HC2NP - 25-30 September 2016 - Puerto de la Cruz, Tenerife

Lattice calculations of the hadronic contributions to the muon g-2

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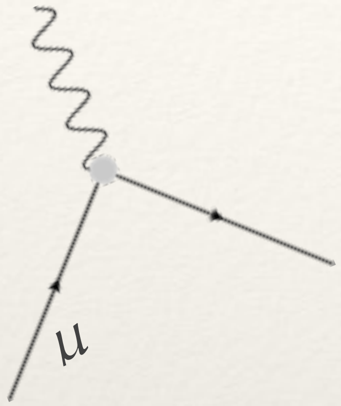
Hadronic Contributions



$$\langle e(\vec{p}')|j_\nu|e(\vec{p})\rangle = -e\,\bar{u}(\vec{p}')\left[F_1(q^2)\gamma_\nu + i\frac{F_2(q^2)}{4m}[\gamma_\nu,\gamma_\rho]q_\rho\right]u(\vec{p})$$

$$F_2(0) = \frac{g-2}{2} \equiv a_\mu$$

	central value x 10^{10}	uncertainty x 10^{10}
QED	11658471.895	0.008
EW	15.4	0.1
QCD LO	692.3	4.2
QCD NLO	-9.84	0.06
QCD NNLO	1.24	0.01
QCD LbL	10.5	2.6
SM TOTAL	11659181.5	4.9
Experiment	11659209.1	6.3

PDG

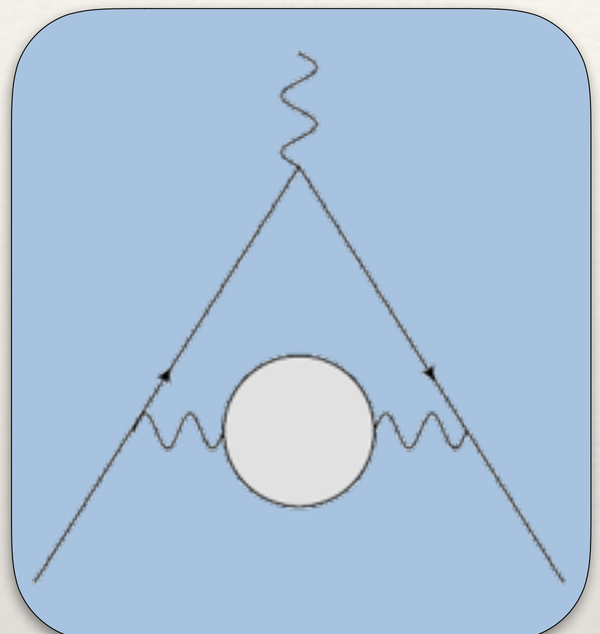
Fermilab **1.6**

J-PARC **4.3 (later ~1)**

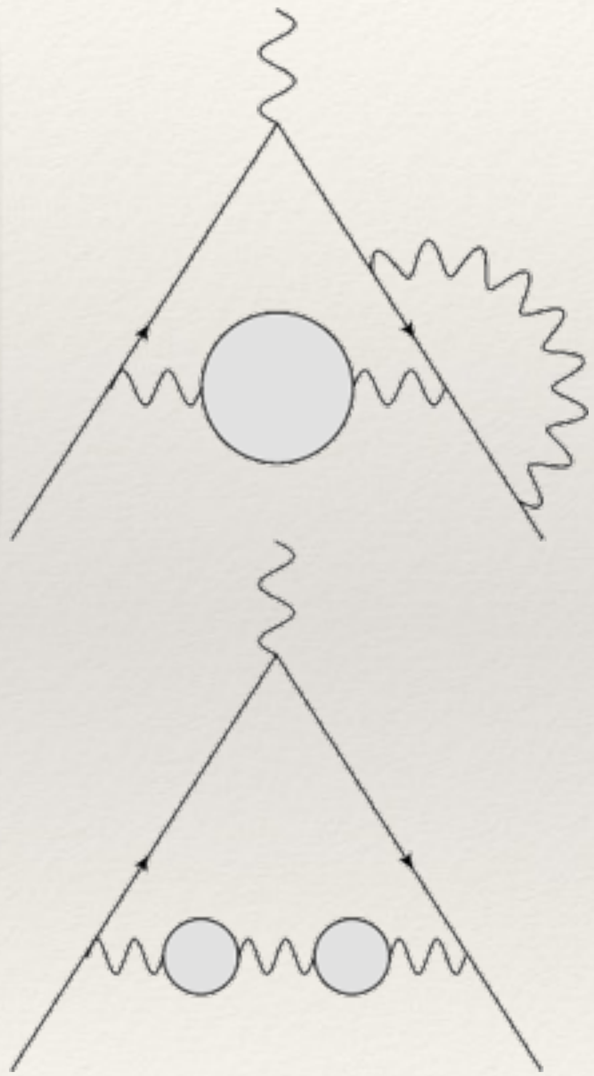
Aim at ~1%(10%) precision for QCD LO(LbL)

Hadronic Contributions

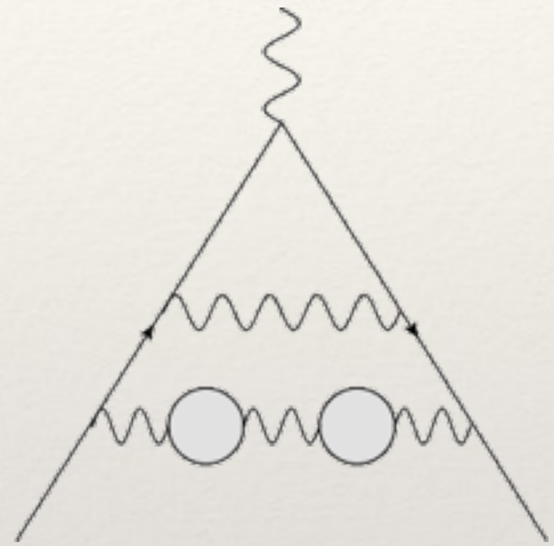
LO HVP



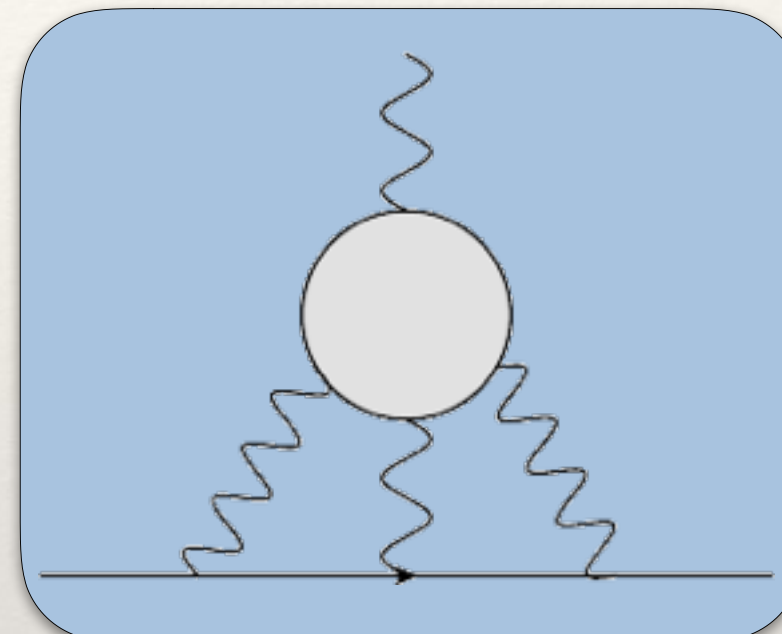
NLO HVP



NNLO HVP



HLbL



LO HVP

$$a_\mu^{\text{LO HAD}} = 4\alpha^2 \int_0^\infty dQ^2 f(Q^2)\left(\Pi(Q^2) - \Pi(0)\right)$$ where Q Euclidean momenta

Lautrup, Peterman, Rafael Nuovo Cim. A1 (1971) 238-242
Blum PRL.91.052001

$$\Pi_{\mu\nu}(Q) = \int d^4x e^{iQ\cdot x}\langle J_\mu(x) J_\nu(0)\rangle = (\delta_{\mu\nu}Q^2 - Q_\mu Q_\nu)\Pi(Q^2)$$

$$J_\mu = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s$$

1. Simulation: compute $\Pi_{\mu\nu}(Q) = a^4 \sum_x e^{iQ\cdot x}\langle J_\mu(x) J_\nu(0)\rangle$
2. Data analysis: determine $\Pi(Q^2)$ and integrate over Q^2

Computing $\Pi_{\mu\nu}(Q^2)$ is text book exercise in principle — but %-level precision for a_μ is very hard

In the following:
- Status of Lattice QCD
- Major difficulties in computing a_μ
  1. Finite volume effects (FVE)
  2. Statistical noise from MCMC
  3. Isospin breaking effects

Lattice QCD

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu} + \sum_f \bar{\psi}_f\left(i\gamma^\mu D_\mu - m_f\right)\psi_f$$

Free parameters:

- gauge coupling $g \to \alpha_s = g^2/4\pi$
- quark masses $m_f = u,d,s,c,b,t$

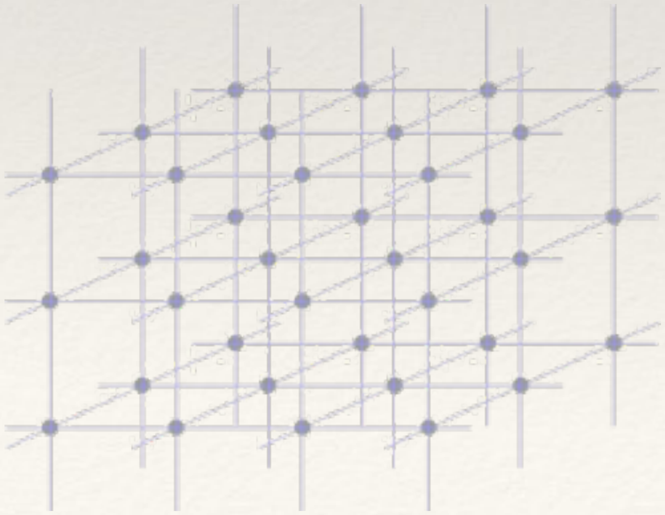
- Lagrangian of massless gluons and *almost massless quarks*
- What experiment sees are bound states, e.g. $m_\pi, m_P \gg m_{u,d}$
- Underlying physics non-perturbative

Path integral quantisation:

$$\langle 0|O|0\rangle = \frac{1}{\mathcal{Z}}\int \mathcal{D}[U,\psi,\bar{\psi}]\,O\,e^{-iS_{\mathrm{lat}}[U,\psi,\bar{\psi}]}$$

$$\langle 0|O|0\rangle = \frac{1}{\mathcal{Z}}\int \mathcal{D}[U,\psi,\bar{\psi}]\,O\,e^{-S_{\mathrm{lat}}[U,\psi,\bar{\psi}]}$$

Euclidean space-time Boltzmann factor



finite volume, space-time grid (IR and UV regulators) $\propto L^{-1}$ $\propto a^{-1}$

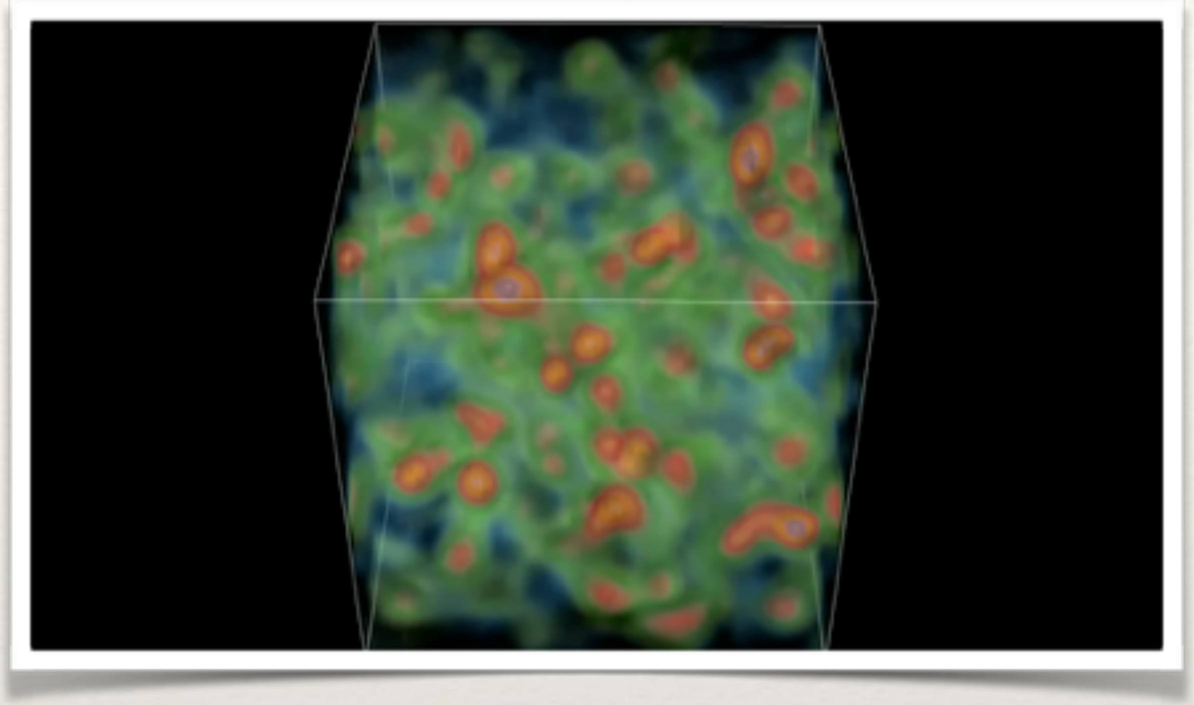
→ Well defined, finite dimensional Euclidean path integral

→ From first principles, solve via MCMC

State of the art of lattice QCD simulations

What we can do

- simulations of QCD with dynamical (sea) *u,d,s,c* quarks with masses as found in nature ➜ $N_f = 2,\ 2+1,\ 2+1+1$
- bottom only as valence quark
- cut-off $a^{-1} \leq 4\mathrm{GeV}$
- volume $L \leq 6fm$



action density of RBC/UKQCD physical point DWF ensemble

Parameter tuning

start from *educated guesses* and compute

- tune light quark mass am_l such that $\frac{am_\pi}{am_P} = \frac{m_\pi^{PDG}}{m_P^{PDG}}$
- tune strange quark mass such that $\frac{am_\pi}{am_K} = \frac{m_\pi^{PDG}}{m_K^{PDG}}$
- determine physical lattice spacing $a = \frac{af_\pi}{f_\pi^{PDG}}$

IMPORTANT:
once the QCD-parameters are *tuned* no further parameters need to be fixed and we can make fully predictive simulations of QCD

HVP tensor on the lattice

$$\Pi_{\mu\nu}(Q) = a^4 \sum_x e^{iQ\cdot x}\langle J_\mu(x) J_\nu(0)\rangle = (\delta_{\mu\nu}Q^2 - Q_\mu Q_\nu)\Pi(Q^2)$$

- For most lattice actions there exists an easily implemented conserved vector current such that $\Delta^*_\mu \langle J^{\rm cons}_\mu O\rangle = \langle \delta O\rangle$

- There are now two possible choices:
  - $O = J^{\rm cons}_\nu$ — this choice leads to a contact term on the r.h.s. of the WI
  - $O = J^{\rm local}_\nu \to \Delta^*_\mu \langle J^{\rm cons}_\mu J^{\rm local}_\nu\rangle = 0$ (local current needs to be renormalised NPly — easy)

- $\Pi_{\mu\nu}(Q)$ from $\langle j^{\rm cons}_\mu j^{\rm loc}_\nu\rangle$ is automatically transverse up to cutoff effects which we remove by applying longitudinal projection resulting in ($p_i = 0$)

$$\Pi(Q^2) \stackrel{\vec{p}=0}{=} \frac{1}{Q^2}\frac{1}{3}\sum_i \Pi_{ii}(Q^2)$$

- There is also a third choice — $\langle j^{\rm loc}_\mu(x) j^{\rm loc}_\nu(0)\rangle$ use only local (not conserved) currents to construct $\Pi_{\mu\nu}$ — there will be a contact terms when $x \to 0$ which needs to be dealt with — see later

HVP - Wick contractions

It is useful to break computation up into components:
individual Wick contractions and Flavour contributions have their independent continuum and finite volume limit AJ, Della Morte *arXiv:0910.3755, JHEP11(2010)154*

allows to fine-tune simulation strategies / precision per contraction / flavour

Break up by Wick contraction

$$\Pi^{\rm conn}_{\mu\nu}(Q) = a^4 \sum_x e^{iQ\cdot x} \langle {\rm tr}\,\{\gamma_\mu S(x,0)\gamma_\nu S(0,x)\}\rangle$$

by far dominant part

$$\Pi^{\rm disc}_{\mu\nu}(Q) = a^4 \sum_x e^{iQ\cdot x} \langle {\rm tr}\,\{\gamma_\mu S(x,x)\}\,{\rm tr}\,\{\gamma_\nu S(0,0)\}\rangle$$

small correction

HVP - Wick contractions

Analytical considerations for Wick contractions:

- Disconnected contribution zero in SU(3) limit

- PQChPT NLO: $$\frac{\Pi^{\rm disc}_{\mu\nu}(Q)}{\Pi^{\rm conn}_{\mu\nu}(Q)} = -\frac{1}{10}$$ confirmed at NNLO

AJ, Della Morte JHEP11(2010)154

Bijnens, Relefors arXiv:1609.01573

Ignores ρ contribution to VP. $\pi\pi$ contribution estimated to be ~10%, would reduce to -1/10*0.1 = 1% effect HPQCD PhysRevD.93.074509 (2016)

→ **Can be more relaxed about precision goal for disconnected contribution**

Break up by flavour

Connected up/down	— strange	— charm contributions
90%	8%	2%

- Unfortunately high precision easier for heavy flavour contribs
- Disconnected contributions mix flavour at source and sink

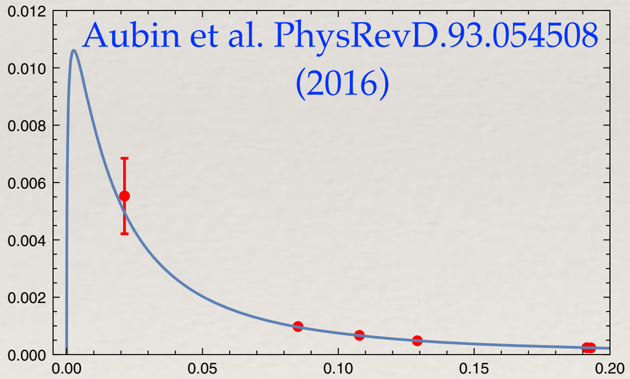
From the HVP to a_μ - I

$$a_\mu^{\mathrm{LO\,HAD}} = 4\alpha^2 \int_0^\infty dQ^2 f(Q^2)\left(\Pi(Q^2) - \Pi(0)\right)$$

There are essentially three different ways for extracting a_μ:

- ***Traditional analysis:***
  - One difficulty is that in a finite volume the $\Pi(0)$ cannot be computed from

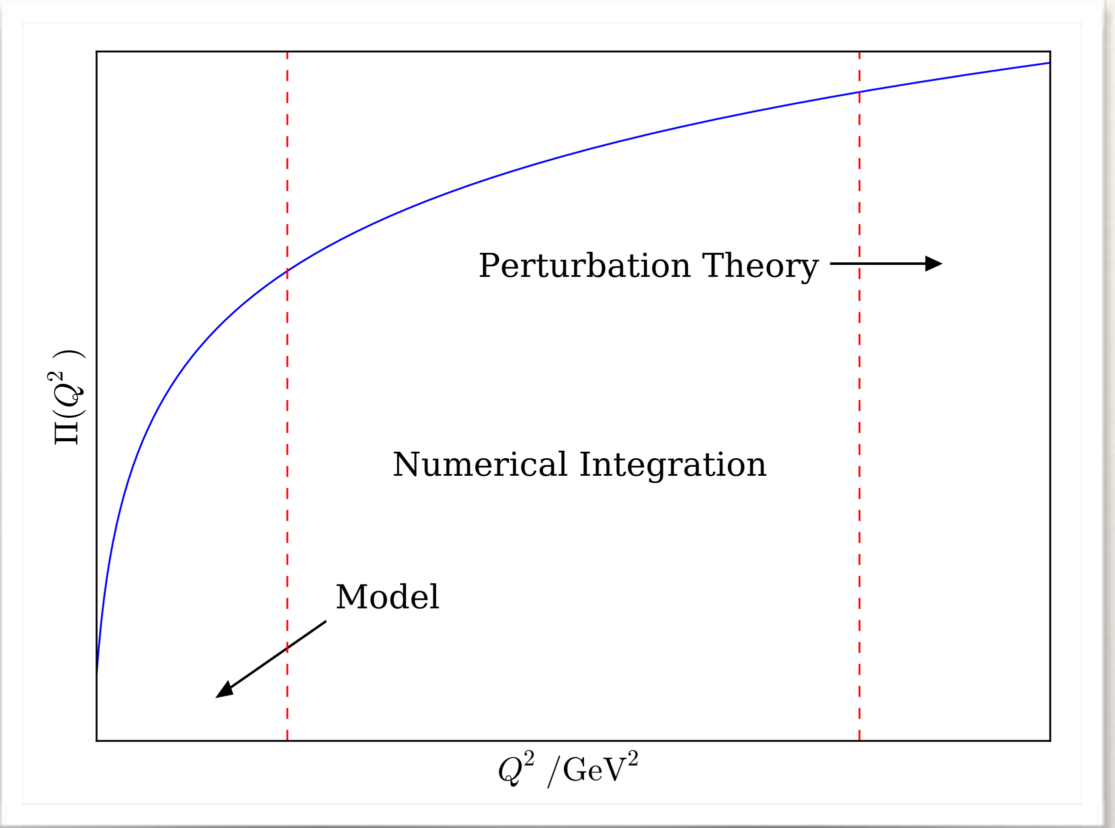
$$\Pi(Q^2) = \Pi_{\mu\nu}(Q)/(\delta_{\mu\nu}Q^2 - Q_\mu Q_\nu)$$



  - Instead one extrapolates from larger Q^2 to $Q^2=0$ using fit ansätze — note that smallest momentum in finite volume ~$2\pi/T$
  - Fit ansätze: conformal polynomials and Padé are widely used
  - Padé approximates provide a model independent description of the data and subsequent orders of Padé bracket the exact solution Aubin et al. PhysRevD.86.054509

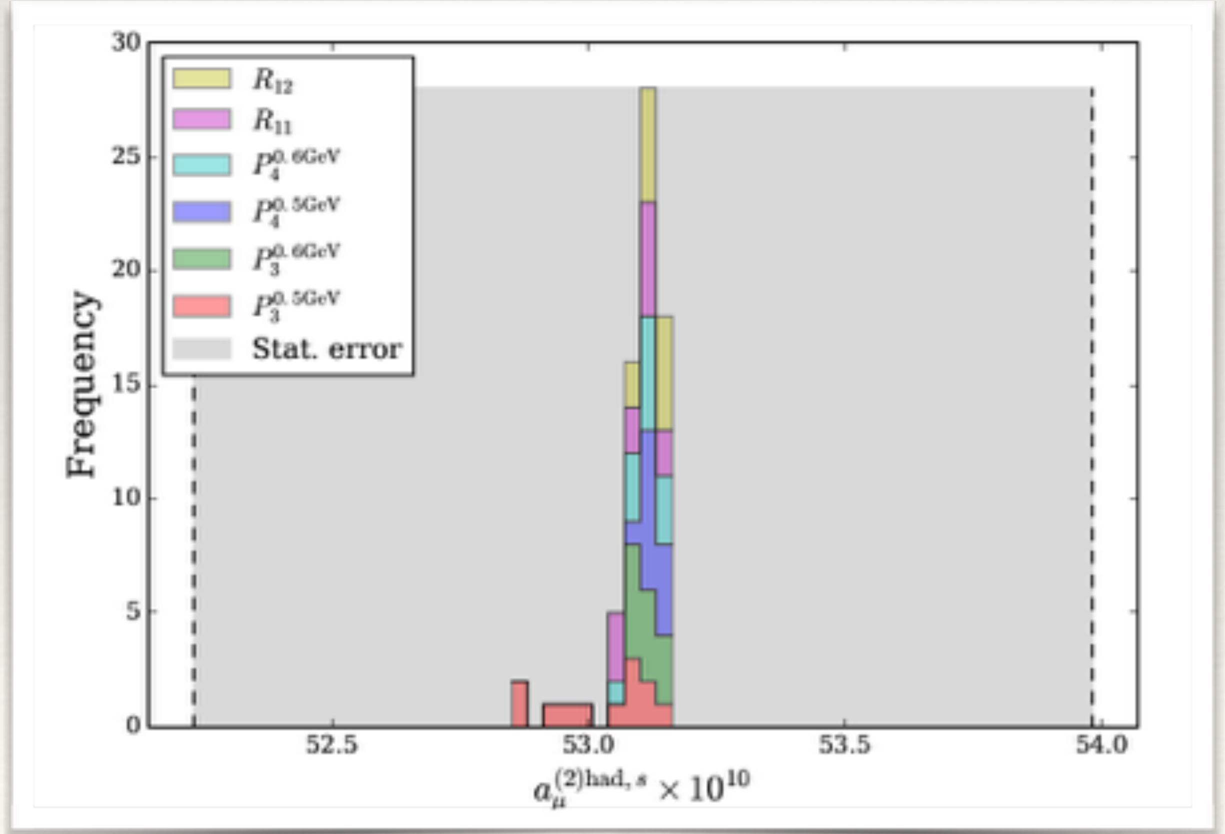
From the HVP to a_μ - II

- ***Traditional analysis (continued):***
  - finite order Padés provide an accurate description only over a limited range in Q^2 → *hybrid analysis*



Spraggs RBC/UKQCD Lattice 2016

Histogram over analyses with different Padés/Conformal Polynomials



Spraggs RBC/UKQCD Lattice 2016

- use various methods/cuts to estimate systematic error

From the HVP to a_μ - III

There are essentially three different ways for extracting a_μ:

- *Traditional analysis*
- *Time moments* HPQCD PRD.89.114501 (2014)

zero momentum projected correlator: $G(t) = a \sum_{\vec{x}} \langle j_i(t,\vec{x}) j_i(0,\vec{0}) \rangle$

$$G_{2n} \equiv a^4 \sum_t t^{2n} G(t) = (-1)^n \frac{\partial^{2n}}{\partial Q^{2n}} Q^2 \hat{\Pi}(Q^2)|_{Q^2=0}$$

$$\hat{\Pi}(Q^2) = \sum_{j=1}^{\infty} Q^{2j} \Pi_j$$

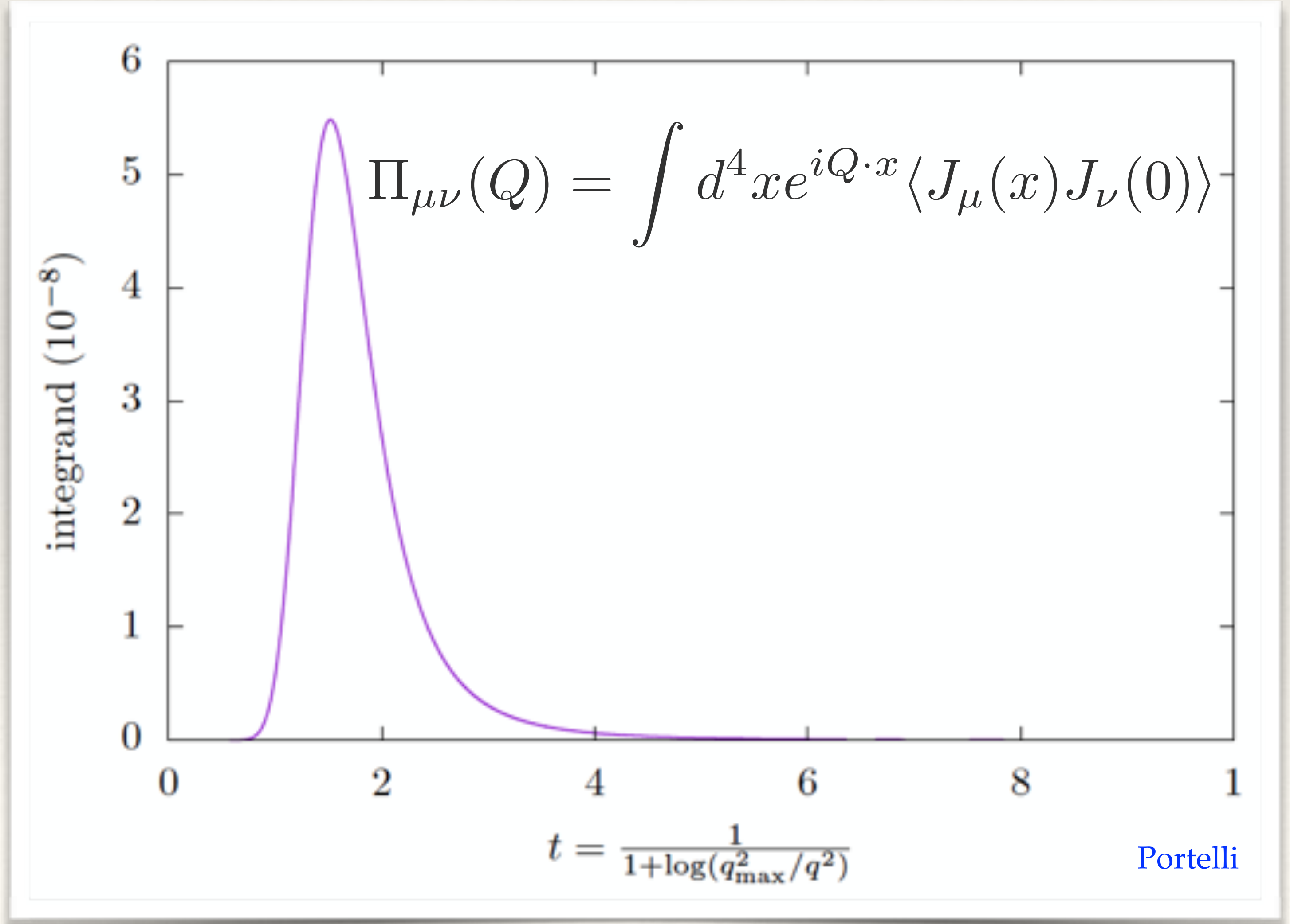
$$\Pi_j = (-1)^{j+1} \frac{G_{2j+2}}{(2j+2)!}$$

- Moments allow to directly solve for coefficients of polynomial/Padé/… approximation for $\Pi(Q^2)$ (no χ^2-minimisation needed)
- t=0 absent → no contact term → can use local currents only

From the HVP to a_μ - IV

There are essentially three different ways for extracting a_μ:

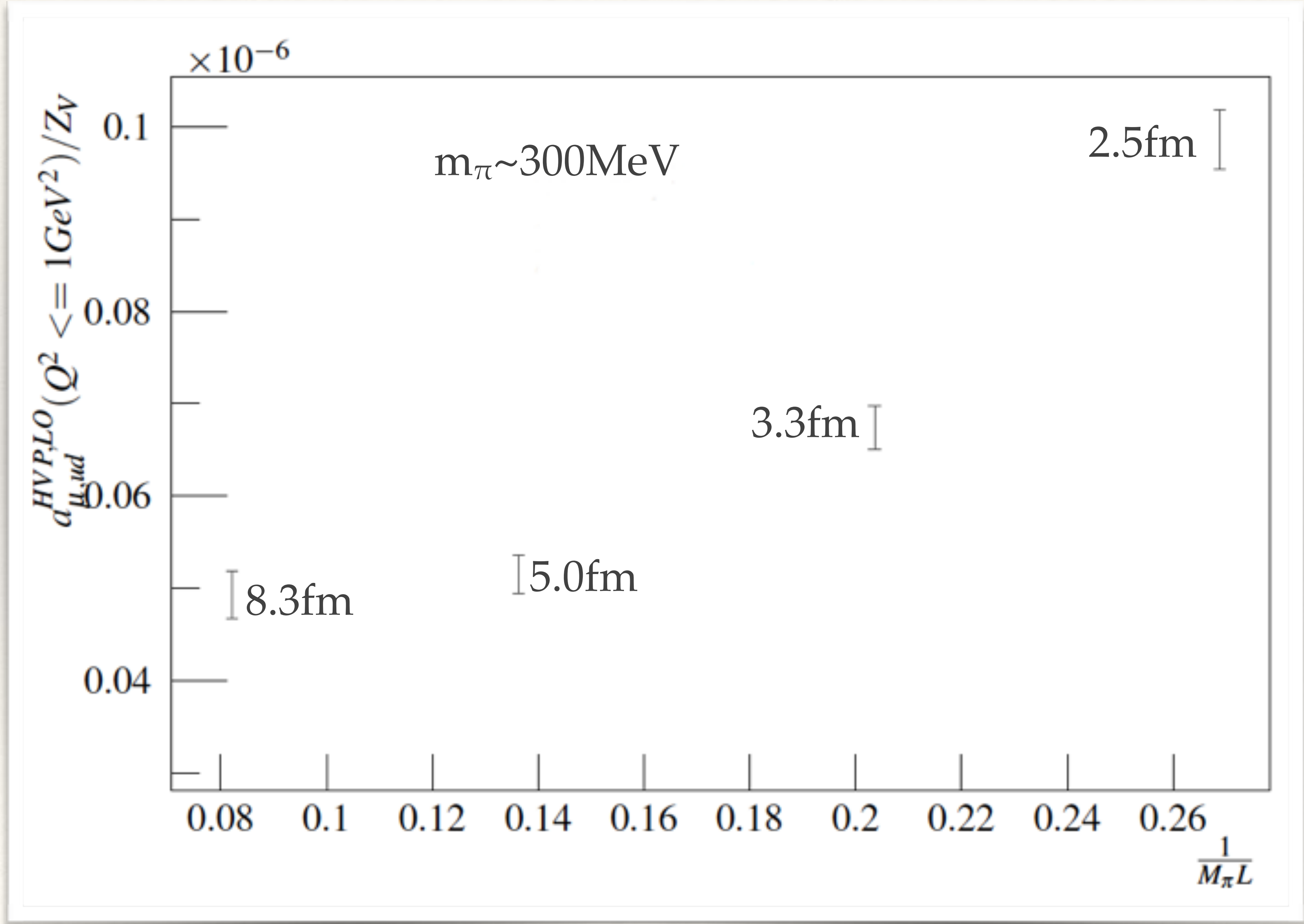
- *Traditional analysis*
- *Time moments* HPQCD PhysRevD.89.114501
- *Sine Cardinal interpolation* — use Fourier transform with continuous momenta
  Feng et al. PhysRevD.88.034505, Bernecker, Meyer epja/i2011-11148-6 , Portelli, Del Debbio in preparation



Nice, since it covers the whole Q^2 range.
According to authors the systematic is an exponentially suppressed finite volume error

Finite Volume Effects

BMW's finite volume scaling study for a_μ



BMW arXiv:1502.02172

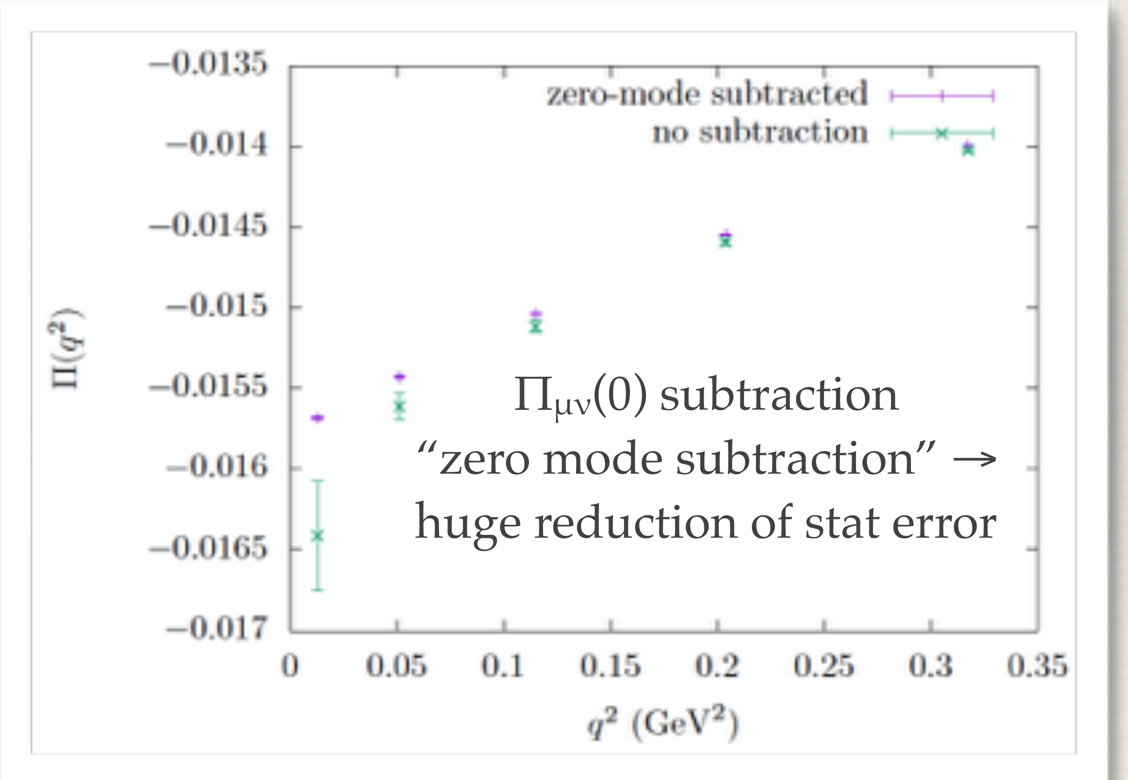
Finite Volume Effects in ChPT

Aubin et al. PhysRevD.93.054508

- In finite volume with L≠T, rotation group broken down to group of cubic rotations
- Finite volume effects in ChPT as per irreducible representation (A_1, A_1^{44}, T_1, T_2, E)

Results:

- $\Pi_{\mu\nu}(0) \neq 0$ in finite volume (known before) — but subtracted VP tensor
$$\overline{\Pi}_{\mu\nu}(Q) = \Pi_{\mu\nu}(Q) - \Pi_{\mu\nu}(0)$$
by an order of magnitude closer to infinite-volume points BMW arXiv:1502.02172
  - confirms previous BMW study
  - further benefit: $\Pi_{\mu\nu}(0)$ and $\Pi_{\mu\nu}(Q^2)$ highly correlated in MCMC data, *subtracting zero* significantly reduces stat. error

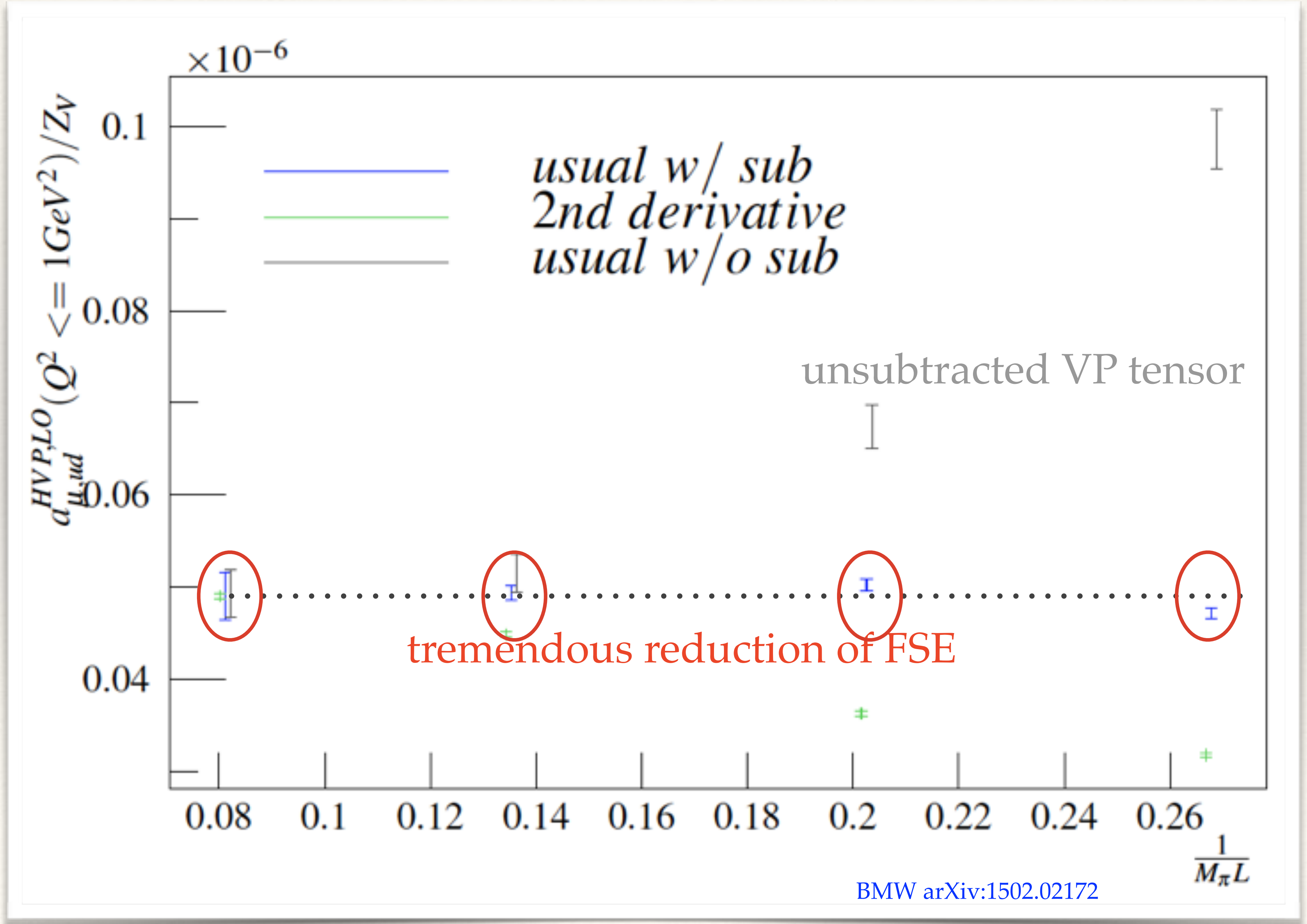


- even for $m_\pi L > 4$ FSE can be of order 10%

- *Conservative* estimate of finite volume errors: infinite volume result lies between result for two different irreps (A_1, A_1^{44})

Finite Volume Effects

data confirms small FVE for subtracted VP tensor $\overline{\Pi}_{\mu\nu}(Q) = \Pi_{\mu\nu}(Q) - \Pi_{\mu\nu}(0)$



Finite Volume Effects

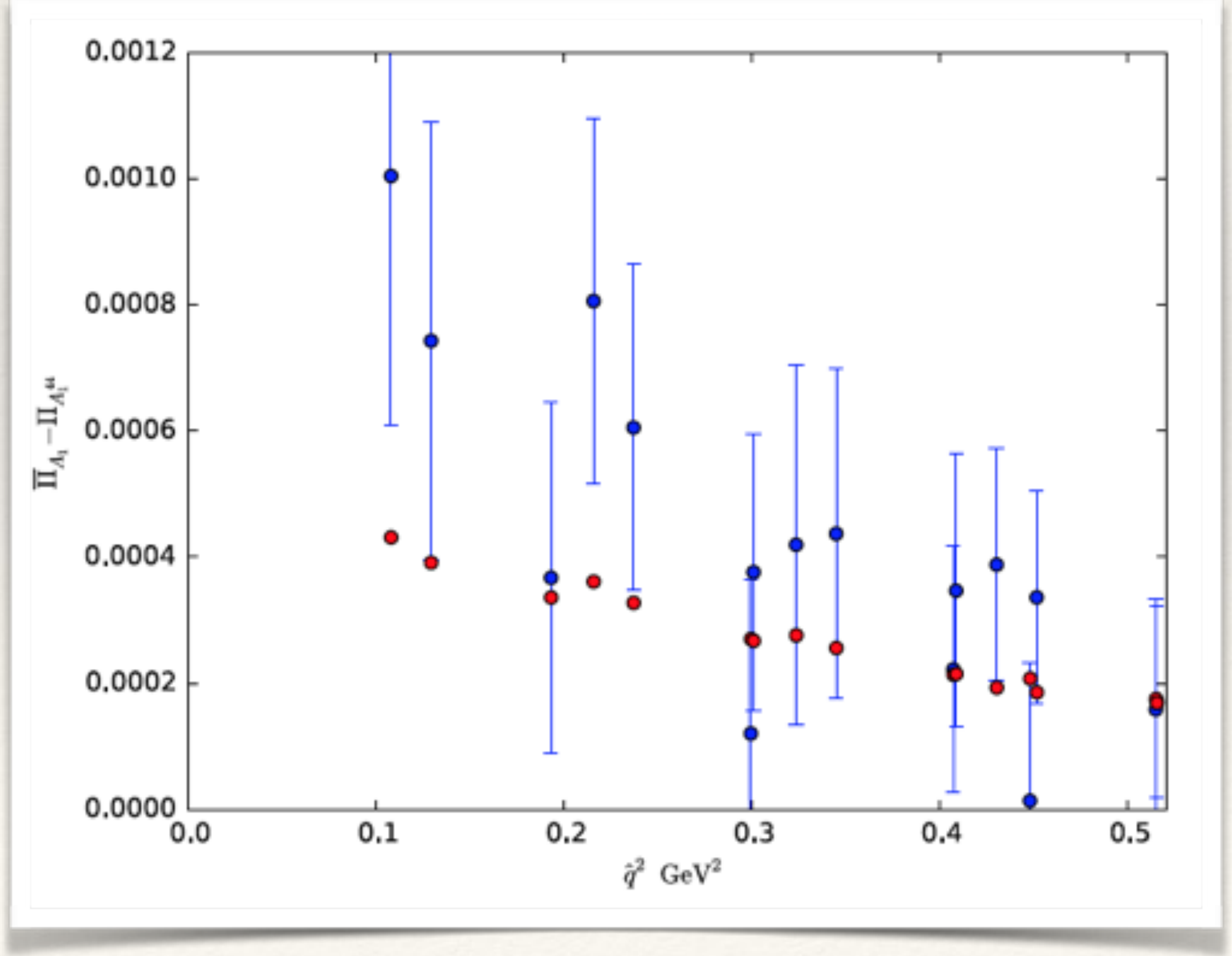
Aubin et al. PhysRevD.93.054508

Does ChPT agree with data?

ChPT only properly describes 2π contribution to FVE (not the ϱ resonance contrib.)

→ consider differences of finite volume effects, e.g. different irreps: A_1-A_1^{44}

(differences of finite volume effects will be dominated by 2π effects)



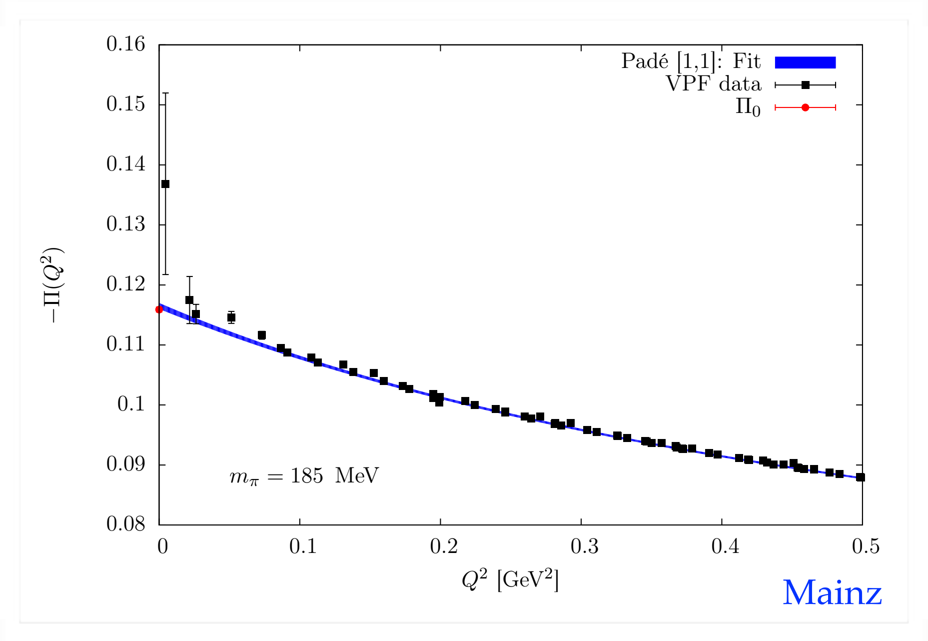
Summary finite volume effects:

- good agreement between eff. theory and lattice data for differences of FVE
- can define estimate of FVE
- hope is that ChPT can be used to control FVE at 1% level but further testing necessary

Signal-To-Noise

$$\Pi_{\mu\nu}(t,\vec{Q}) = \int d^4x e^{i\vec{Q}\vec{x}} \langle J_\mu(t,\vec{x}) J_\nu(0) \rangle$$

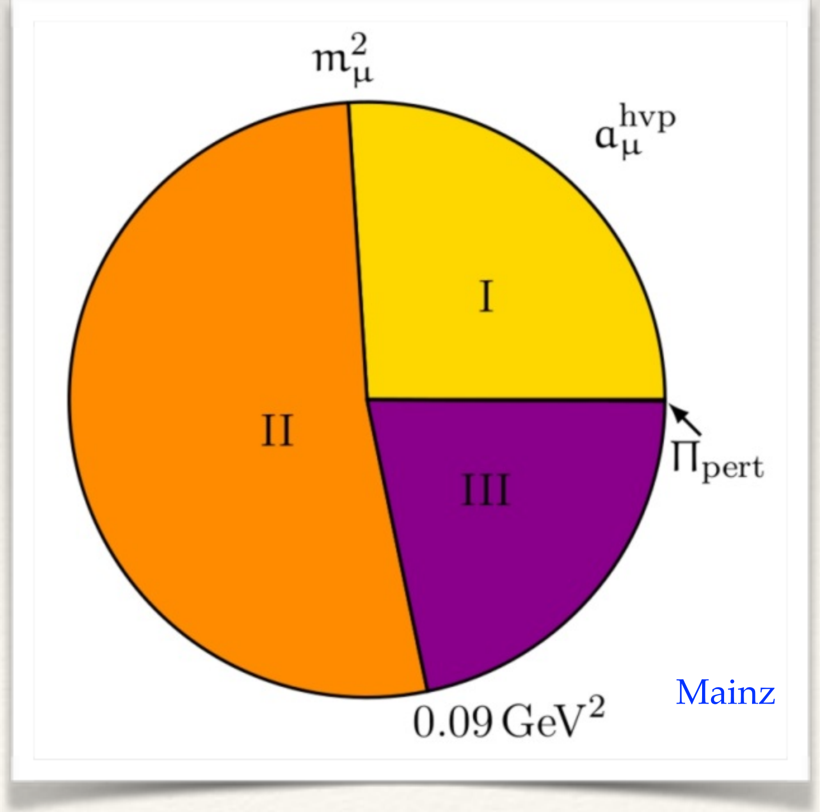
- Correlation function easy to compute but signal-to-noise deteriorates for small momenta. This is expected due to the understood exponential deterioration of the signal-to-noise ratio at large distance in the vector correlator



- This is really bad since the Kernel of

$$a_\mu^{\text{LO HAD}} = 4\alpha^2 \int_0^\infty dQ^2 f(Q^2)\left(\Pi(Q^2) - \Pi(0)\right)$$

receives dominant contribution from low Q^2 region



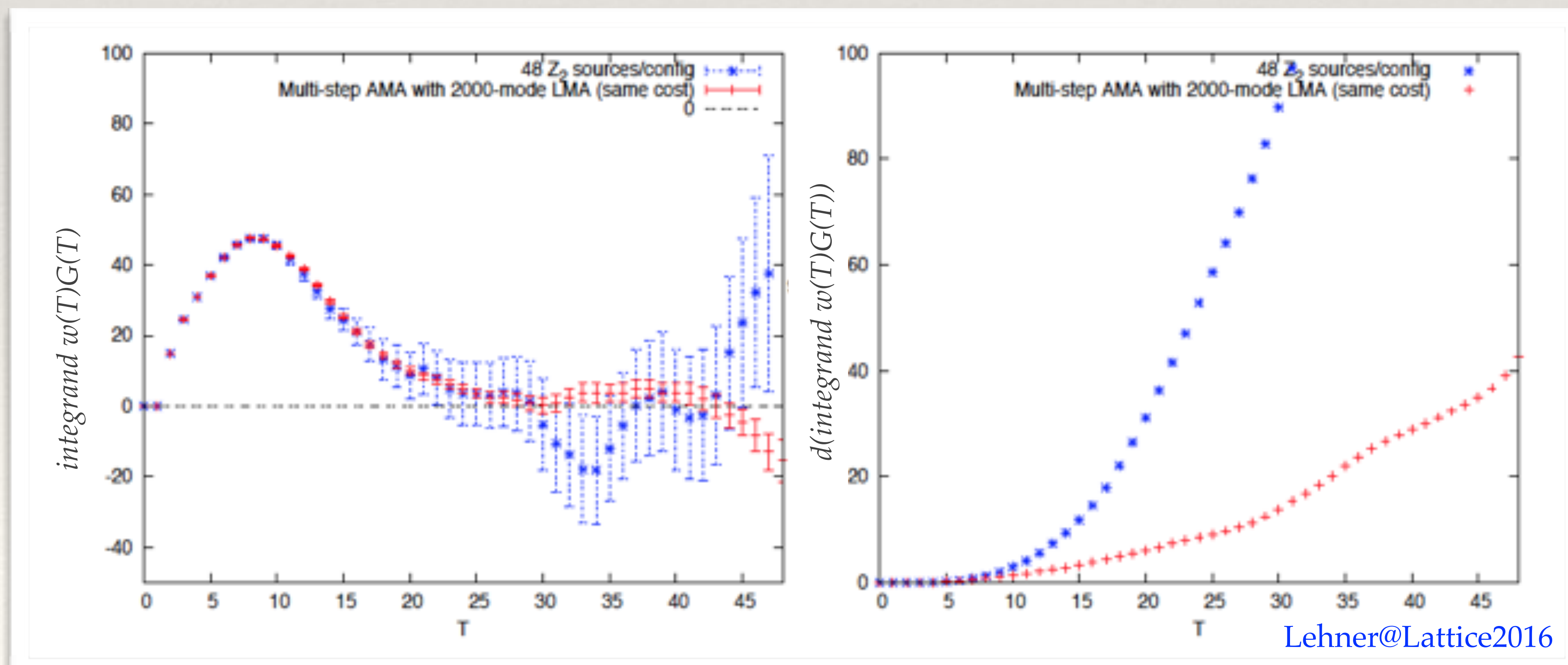
Signal-To-Noise

Let's go to *time-momentum representation* ($\vec{Q}=0$)

$$\Pi(Q^2)-\Pi(0)=\frac{1}{3}\sum_t\left(\frac{\cos(Q_t t)-1}{q^2}+\frac{1}{2}t^2\right)G(t) \qquad a_\mu^{\mathrm{LO\,HVP}}=\sum_{t=0}^{\infty}w(t)G(t)$$

Bernecker, Meyer epja/i2011-11148-6

- HVP automatically renormalised
- incorporates zero mode subtraction which significantly reduces the stat error on $\Pi_{\mu\nu}$
- large-t signal-to-noise deteriorating (as expected)
- intense algorithmic investigations going on (use physics intuition)



Signal-To-Noise

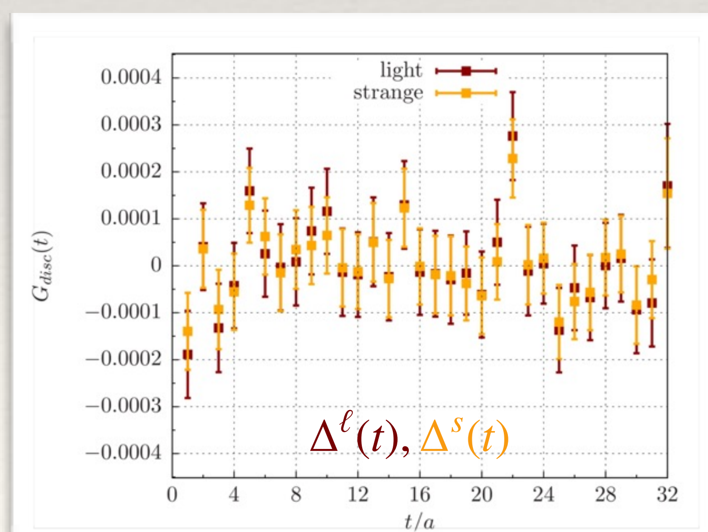
Example for how we are currently dealing with signal-to-noise issue:
RBC/UKQCD's computation of quark-disconnected contribution on Domain Wall Fermion ensembles with physical sea pions RBC/UKQCD PhysRevLett.116.232002
(example for connected analysis later in C. Davies' talk)

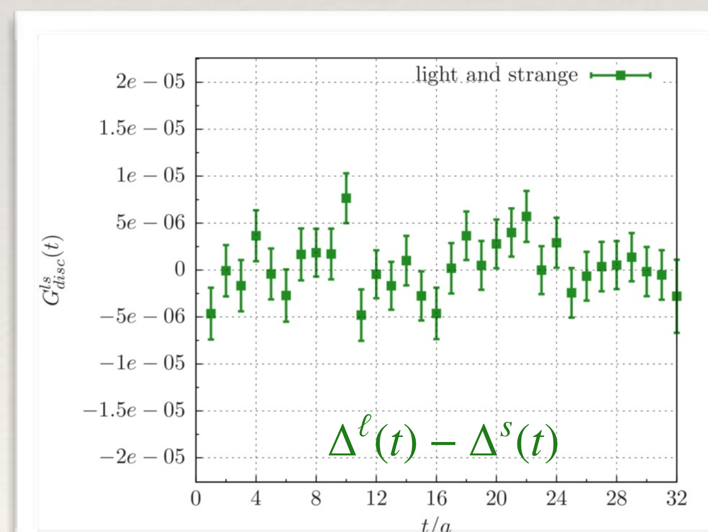
consider disconnected correlator:

$$C(t) = \frac{1}{3V} \sum_{i,t'} \langle \mathcal{V}_i(t) \mathcal{V}_i(0) \rangle \text{ and } \mathcal{V}_i = \frac{1}{3} \left(\mathcal{V}_i^{u/d} - \mathcal{V}_i^s \right)$$

$$\mathcal{V}_i^f(t) = \sum_{\vec{x}} \mathrm{ImTr} \left(S^f(x,x) \gamma_i \right)$$

- Mainz group observed: stat. fluctuations of s- and u/d quarks anti-correlated
  →statistical error in difference of s and l quarks cancel Gülpers Lattice 2014





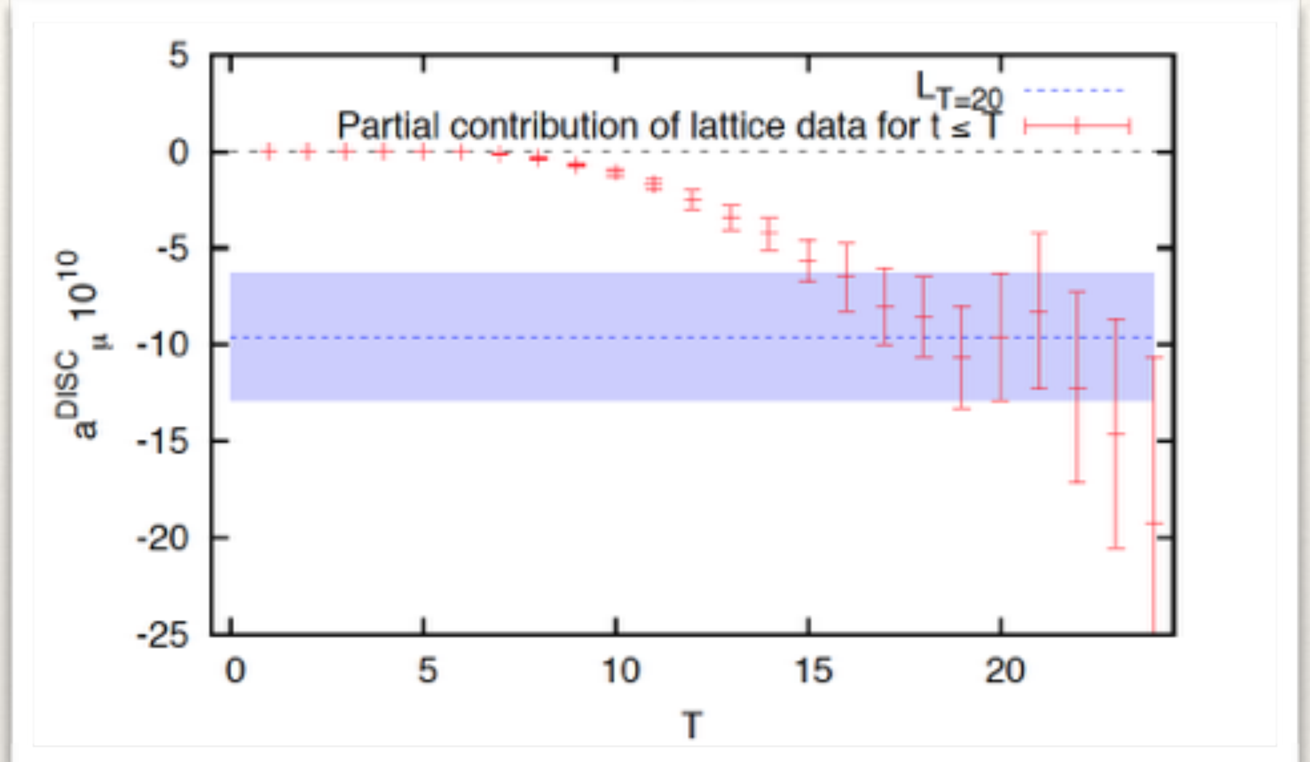
we use this by computing eigenmodes of Dirac operator up to the strange mass exactly
(best use of correlations, contains dominant part of the signal)

Signal-To-Noise

Consider partial sum up to time-extent T

$$L_T = \sum_{t=0}^{T} w(t)G(t)$$

- Signal-To-Noise issue clearly visible
- G(t) consistent with zero for t ≥ 15



Idea: use $G(t) = \begin{cases} G(t)^{\rm data}, & t \le t^{\rm cut} \\ G(t)^{\rm model}, & t > t^{\rm cut} \end{cases}$

Signal-To-Noise

RBC/UKQCD PhysRevLett.116.232002

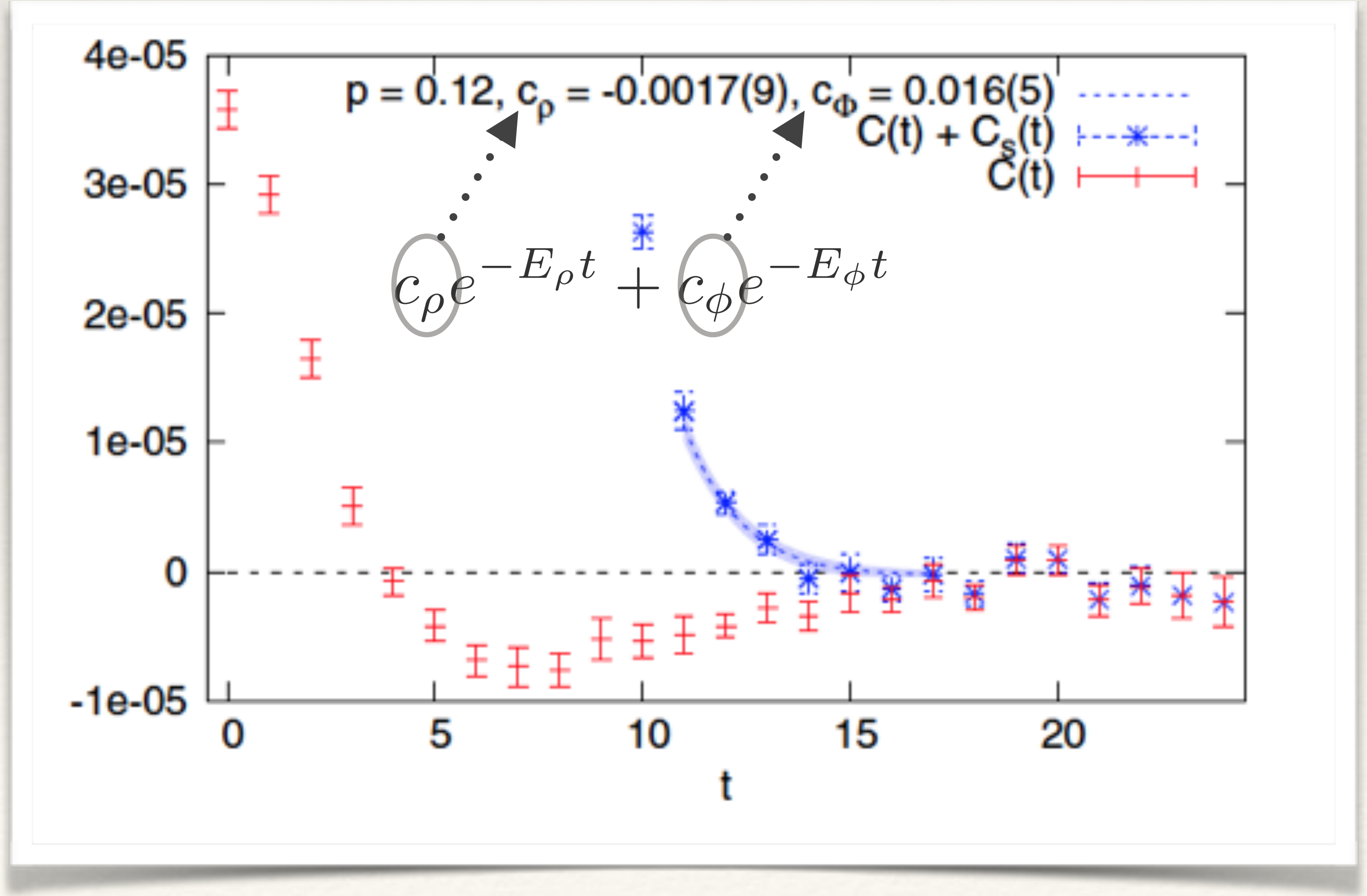
- using isospin and flavour algebra we can write the light-disconnected contribution as a correlation function with a continuum and infinite volume limit

$$\langle V_\mu^{uu} V_\nu^{uu} \rangle - \langle V_\mu^{ud} V_\nu^{du} \rangle$$

AJ, Della Morte JHEP11(2010)154

- not possible for strange contribution but consider instead

$$\langle \left(V_\mu^{uu} - V_\mu^{ss} \right) \left(V_\nu^{uu} - V_\nu^{ss} \right) \rangle - \langle V_\mu^{ud} V_\nu^{du} \rangle = C(t) + C_s(t) = \sum_m c_m e^{-E_m t}$$



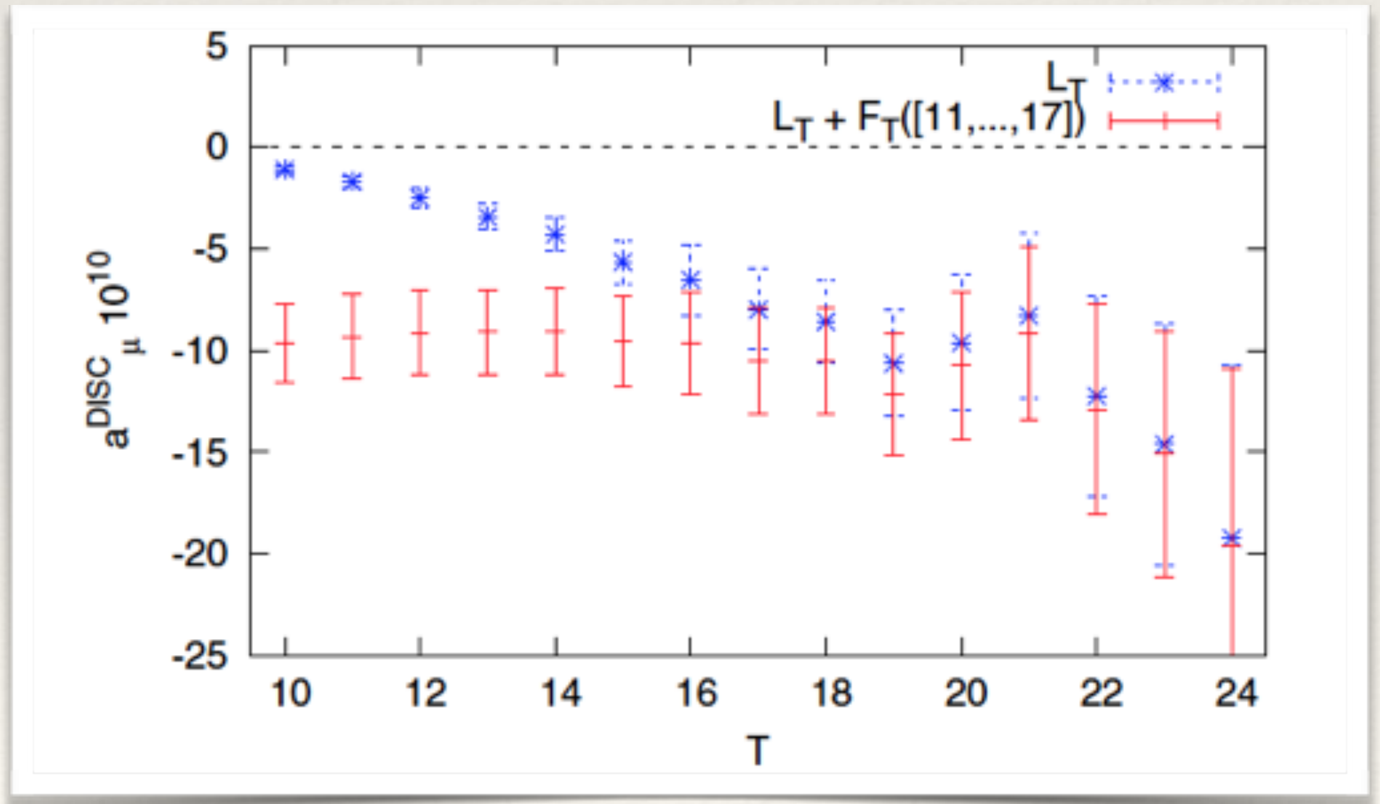
Signal-To-Noise

$$L_T = \sum_{t=0}^{T} w(t) G(t)$$

$$F_T(r) = \sum_{t=T+1}^{t_{\max}} w(t) \left(c_\rho^r e^{-E_\rho t} + c_\phi^r e^{-E_\phi t} - C_s(t) \right)$$

strange quark connected

- E_ρ, E_ϕ from experiment, c_ρ, c_ϕ from fit
- **central value for a^{DISC} from L_T**
- **systematic error due to cut from F_T**



final result from T=20

$a^{DISC} = -9.6(3.3)\times10^{-10}$

Systematics:

- Finite T effects
- Finite volume errors ($\pi\pi$ in ChPT)
- Cutoff effects
- Variations in fit range to C+C_s:

$$a^{DISC} = -9.6(3.3)(2.3)\times10^{-10}$$

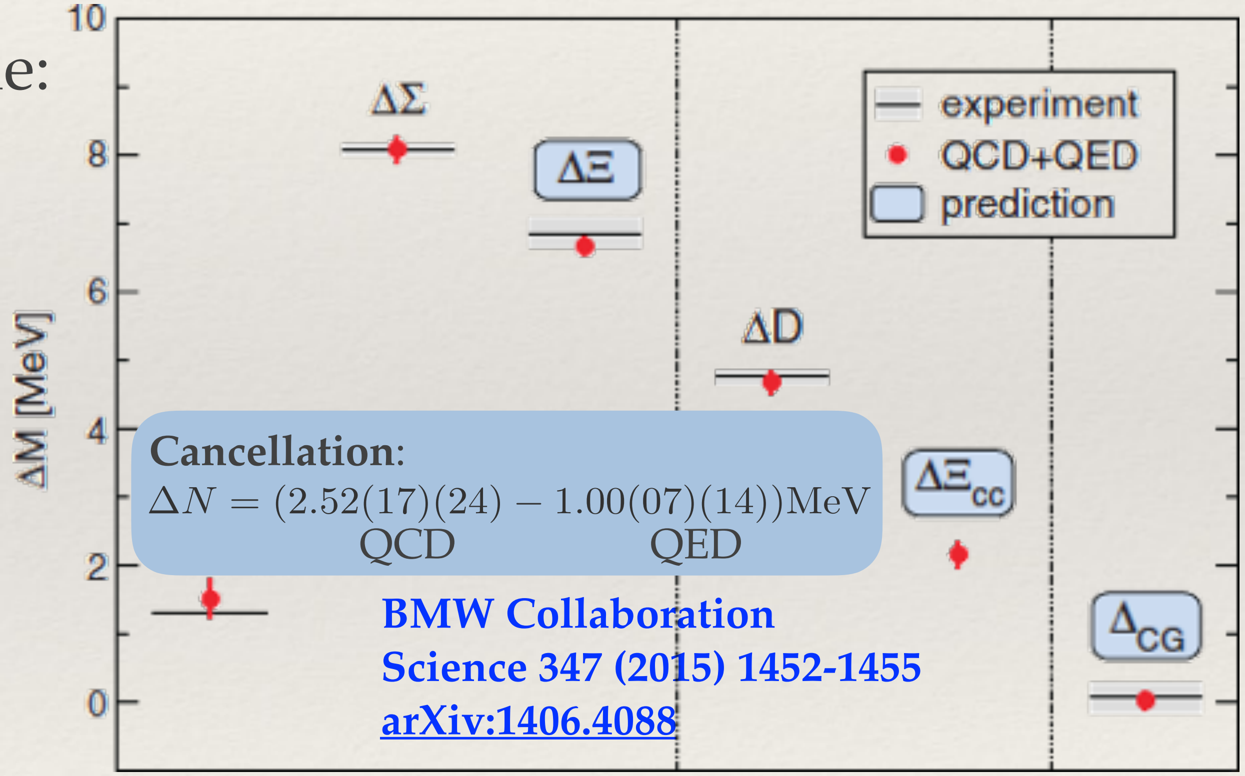
This is our result ($N_f = 2+1$) for physical pion mass!!!

Isospin Breaking Effects

- Most current simulations $N_f = 2+1(+1)$ flavour
  $m_u = m_d$, α_{EM}

- QED effects in HVP expected to be ~1% — needs to be taken seriously

- L(QED+QCD) has become quite fashionable:
  - post/predicting hadron spectra/mass splittings
  - including QED for matrix elements theoretically/technically challenging — IR divergences (Bloch-Nordsieck)
    Carrasco et al. PRD 91 074506 (2015) arXiv:1502.00257



Cancellation:
$\Delta N = (2.52(17)(24) - 1.00(07)(14))\text{MeV}$
QCD QED

BMW Collaboration
Science 347 (2015) 1452-1455
arXiv:1406.4088

- a_μ is special — no IR divergences

should be doable modulo finite volume effects due to the photon (later)

Isospin Breaking Effects - FSE

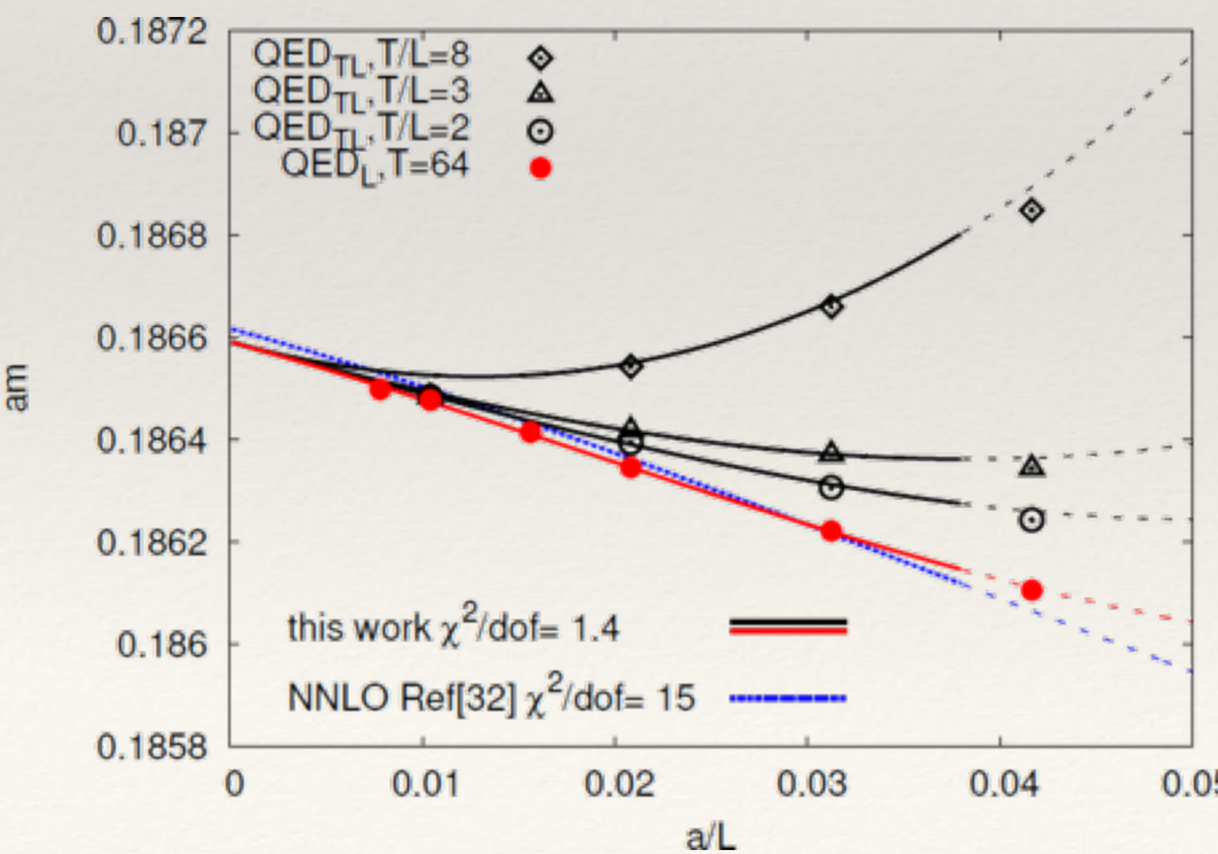
Example: FV correction to mass of a spin-1/2 particle in QED

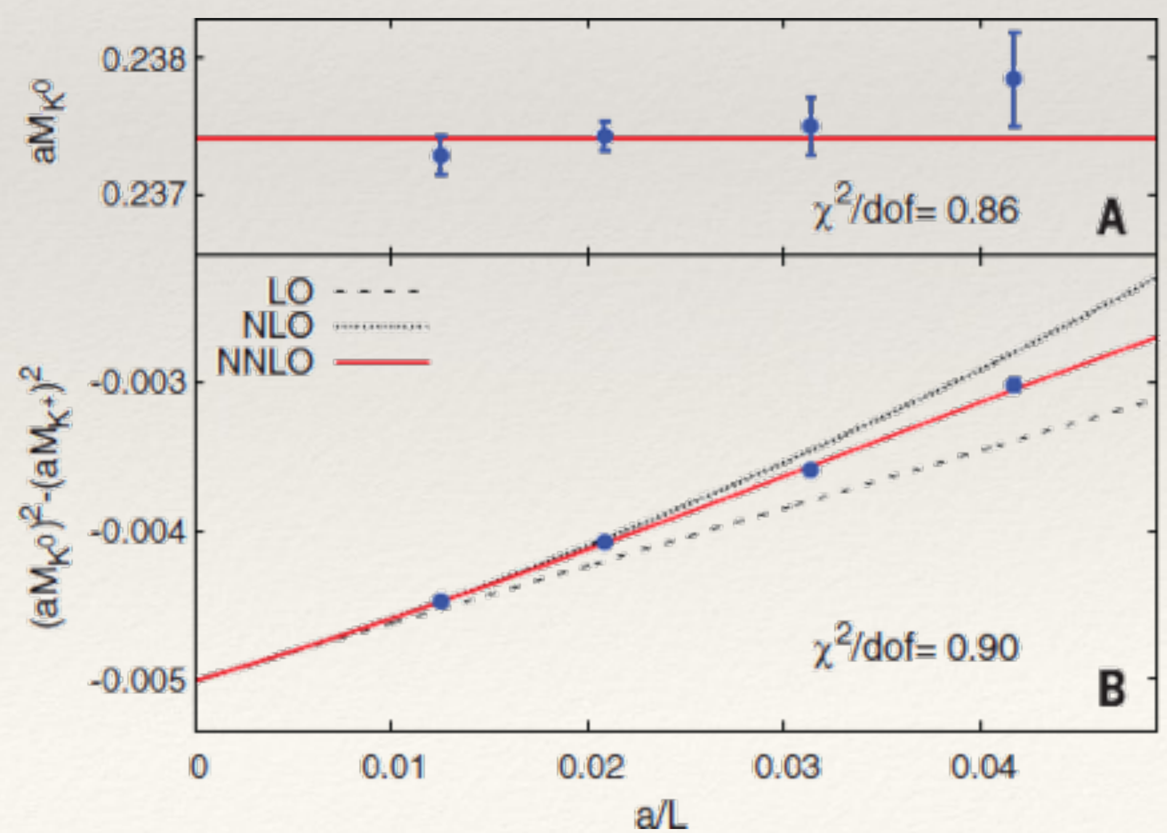
BMW Collaboration
Science 347 (2015) 1452-1455
arXiv:1406.4088

analytically compute the difference of the *finite volume* and *infinite volume* self energies Σ:

$$m^2(T,L) \overset{T,L\to\infty}{\propto} m^2 \left\{ 1 - q^2\alpha \left[\frac{\kappa}{2mL}\left(1+\frac{2}{mL}\right) - \frac{3\pi}{(mL)^3} \right] \right\}$$

leading behaviour universal in *κ* (structure- and spin-independent)





Isospin Breaking Effects

Stochastic method Duncan PhysRevLett.76.3894

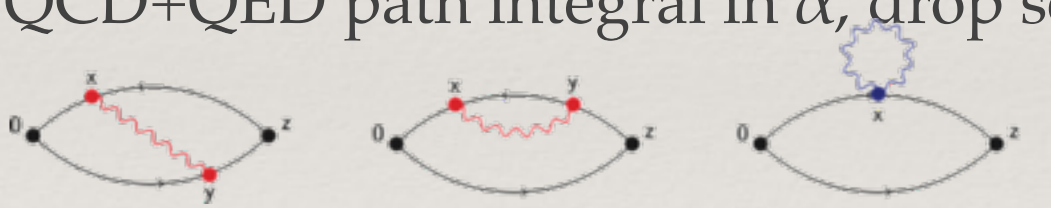
- QCD+quenched QED
- generate U(1) gauge configs
- Promote SU(3) gauge links to U(3)

$$U_\mu^{U(3)}(x) = e^{iq_{\mathrm{em}}A_\mu(x)}U_\mu^{SU(3)}(x)$$

- γ zero-mode subtracted
- Feynman or Coulomb gauge

Perturbative method Rome123 PhysRevD.87.114505

- expand QCD+QED path integral in α, drop sea quark contribution
- $O(\alpha)$:



- insert Feynman/Coulomb gauge photon propagator

The Southampton group is computing isospin breaking effects using both techniques (see Harrison's and Gulper's talks at Lattice 2016)

LO HVP

arXiv:1601.03071 — JHEP 1604 (2016) 063 arXiv:1602.01767 arXiv:1512.09054

a_μ x 10^{10}	HPQCD	RBC/UKQCD
light	598(11)	work in progress
strange	53.4(6)	52.4(2.1)
charm	14.4(4)	work in progress
disconnected	0(9)	−9.6(3.3)(2.3)
all	666(6)(12)	—
SM OK exp all	720(7)	720(7)

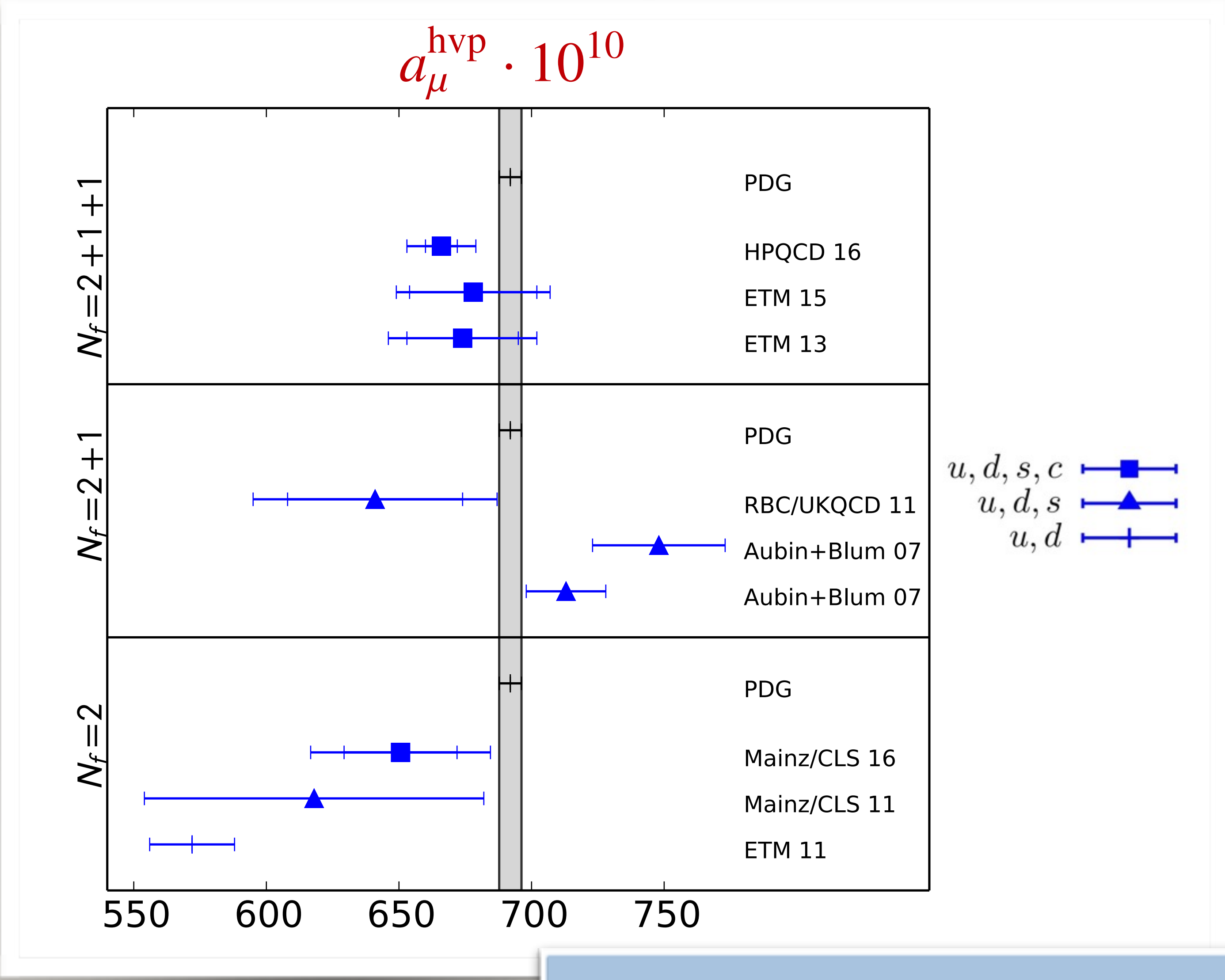
- strange, charm and bottom sufficiently precisely known
- getting the disconnected in full LQCD was a big achievement (previously considered show stopper)

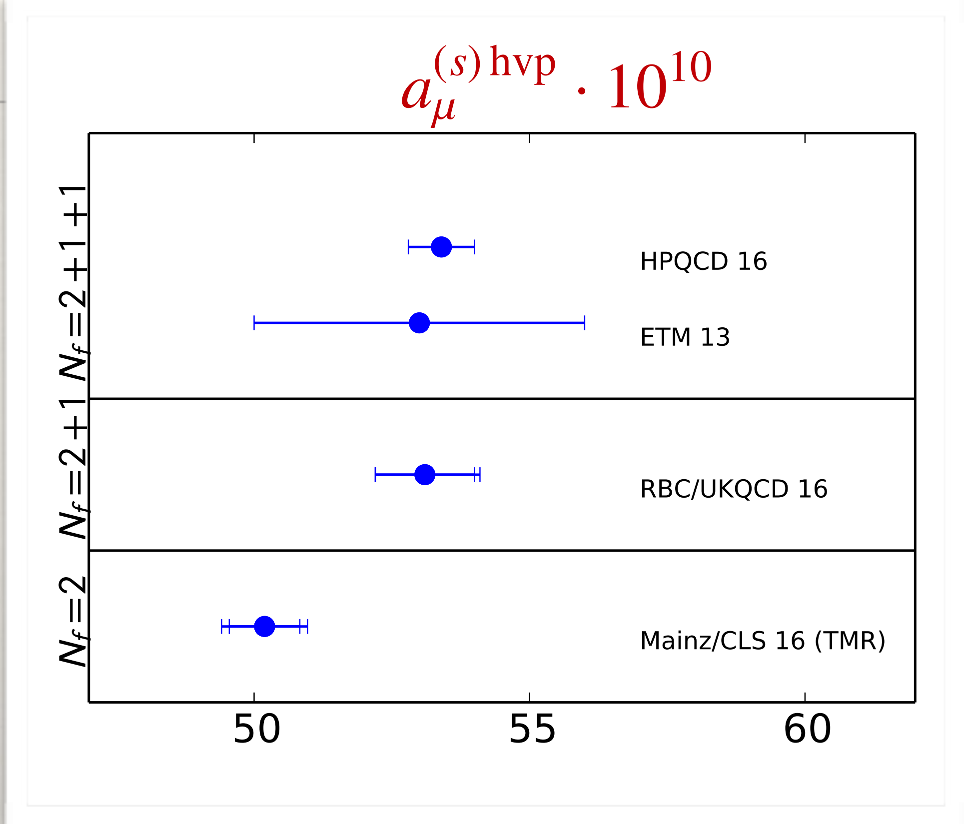
- first results (HPQCD) indicate tension confirmed

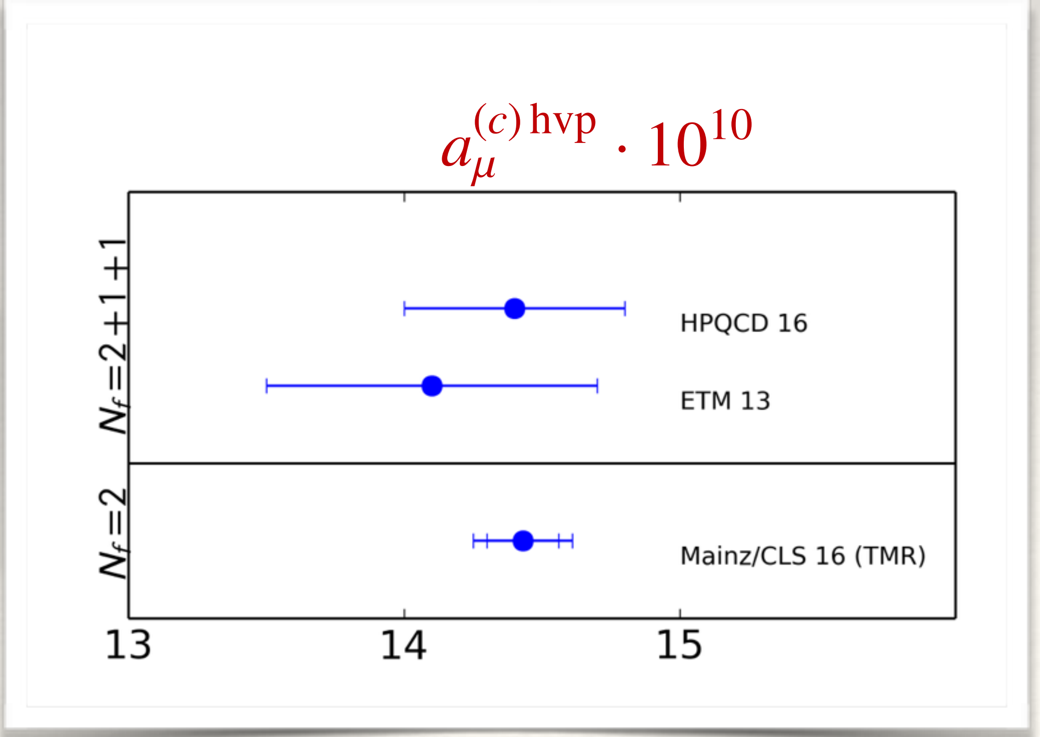
Need to concentrate on:

- stat. error on light contribution
- strong and elm. isospin breaking effects

Results

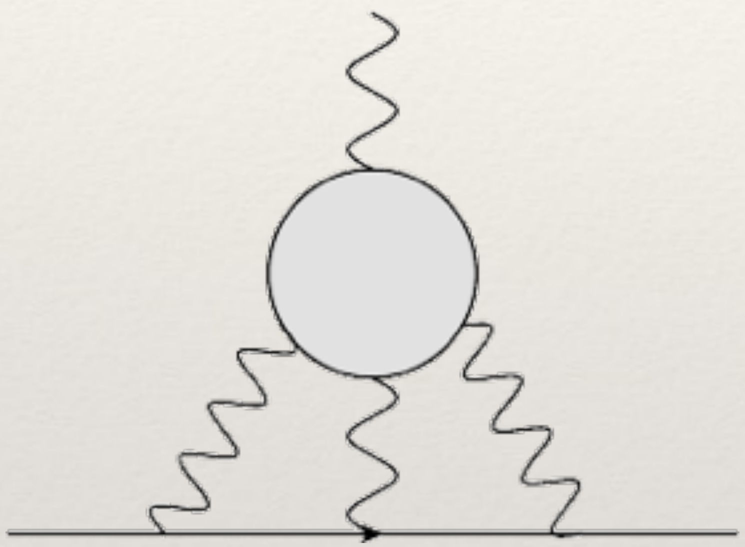






Plots from H. Wittig's Lattice 2016 plenary

Light-by-Light Scattering



Several approaches on the market

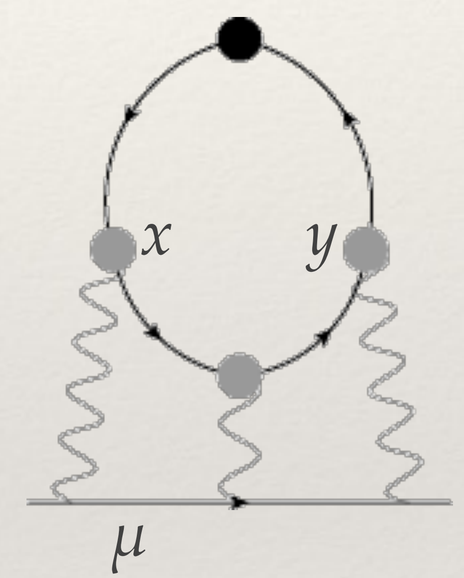
- QCD + QED simulations
  Blum et al. PhysRevLett.114.012001
- QCD + stochastic/exact QED
  Blum et al. PhysRevD.93.014503
- LbL 4pt function
  Green et al. PhysRevLett.115.222003
- computation of sub-processes $\pi^0 \to \gamma^* \gamma^*$
  Feng et al. PhysRevLett.109.182001 Gérardin et al. arXiv:1607.08174

LbL via exact photon propagators

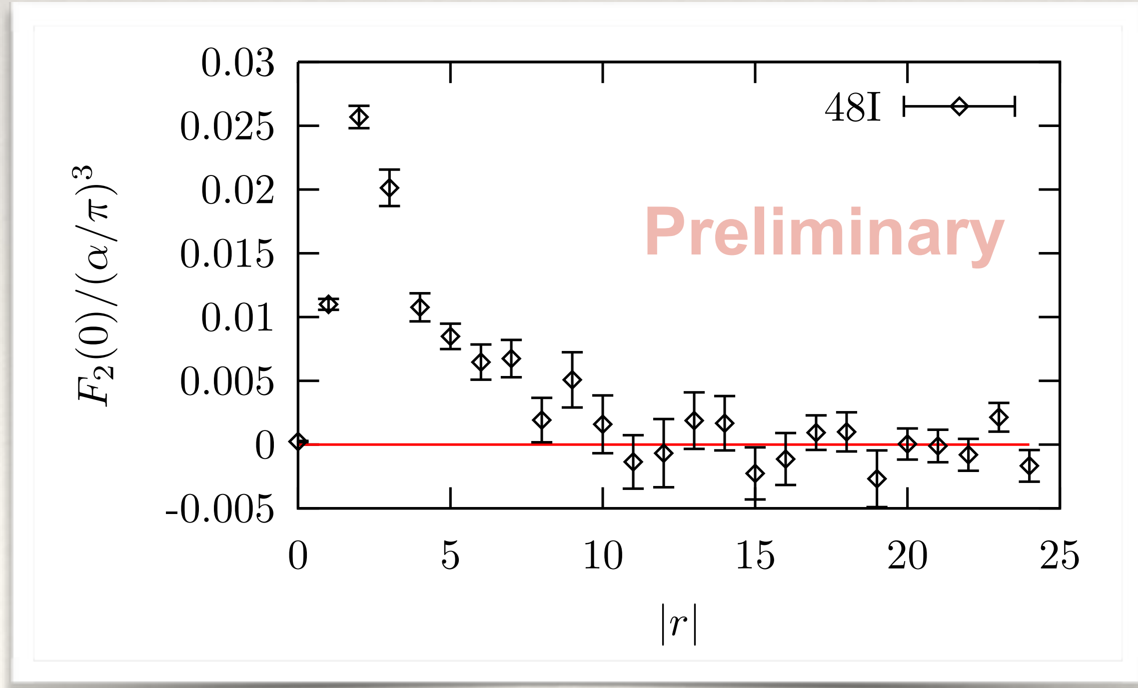
Blum et al. PhysRevD.93.014503

- similar to HVP, moment based approach $(g_\mu - 2)_{cHLbL} \vec{\sigma}_{s's} \propto \int d^3r \left[\vec{r} \times \langle \mu(s') | \vec{J}(\vec{r}) | \mu(s) \rangle \right]$
- perturbative construction including (free) muon propagators
- three Feynman Gauge photon propagators inserted explicitly

$$G_{\mu\nu}(x,y) = \frac{1}{VT} \delta_{\mu\nu} \sum_{k, |\vec{k}| \neq 0} \frac{e^{ik(x-y)}}{\hat{k}^2}$$



- weighted stochastic sampling of x and y position with $r=|x-y|$



L. Jin @ Lattice 2016

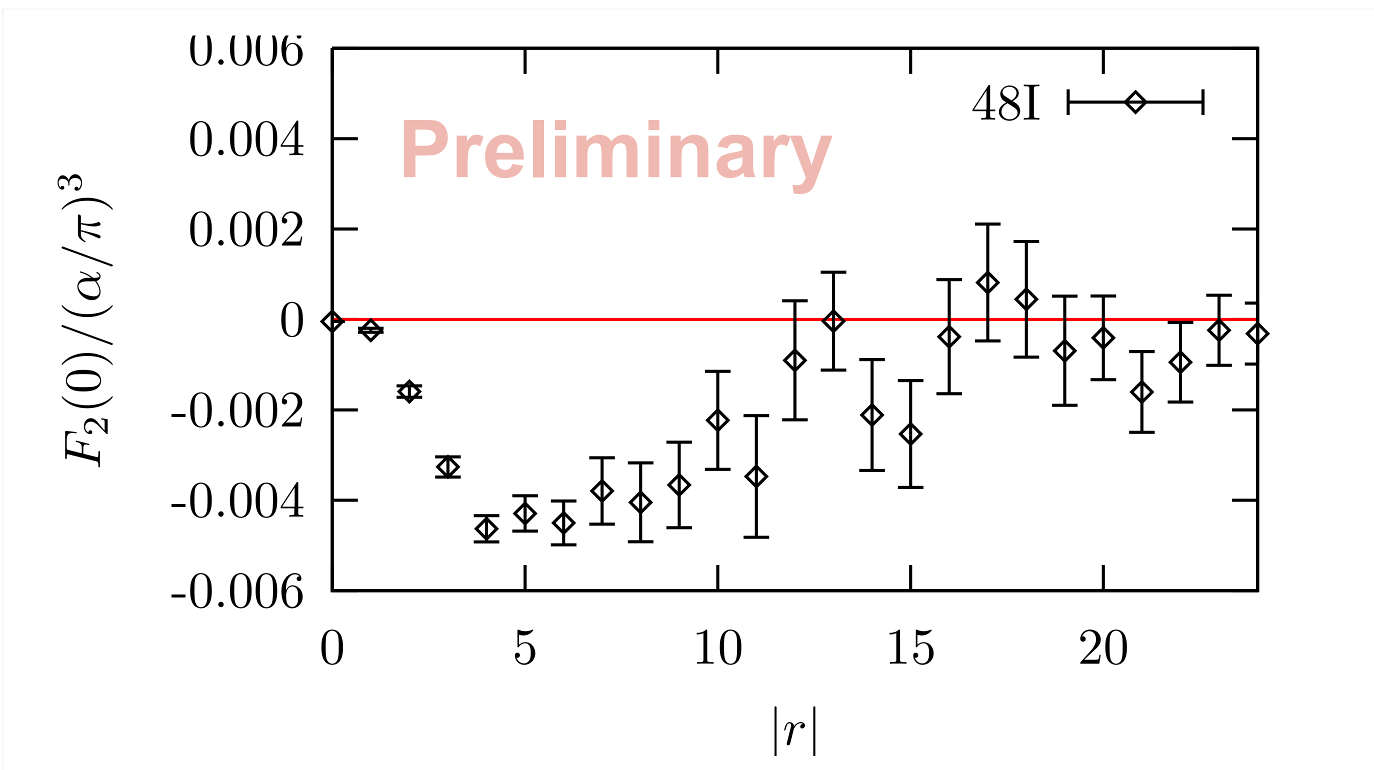
$$a_\mu^{cHLbL} = \begin{cases} 132.1(6.8) \times 10^{-11} & m_\pi = 171\text{MeV} \\ 116.1(9.1) \times 10^{-11} & m_\pi = 139\text{MeV} \end{cases}$$

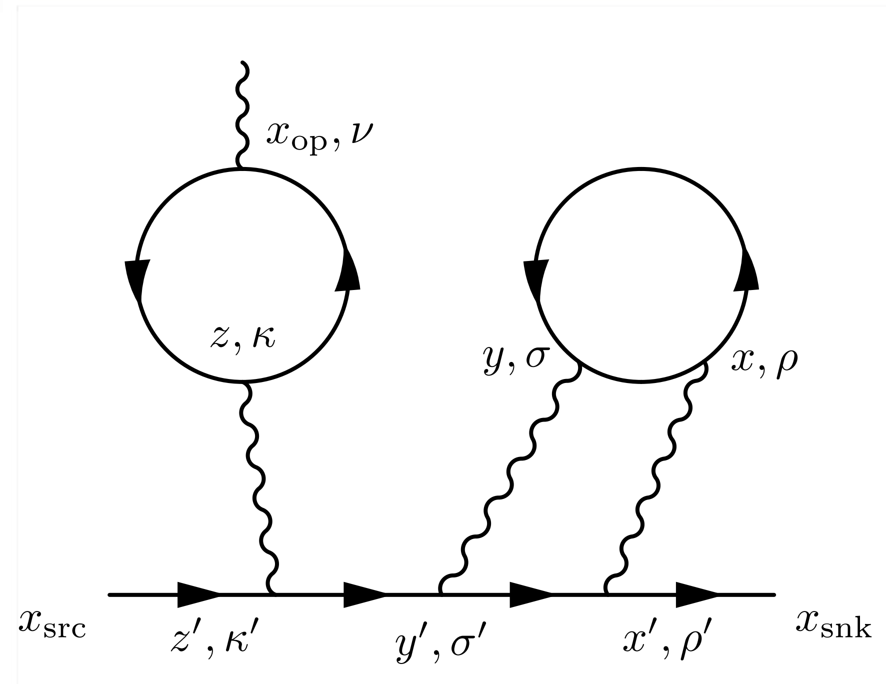
Preliminary result, connected only, further analysis needed

LbL via exact photon propagators

Blum et al. PhysRevD.93.014503

Work on disconnected diagrams under way: L. Jin @ Lattice 2016





$$a_\mu^{dHLbL} = -56.0(12.6) \times 10^{-11} \quad \text{(stat. error only)}$$

There is a clear signal for LbL both connected and disconnected contribs, further work on disconnected, finite volume etc. needed but on track…

Summary

- The hadronic contributions to the muon *g*-2 are now a big topic in L(QCD+QED)
- Physical quark mass simulations have allowed for a real breakthrough in reliability
- Tremendous theoretical/algorithmic /computational progress has been made and the prospect of new experimental results keeps the pressure up
- Most concerned about signal-to-noise (long distance) and finite volume effects
- New techniques developed with impact on applications beyond *g*-2
- 1%(10%)-level precision on LO HVP(LbL) are feasible and we will be able to go beyond
- Very exciting times!!!!!!