathbb{R}$$

Analogously for p_1



$p_1 = m_\pi$. Branch points at $\pm(p_1 \pm i m_\pi)$

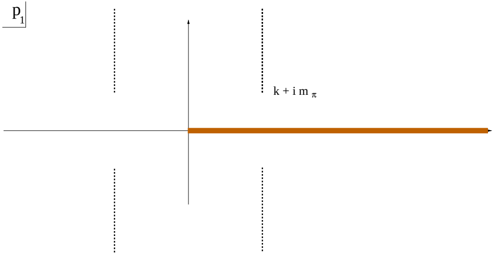
Deforming the integration contour in the LS equation

$k, \ k' \in \mathbb{R}$ in the half-off-shell T-matrix $t(k,k';{k'}^2/m)$,

$$t(k,k';\frac{{k'}^2}{m}) = v(k,k') + \frac{m}{2\pi^2}\int_0^\infty \frac{dp_1 p_1^2}{p_1^2 - {k'}^2} v(k,p_1) t(p_1,k';\frac{{k'}^2}{m}) \ ,$$

$v(k,p_1)$ implies the vertical cuts

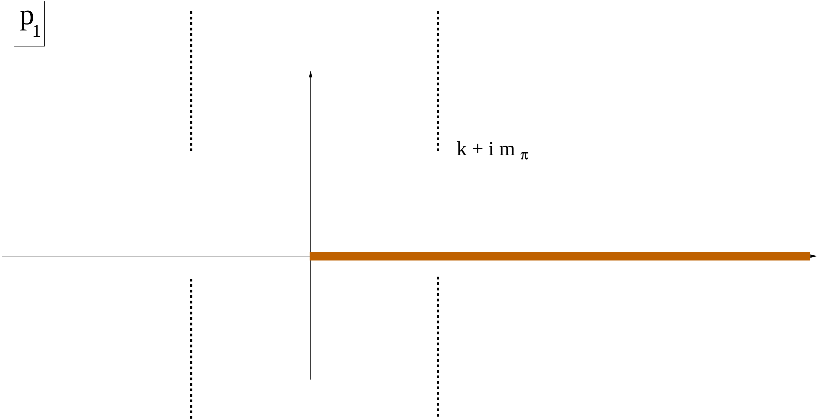
$$p_1 = \pm(k \pm i\sqrt{m_\pi^2 + x^2}) \ x \in \mathbb{R}$$



We add an increasing positive imaginary part to k

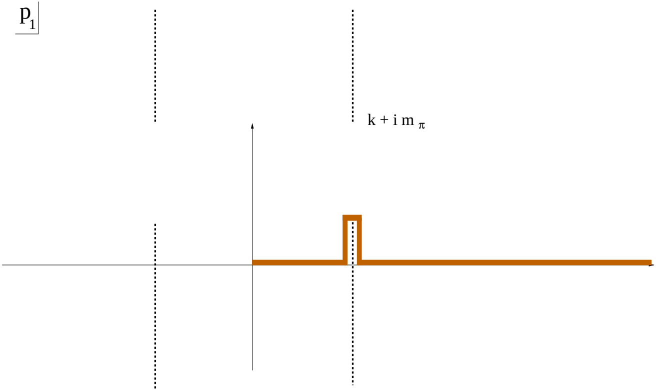
$$k = k_r + i\,k_i \;,\; k_i > 0$$

$$p_1 = \pm\,(k_r + i\,k_i \pm i\sqrt{m_\pi^2 + x^2})\;\; x \in \mathbb{R}$$

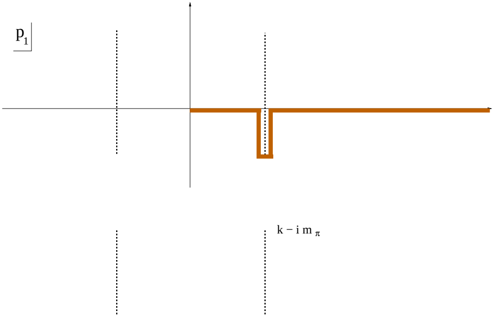


We add an increasing positive imaginary part to k

$$k = k_r + i\, k_i \ , \ k_i > m_\pi$$

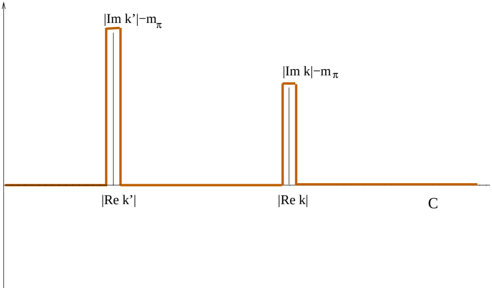


$$k_r > 0 \ , \ k_i > m_\pi$$
$$k_r < 0 \ , \ k_i < -m_\pi$$



$$k_r > 0\ ,\ \ k_i < -m_\pi$$
$$k_r < 0\ ,\ \ k_i > m_\pi$$

- $t(p_1, k'; {k'}^2/m)$ follows the same pattern in terms of k'.



Calculation of $\Delta(-k^2)$: Discontinuity across the LHC

On-shell scattering $t(k,k;k^2/m)$
LHC:

$$p = -p \pm i\sqrt{m_\pi^2 + x^2} \longrightarrow p = \pm\frac{i}{2}\sqrt{m_\pi^2 + x^2}$$

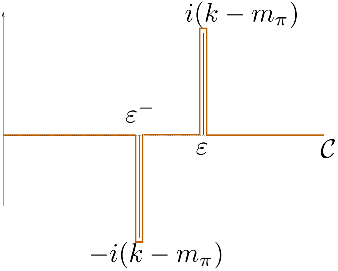
$$p^2 = -\frac{1}{4}(m_\pi^2 + x^2) \longrightarrow p^2 \in]-\infty, L]\ ,\ L = -m_\pi^2/4$$

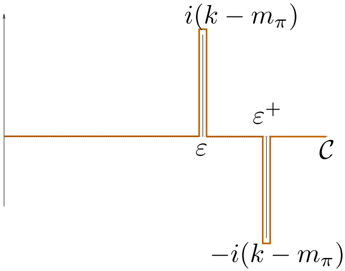
$$2i\Delta(-k^2) = t(ik + i\varepsilon, ik + i\varepsilon) - t(ik - i\varepsilon, ik + i\varepsilon)$$

$$= (-1)^\ell \left\{ t(-ik + \varepsilon^-, ik + \varepsilon) - t(-ik + \varepsilon^+, ik + \varepsilon) \right\}$$

$$\varepsilon^- < \varepsilon < \varepsilon^+$$

$$t(-ik+\varepsilon^-, ik+\varepsilon) \qquad\qquad t(-ik+\varepsilon^+, ik+\varepsilon)$$





$$
\begin{aligned}
&\mathrm{Im}\, t(-ik+\varepsilon^-, ik+\varepsilon) - \mathrm{Im}\, t(-ik+\varepsilon^+, ik+\varepsilon) \\
&= \mathrm{Im}\, v(i\nu+\varepsilon^-, ik+\varepsilon) - \mathrm{Im}\, v(i\nu+\varepsilon^+, ik+\varepsilon) \\
&+\theta(k-\nu-2m_\pi)\frac{m}{2\pi^2}\int_{-k+m_\pi}^{k-m_\pi}\frac{d\nu_1\nu_1^2}{k^2-\nu_1^2} \\
&\times\left[\mathrm{Im}\, v(i\nu+\varepsilon^-, i\nu_1+\varepsilon) - \mathrm{Im}\, v(i\nu+\varepsilon^+, i\nu_1+\varepsilon)\right] \\
&\times\left[\mathrm{Im}\, t(i\nu_1+\varepsilon-\delta, ik+\varepsilon) - \mathrm{Im}\, t(i\nu_1+\varepsilon+\delta, ik+\varepsilon)\right] .
\end{aligned}
$$

- **One needs to know**

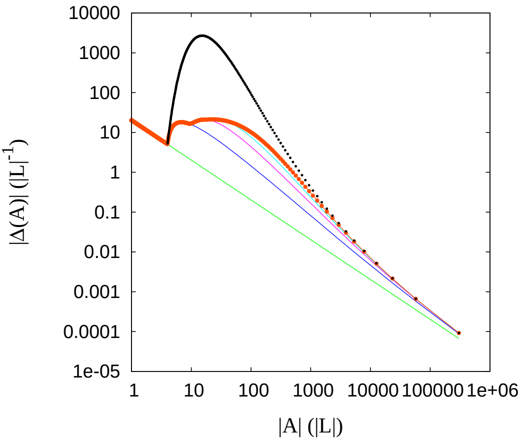
$$
\begin{aligned}
&\mathrm{Im}\, t(i\nu+\varepsilon^-, ik+\varepsilon) - \mathrm{Im}\, t(i\nu+\varepsilon^+, ik+\varepsilon) \\
&-k+m_\pi < \nu < k-m_\pi
\end{aligned}
$$

Proceeding in the same way
Integral Equation $-k+m_\pi<\nu<k-m_\pi$:

$$\begin{aligned}\mathfrak{f}(\nu)\equiv&\operatorname{Im}t(i\nu+\varepsilon^-,ik+\varepsilon)-\operatorname{Im}t(i\nu+\varepsilon^+,ik+\varepsilon)\\ =&\operatorname{Im}v(i\nu+\varepsilon^-,ik+\varepsilon)-\operatorname{Im}v(i\nu+\varepsilon^+,ik+\varepsilon)\\ &+\theta(k-\nu-2m_\pi)\frac{m}{2\pi^2}\int_{\nu+m_\pi}^{k-m_\pi}\frac{d\nu_1\nu_1^2}{k^2-\nu_1^2}\\ &\times\left[\operatorname{Im}v(i\nu+\varepsilon^-,i\nu_1+\varepsilon)-\operatorname{Im}v(i\nu+\varepsilon^+,i\nu_1+\varepsilon)\right]\\ &\times\left[\operatorname{Im}t(i\nu_1+\varepsilon-\delta,ik+\varepsilon)-\operatorname{Im}t(i\nu_1+\varepsilon+\delta,ik+\varepsilon)\right]\ .\end{aligned}$$

$$\Delta(A)=(-1)^\ell\frac{\mathfrak{f}(-k)}{2}$$

log-log plot for $^1S_0$ (Yukawa Pot.) $\Delta(A)$; $g_A = 6.80$



- $|L| = m_\pi^2/4$; $\Delta_{1\pi}$, $\Delta_{2\pi}$, $\Delta_{3\pi}$, $\Delta_{4\pi}$,
  Asymptotic sol. (dots) $|A| \gg m_\pi^2$
- Full solution $\Delta(A)$

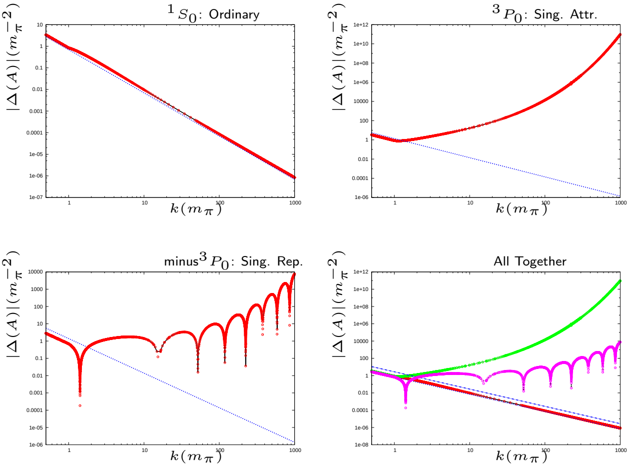
Yukawa potential

- Asymptotic solution for $k \gg m_\pi$

$$\frac{\mathfrak{f}'(\nu)}{\mathfrak{f}(\nu)} = -\lambda \frac{\theta(k - 2m_\pi - \nu)}{k^2 - (m_\pi + \nu)^2}$$

$$\begin{aligned} \Delta(A) &= \frac{\lambda \pi^2}{M_N A} e^{\frac{2\lambda}{\sqrt{-A}} \operatorname{arctanh}\left(1 - \frac{m_\pi}{\sqrt{-A}}\right)} \\ \lambda &= \frac{g M_N}{2\pi} \end{aligned}$$

$^3P_0$: singular attractive potential; m^3P_0:singular repulsive potential ($g \to -g$)



$k \to +\infty$:
$^3P_0$:"Exponential" growth
m$^3P_0$:Oscillatory "Exponential" growth

$^1S_0$:Vanishes

Qualitative difference

• Ordinary Potential $\Delta(A)$ vanishes for $A \to -\infty$

• For the attractive singular potentials $|\Delta(A)|$ grows faster than any power

• **The full non-perturbative solution for a singular and ordinary potentials are qualitatively different**

With singular potentials short- and long-range physics are interrelated

Nonperturbative Solutions: Any number of counterterms are not effective

N/D method with non-perturbative $\Delta(A)$

Once we now the exact $\Delta(A)$ for a given potential we can use S-matrix theory to solve the LS: N/D **method with the full** $\Delta(A)$

$$T_{J\ell S}(A) = \frac{N_{J\ell S}(A)}{D_{J\ell S}(A)}$$

$N_{J\ell S}(A)$ has Only LHC

$D_{J\ell S}(A)$ has Only RHC

$$\mathrm{Im} D_\ell(A) = -N_\ell(A)\rho(A)\ , \ A > 0\ (\mathbf{RHC - Unitarity})$$

$$\mathrm{Im} N_\ell(A) = D_\ell(A)\,\Delta(A)\ , \ A < L\ (\mathbf{LHC})$$

(m_1, m_2) N/D equations for $D(A)$ and $N(A)$

$$N/D_{m_1\, m_2}$$

$$N(A) = \sum_{i=1}^{m_1} \nu_i (A-C)^{m_1-i} + \frac{(A-C)^{m_1}}{\pi} \int_{-\infty}^{L} dk^2 \frac{\Delta(k^2) D(k^2)}{(k^2-A)(k^2-C)^{m_1}}$$

$$D(A) = \sum_{i=1}^{m_2} \delta_i (A-C)^{m_2-i} - \frac{(A-C)^{m_2}}{\pi} \int_{0}^{\infty} dq^2 \frac{\rho(q^2) N(q^2)}{(q^2-A)(q^2-C)^{m_2}}$$

- $N(A)$ is substituted in $D(A)$
- Linear IE for $D(A)$ arises
- $D(0) = 1$. To fix a floating constant in the ratio $T(A) = N(A)/D(A)$

Regular interactions

- N/D_{01}: **Regular solution for an ordinary potential**

Scattering is completely fixed by the potential

$$D(A) = 1 - \frac{i\mu\sqrt{A}}{2\pi^2}\int_{-\infty}^{L} d\omega_L \frac{\Delta(\omega_L)D(\omega_L)}{\sqrt{\omega_L}\left(\sqrt{\omega_L}+\sqrt{A}\right)}$$

- N/D_{11}: **Additional subtraction in $N(A)$ is fixed in terms of scattering length**

$$D(A) = 1 + ia\sqrt{A} + i\frac{M_N}{4\pi^2}\int_{-\infty}^{L} d\omega_L \frac{D(\omega_L)\Delta(\omega_L)}{\omega_L}\frac{A}{\sqrt{A}+\sqrt{\omega_L}}$$

Effective Range Expansion (ERE)

$$k\cot\delta(k) = -\frac{1}{a} + \frac{1}{2}rk^2 + \sum_{i=2} v_i k^{2i}$$

- **N/D_{12}: Additional subtraction in $D(A)$, r is fixed**

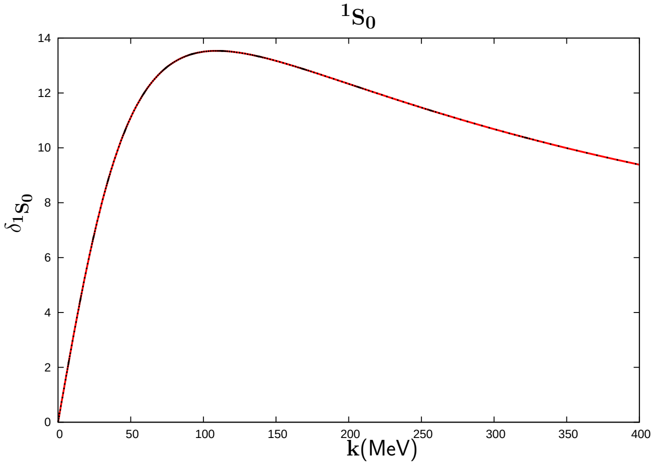
$$D(A) = 1 + ia\sqrt{A} - \frac{ar}{2}A - i\frac{M_N A}{4\pi^2}\int_{-\infty}^{L} d\omega_L \frac{D(\omega_L)\Delta(\omega_L)}{\omega_L} \times \left[\frac{\sqrt{A}}{(\sqrt{\omega_L}+\sqrt{A})\sqrt{\omega_L}} - \frac{i}{a\omega_L}\right]$$

- **N/D_{22}: v_2 is fixed additionally**

$$D(A) = (1 - \frac{2v_2}{r}A)(1 + ia\sqrt{A}) - \frac{ar}{2}A + i\frac{M_N}{4\pi^2}A\int_{-\infty}^{L} d\omega_L \frac{D(\omega_L)\Delta(\omega_L)}{\omega_L^2} \times \left[\frac{A}{\sqrt{A}+\sqrt{\omega_L}} + i\frac{2}{ra^2\omega_L}(1+ia\sqrt{\omega_L})(1+ia\sqrt{A})\right]$$

The results are just dependent on $\Delta(A)$ (input potential) and experimental ERE parameters

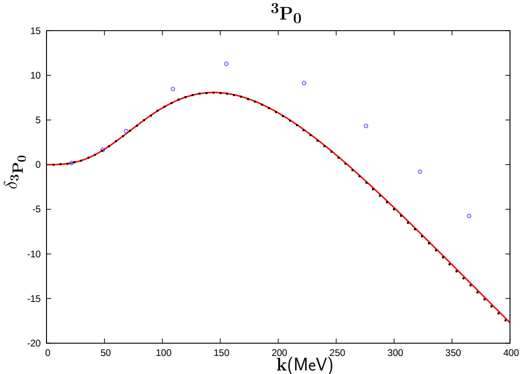
Example: Regular case. $^1S_0$ Yukawa potential



N/D_{01}; **LS** (black dots)

Analytical properties determine the solutions for singular potentials

Attractive singular interaction: $^3P_0$
$\mathbf{N/D_{12}}$ $T(A)=0$ (N/D_{11} does not converge)
At least one parameter is needed The scattering volume is fixed



We compare with LS renormalized with one contact term C_1:

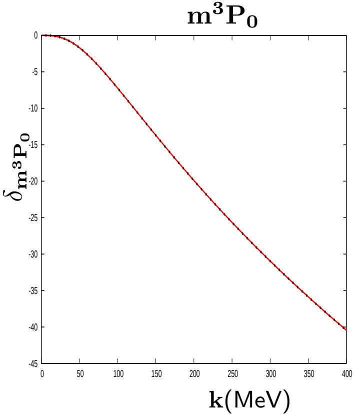
$$V(p_1,p_2) \to V(p_1,p_2) + C_1 p_1 p_2$$

N/D_{12};
LS (black dots);
Phase shifts: Granada analysis

Repulsive singular interaction: Minus-$^3P_0$ ($g \to -g$)

$\mathbf{N/D_{01}}$ No free parameters

Repulsive Singular Potential: LS is insensitive to all C_i



N/D_{12};
LS (black dots);

Attractive singular interaction: Sensitivity to scatter inner structure

One can go beyond the case of just one counterterm

Example: NN $^1S_0$ partial wave

LS renormalized with contact terms:

$$V(p_1, p_2) \to V(p_1, p_2) + C_0 + C_1(p_1^2 + p_2^2) + \ldots$$

LS is insensitive or not convergent when including C_i, $i > 0$
Entem, Arriola, Pavon, Machleidt PRC77,044006('08)

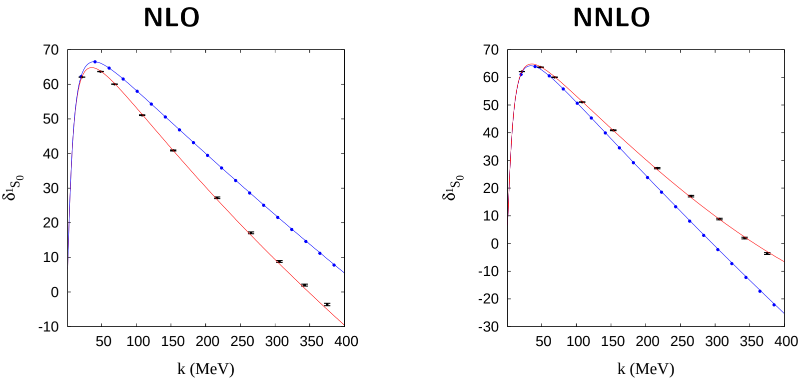
NLO and NNLO ChPT NN potentials

Full spectral decomposition of the potential

- **Not all the N/D types of IE's converge for sing. pot. (they do for ord. pot.)**

- N/D_{11} (a) and N/D_{22} (a, r, v_2) are convergent

- N/D_{01} (at least one parameter is needed) and N/D_{12} are not convergent



N/D_{11} , N/D_{22}

Subtractive-renormalized LS: points

One counterterm *It cannot reproduce* r

Yang,Elster,Phillips, PRC80,044002('09)

$a = -23.75$, $r = 2.655$ fm , $v_2 = -0.6265$ fm^3

$T(A)$ in the complex plane

• As a **bonus** the non-perturbative-Δ N/D method allows to calculate $T(A)$ for $A \in \mathbb{C}$ in the 1st/2nd Riemann sheet

This is not trivial with LS

Look for and study resonances, virtual states and bound states

For bound states one does not need to solve the full-off-shell LS equation or Schrödinger equation

Bound State $p = ik$, $A = -k^2$

Binding energy of near threshold bound state, $g_A = 7.45$

One does not need to solve Schrödinger equation

Poles of $T(A) \leftrightarrow$ zeros of $D(A)$

• As a **bonus** the non-perturbative-Δ N/D method allows to calculate $T(A)$ for $A \in \mathbb{C}$ in the 1st/2nd Riemann sheet

This is not trivial with LS

Binding energy of near threshold bound state, $g_A = 7.45$

One does not need to solve Schrödinger equation

Poles of $T(A) \leftrightarrow$ zeros of $D(A)$

$A = (ik)^2$	N/D_{01}	N/D_{11}	Schrödinger
$\Delta_{1\pi}$		2.02	
$\Delta_{2\pi}$		2.18	
$\Delta_{3\pi}$		2.21	
$\Delta_{4\pi}$	0.89	2.22	
Non-perturbative	2.22	2.22	2.22

- **Anti-bound (virtual) state for $^1S_0$**

$$
\begin{aligned}
T_{II}^{-1}(A) &= T_I^{-1}(A) + 2i\rho(A) \\
&= \frac{D_I + N_I\, 2i\rho(A)}{N_I}\ ,\ \mathrm{Im}\sqrt{A} \geq 0
\end{aligned}
$$

Look for zero of $D_{II}(A)$. $E = A/M_N =$

N/D_{11}:
-0.070 (LO) , -0.067 (NLO,NNLO) MeV

For the other $N/D_{m_1 m_2}$: -0.066 MeV always

G.E. Brown, A.D. Jackson "The Nucleon-Nucleon interaction", North-Holland, 1976. Page 86: *"In practice, of course, we do not know the exact form of $\Delta(p^2)$ for a given potential and the N/D equations do not represent a practical alternative to the exact solution of the LS equation for potential scattering..."*

Now this statement is superseded

Furthermore: This method is superior for singular potentials.

Conclusions

- A new non-singular IE allows to calculate the exact $\Delta(A)$ in potential scattering for a given potential
- One can calculate the scattering amplitude for regular/singular potentials from its analytical/unitarity properties.
- Any proper solution for singular potentials can be found with this method
- We reproduce the LS outcome with/without one counterterm
- One can go go beyond LS+one counterterm for an attractive singular potential.

- It can be straightforwardly used in the whole complex plane (bound states, resonances, virtual states)
- Including as well higher order chiral NN potentials.