The Lamb shift in muonic hydrogen and the proton radius from effective field theories

Based on:
Pineda, hep-ph/0210210, hep-ph/0308193, hep-ph/0412142
Nevado&Pineda, 0712.1294
Peset&Pineda, 1403.3408, 1406.4524, 1508.01948

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Muonic hydrogen

Scales:

$m_p \sim \Lambda_\chi$

$m_\mu \sim m_\pi \sim m_r = \frac{m_\mu m_p}{m_p+m_\mu}$

$m_r\alpha \sim m_e$

Expansion parameters, ratios between scales, mainly:

$$\frac{m_\pi}{m_p} \sim \frac{m_\mu}{m_p} \sim \frac{1}{9}$$

$$\frac{m_r\alpha}{m_r} \sim \frac{m_r\alpha^2}{m_r\alpha} \sim \alpha \sim \frac{1}{137}$$

Needed precision: $m_r\alpha^5$ (Heavy Quarkonium)

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Tool:

Effective Field Theories ≡ Factorization

Why?: There is a hierarchy of different scales (hard, soft and ultrasoft).

$$m >> mv >> mv^2 , \quad (\Lambda_{QCD})$$

EFTs are especially useful in these situations.

1) Perturbative calculations much easier and systematic.

2) Nonperturbative information is parameterized in a model independent way.

3) Power counting.

4) Connection between non-relativistic (NR) Quantum Mechanics and Quantum Field Theories.

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Theoretical setup (muonic hydrogen)

We use an effective field theory, Potential Non-Relativistic QED, which describes the muonic hydrogen dynamics and profits from the hierarchy $m_\mu \gg m_\mu\alpha \gg m_\mu\alpha^2$

$$\left.\begin{array}{l}\left(i\partial_0 - \dfrac{\mathbf{p}^2}{2m_r} - \dfrac{\alpha}{r}\right)\psi(\mathbf{r}) = 0 \\ +\text{corrections to the potential} \\ +\text{interaction with ultrasoft photons}\end{array}\right\} \text{potential NRQED} \qquad E \sim mv^2$$

$$HBET(m_\pi/m_\mu) \rightarrow NRQED(m_\mu\alpha) \rightarrow pNRQED$$

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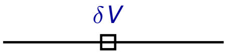
$$HBET(m_\pi/m_\mu) \rightarrow NRQED(m_\mu\alpha) \rightarrow pNRQED$$

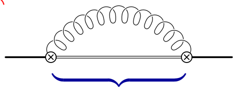
Summary (muonic hydrogen)

1) Matching HBET to NRQED. Integrating out the hard scale, $m_\mu \sim m_\pi$
HBET Feynman diagrams $\leftarrow$
2) Matching NRQED to pNRQED. Integrating out the soft scale, $m_\mu v$
Potential = Wilson loops = HQET-like Feynman diagrams $\leftarrow$
3) Observable: Spectrum or decays
Corrections to the Green Function ($h_s^{(0)} = \mathbf{p}^2/m + V_s^{(0)}$)

$$G_s(E) = P_s \frac{1}{h_s^{(0)} - H_I - E} P_s = G_s^{(0)} + \delta G_s \qquad G_s^{(0)}(E) = \frac{1}{h_s^{(0)} - E}$$

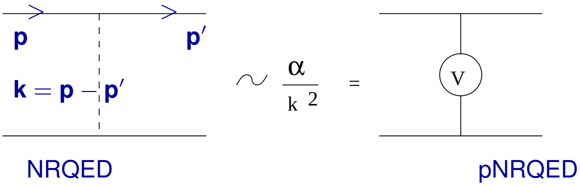
A) Ultrasoft loops (lamb shift-like): $\mathbf{x} \cdot \mathbf{E} \leftarrow$
B) Quantum mechanics perturbation theory$\leftarrow$



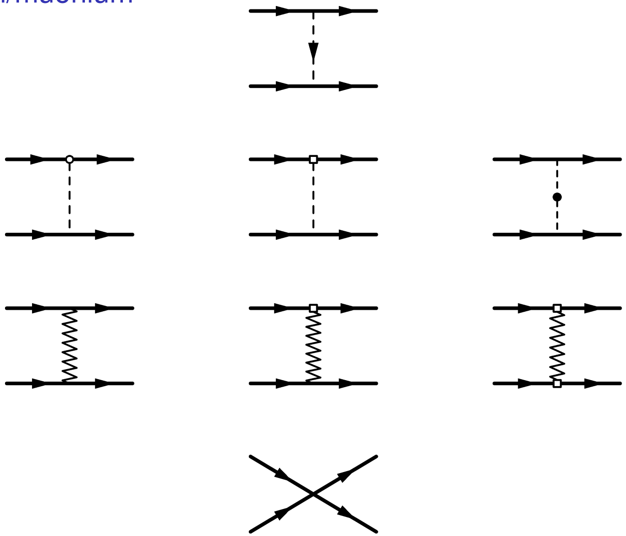


$1/(E - V_s^{(0)} - \mathbf{p}^2/m)$

Matching NRQED to pNRQED



Positronium/muonium
Tree level



Order $1/m^2$

$$\tilde{V}^{(b)} = \frac{\pi\alpha}{2}\left[Z_p \frac{c_D^{(\mu)}}{m_\mu^2} + Z_\mu \frac{c_D^{(p)}}{m_p^2}\right],$$

$$\tilde{V}^{(c)} = -i2\pi\alpha \frac{(\mathbf{p}\times\mathbf{k})}{\mathbf{k}^2}\cdot\left\{Z_p \frac{c_S^{(\mu)}\mathbf{s}_1}{m_\mu^2} + Z_\mu \frac{c_S^{(p)}\mathbf{s}_2}{m_p^2}\right\},$$

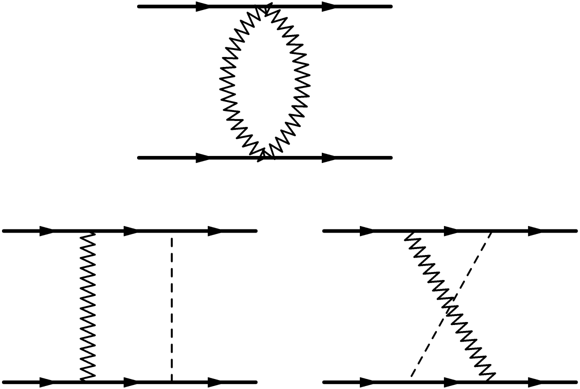
$$\tilde{V}^{(d)} = -Z_\mu Z_p 16\pi\alpha \left(\frac{d_2^{(\mu)}}{m_\mu^2} + \frac{d_2^{(\tau)}}{m_\tau^2} + \frac{d_{2,NR}}{m_p^2}\right),$$

$$\tilde{V}^{(e)} = -Z_\mu Z_p \frac{4\pi\alpha}{m_\mu m_p}\left(\frac{\mathbf{p}^2}{\mathbf{k}^2} - \frac{(\mathbf{p}\cdot\mathbf{k})^2}{\mathbf{k}^4}\right),$$

$$\tilde{V}^{(f)} = -\frac{i4\pi\alpha}{m_\mu m_p}\frac{(\mathbf{p}\times\mathbf{k})}{\mathbf{k}^2}\cdot\left(Z_p c_F^{(\mu)}\mathbf{s}_1 + Z_\mu c_F^{(p)}\mathbf{s}_2\right),$$

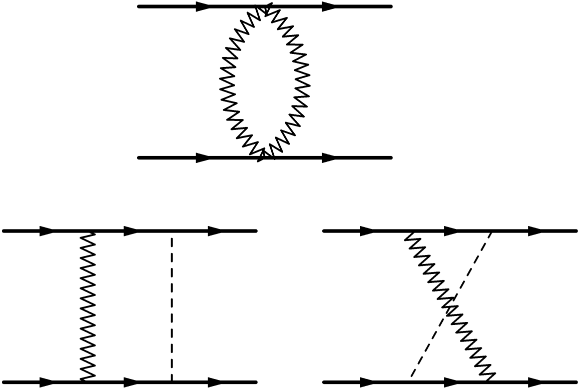
$$\tilde{V}^{(g)} = \frac{4\pi\alpha c_F^{(1)} c_F^{(2)}}{m_\mu m_p}\left(\mathbf{s}_1\cdot\mathbf{s}_2 - \frac{\mathbf{s}_1\cdot\mathbf{k}\mathbf{s}_2\cdot\mathbf{k}}{\mathbf{k}^2}\right),$$

$$\tilde{V}^{(h)} = -\frac{1}{m_p^2}\left\{(c_3^{pl_i} + 3c_4^{pl_i}) - 2c_4^{pl_i}\mathbf{S}^2\right\}.$$



$$\tilde{V}^{(a)}_{1\,loop} = \frac{Z_\mu^2 Z_p^2 \alpha^2}{m_\mu m_p}\left(\log\frac{\mathbf{k}^2}{\mu^2} - \frac{8}{3}\log 2 + \frac{5}{3}\right),$$

$$\tilde{V}^{(b,c)}_{1\,loop} = \frac{4 Z_\mu^2 Z_p^2 \alpha^2}{3 m_\mu m_p}\left(\log\frac{\mathbf{k}^2}{\mu^2} + 2\log 2 - 1\right).$$



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Muonic Hydrogen: electron vacuum polarization

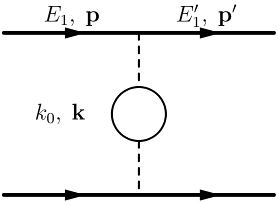

Figure: *Leading correction to the Coulomb potential due to the electron vacuum polarization.* $\mathbf{k} = \mathbf{p} - \mathbf{p}'$ *and* $k_0 = E_1 - E_1'$.

$$\tilde{V}^{(0)} \equiv -4\pi Z_\mu Z_p \alpha_V(k) \frac{1}{\mathbf{k}^2},$$

$$\alpha_{eff}(k) = \alpha \frac{1}{1 + \Pi(-\mathbf{k}^2)},$$

where

$$\Pi(k^2) = \alpha \Pi^{(1)}(k^2) + \alpha^2 \Pi^{(2)}(k^2) + \alpha^3 \Pi^{(3)}(k^2) + \ldots$$

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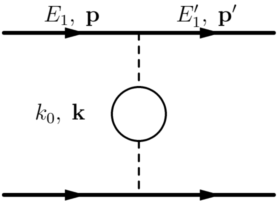

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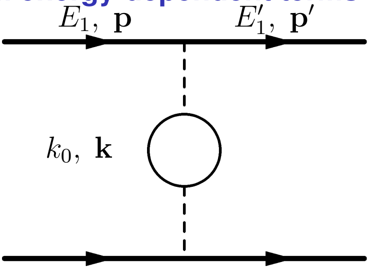
Order $1/m^2$ from energy-dependent terms



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$$u(q^2) = \frac{1}{3}\sqrt{1 - \frac{4}{q^2}}\left(1 + \frac{2}{q^2}\right) \,.$$

Theoretical setup (muonic hydrogen Lamb shift: $\Delta E_L \equiv E(2P_{3/2}) - E(2S_{1/2})$)

$$L_{pNRQED} = \int d^3\mathbf{r} d^3\mathbf{R} dt S^\dagger(\mathbf{r},\mathbf{R},t)\Bigg\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r}$$

$$-V(\mathbf{r},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) + e\mathbf{r}\cdot\mathbf{E}(\mathbf{R},t)\Bigg\} S(\mathbf{r},\mathbf{R},t) - \int d^3\mathbf{r}\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,,$$

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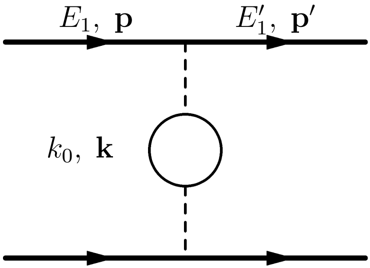
Vacuum polarization effects: $\mathcal{O}(m_r\alpha^3)$



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1-loop static potential

$$E_{LO} = \langle n|\delta V|n\rangle = 205.0074\ \mathrm{meV} = \mathcal{O}(m_r\alpha^3)$$

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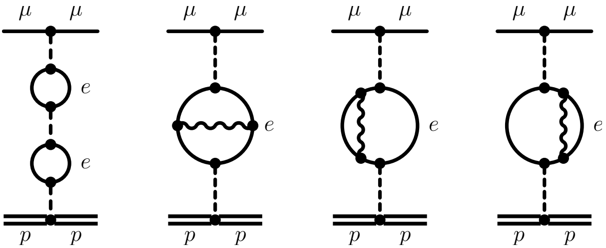


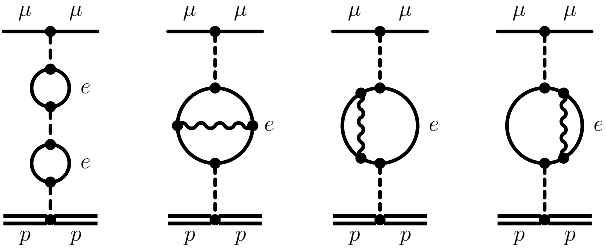
Pachuki/Borie
2-loop static potential is the same as two-loop vacuum polarization iterations
(*two loop vacuum polarization*)

$$\delta E = \langle n|\delta V|n\rangle = 1.5079 \text{ meV} = \mathcal{O}(m_r\alpha^4)$$

Quantum mechanics perturbation theory (*iteration one-loop*)

$$\delta E \sim \langle n|\delta V \frac{1}{H_C - E_n}\delta V|n\rangle = 0.151 \text{ meV} = \mathcal{O}(m_r\alpha^4)$$

Vacuum polarization effects: $\mathcal{O}(m_r\alpha^4)$



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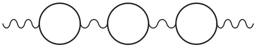
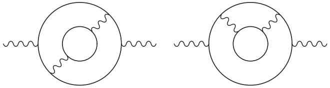
Vacuum polarization effects: $\mathcal{O}(m_r\alpha^5)$

3-loop static potential (three loop vacuum polarization, Kinoshita-Nio)

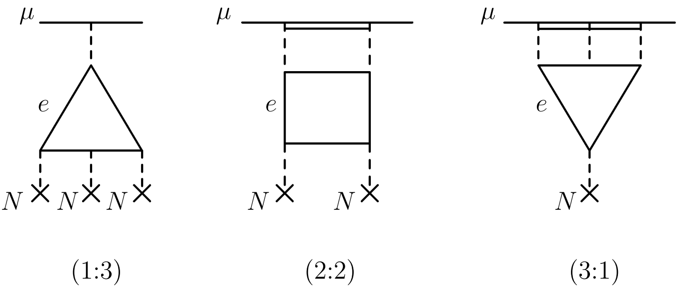
$$0.00752\ \mathrm{meV} = \mathcal{O}(m_r\alpha^5)$$

Slightly corrected by Ivanov et al.





Static potential, not vacuum polarization: $\mathcal{O}(m_r\alpha^5)$



Light-by-light (Wichmann-Kroll and Delbrück) contribution very small (Karshenboim et al.)

$$\Delta E \simeq -0.0009 \text{ mev} = \mathcal{O}(m_r\alpha^5)$$

Earlier work by Borie

$1/m$ potential

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x},\mathbf{X},t)\left\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) + e\mathbf{x}\cdot\mathbf{E}(\mathbf{X},t)\right\} S(\mathbf{x},\mathbf{X},t) - \int d^3\mathbf{x}\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,,$$

$$V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

$$\frac{V^{(1)}(r)}{m_\mu} \rightarrow \mathcal{O}(m_r\alpha^6)$$

relativistic corrections+vacuum polarization

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$$\frac{V^{(2)}(r)}{m_\mu^2} \to \mathcal{O}(m_r \alpha^4, \alpha^5)$$

$\mathcal{O}(m\alpha^4 \times \frac{m^2}{m_p^2})$ 0.0575 (purely relativistic)

$\mathcal{O}(m\alpha^5)$ 0.0059 (Pachucki)

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relativistic corrections+vacuum polarization

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x},\mathbf{X},t)\left\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) + e\mathbf{x}\cdot\mathbf{E}(\mathbf{X},t)\right\} S(\mathbf{x},\mathbf{X},t) - \int d^3\mathbf{x}\frac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

$$V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

$$\frac{V^{(2)}(r)}{m_\mu^2} \to \mathcal{O}(m_r\alpha^4, \alpha^5)$$

$\mathcal{O}(m\alpha^4 \times \frac{m^2}{m_p^2})$ 0.0575 (purely relativistic)

$\mathcal{O}(m\alpha^5)$ 0.0059 (Pachucki)

relativistic corrections+vacuum polarization

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$$-V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) + e\mathbf{x}\cdot\mathbf{E}(\mathbf{X},t)\Big\} S(\mathbf{x},\mathbf{X},t) - \int d^3\mathbf{x}\frac{1}{4}F_{\mu\nu}F^{\mu\nu} ,$$

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$\mathcal{O}(m\alpha^5)$ 0.0169 (Pachucki and Veitia; Borie)

relativistic corrections+vacuum polarization

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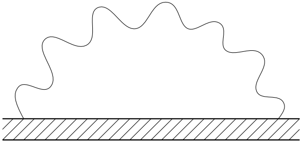
$\mathcal{O}(m\alpha^4 \times \frac{m^2}{m_p^2})$ 0.0575 (purely relativistic)

$\mathcal{O}(m\alpha^5)$ 0.018759 (Jentschura; Karshenboim&Ivanov&Korzinin; Peset&Pineda)

Ultrasoft effects: $\mathcal{O}(m\alpha^5)$



$$\Delta E = -0.6677 \; meV$$

$$\mathcal{O}(m\alpha^5 \frac{m_\mu}{m_p}) : \qquad \Delta E = -0.045 \; meV$$

All (soft+ultrasoft):

$$\Delta E = -0.71896 \; \mathrm{meV}.$$

Start the overlap with hadronic effects.

Hadronic corrections

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x},\mathbf{X},t)\Big\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r}$$

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$$\frac{\delta V^{(2)}(r)}{m_\mu^2} \to \frac{1}{m_p^2} D_d^{had.} \delta^3(\mathbf{r}) \to \Delta E \sim \frac{1}{m_p^2} D_d^{had.} (m_r\alpha)^3$$

$$D_d^{had.} = -c_3 - 16\pi\alpha d_2 + \frac{\pi\alpha}{2} c_D$$

c_3, d_2, c_D, ... matching coefficients of NRQED.

$$HBET(m_\pi/m_\mu) \to NRQED(m_\mu\alpha) \to pNRQED$$

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Hadronic corrections

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x},\mathbf{X},t)\left\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r} \right.$$

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HBET (m_π)

$$\mathcal{L}_{HBET} = \mathcal{L}_\gamma + \mathcal{L}_l + \mathcal{L}_\pi + \mathcal{L}_{l\pi} + \mathcal{L}_{(N,\Delta)} + \mathcal{L}_{(N,\Delta)l} + \mathcal{L}_{(N,\Delta)\pi} + \mathcal{L}_{(N,\Delta)l\pi},$$

$$\mathcal{L}_\gamma = -\frac{1}{4}F^2 + \frac{d_2}{m_p^2}F_{\mu\nu}D^2F^{\mu\nu} + \cdots$$

$$\mathcal{L}_\pi = \frac{F_\pi^2}{4}\mathrm{Tr}\left[D_\mu U D^\mu U\right] + \cdots \qquad U = u^2 = e^{i\frac{\Pi}{F_\pi}}$$

$$\mathcal{L}_N = N^\dagger(iv^\mu\nabla_\mu + g_A u_\mu S^\mu)N + \cdots + (\Delta) + \cdots - e\frac{c_D}{m_p^2}N_p^\dagger \boldsymbol{\nabla}\cdot\mathbf{E}N_p$$

$$D_\mu = \partial_\mu + ieQA_\mu \qquad \nabla_\mu = \partial_\mu + \Gamma_\mu \qquad u_\mu = iu^\dagger(\nabla_\mu U)u$$

$$\Gamma_\mu = \frac{1}{2}\left\{u^\dagger(\partial_\mu + ieQA_\mu)u + u(\partial_\mu + ieQA_\mu)u^\dagger\right\}$$

$$\mathcal{L}_{N,l} = \frac{1}{m_p^2}\sum_i c_{3,R}^{pl_i}\bar{N}_p\gamma^0 N_p \bar{l}_i\gamma^0 l_i + \frac{1}{m_p^2}\sum_i c_{4,R}^{pl_i}\bar{N}_p\gamma^j N_p\bar{l}_i\gamma_j l_i$$

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Hadronic vacuum polarization effects

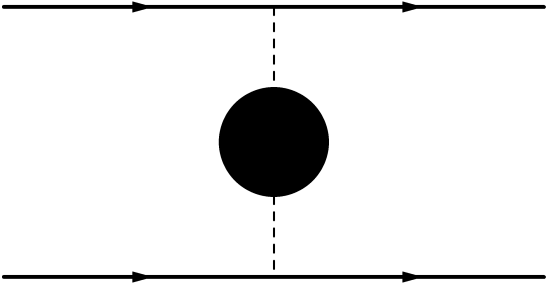
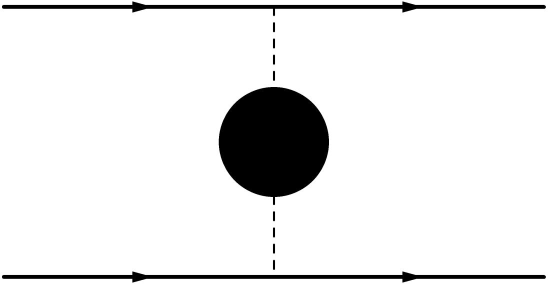


Figure: *Leading correction to the Coulomb potential due to the hadronic vacuum polarization.*

$d_2 \to$ hadronic vacuum polarization

$$\Delta E = 0.011 \text{ meV}$$

Hadronic vacuum polarization effects



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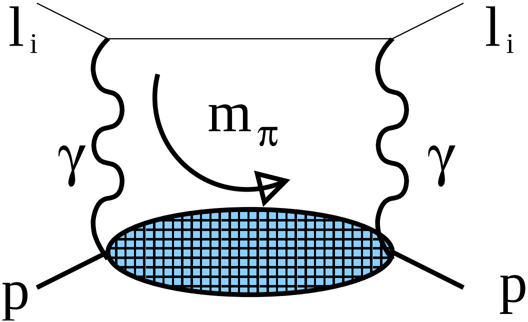
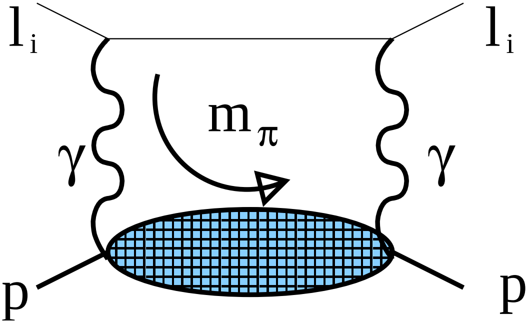
$c_3^{\rm had} =$



$$T^{\mu\nu} = i\int d^4x\, e^{iq\cdot x}\langle p,s|TJ^\mu(x)J^\nu(0)|p,s\rangle\,,$$

$$T^{\mu\nu} = \left(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right)S_1(\rho,q^2)+\frac{1}{m_p^2}\left(p^\mu-\frac{m_p\rho}{q^2}q^\mu\right)\left(p^\nu-\frac{m_p\rho}{q^2}q^\nu\right)S_2(\rho,q^2)$$

$S_1 =??$ $\qquad S_2 =??$

$c_3^{\rm had} =$ 

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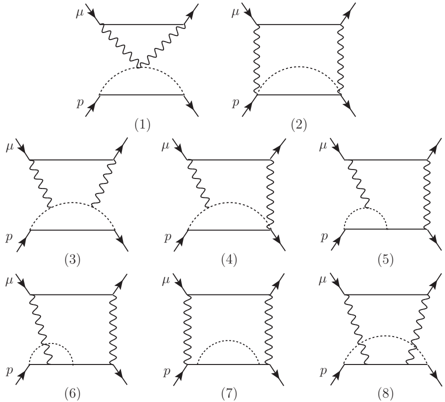


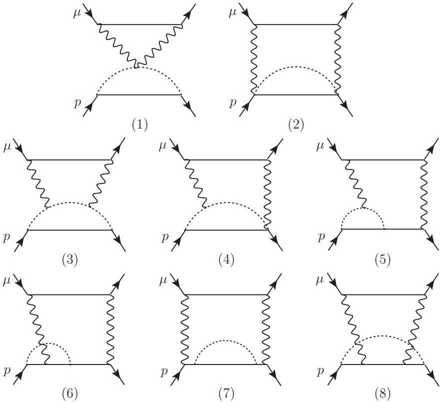
m_μ extra suppression+χPT (Model independent)

Power-like chiral enhanced ($\rightarrow \chi$PT can predict the leading order!)

$$c_3^{\rm had} \sim \alpha^2 \frac{m_\mu}{m_\pi} + \mathcal{O}\left(\alpha^2 \frac{m_\mu}{\Lambda_{QCD}}\right) \qquad \delta E \sim \mathcal{O}(m_\mu \alpha^5 \times \frac{m_\mu^2}{\Lambda_\chi^2} \times \frac{m_\mu}{m_\pi})$$

Error ($\Delta = M_\Delta - M_p \sim 300$ MeV): LO $\times \frac{m_\pi}{\Delta} \simeq$ LO $\times \frac{1}{2}$

$\rightarrow c_3^{\rm had} = \alpha^2 \frac{m_\mu}{m_\pi} 47.2(23.6)$



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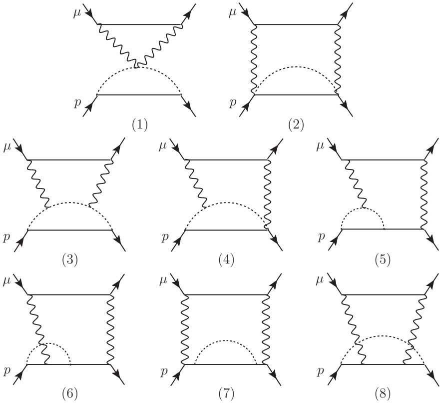

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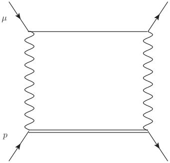
Large N_c. Including the Δ particle

Error:

$$\frac{m_\mu}{\Delta} \sim N_c \frac{m_\mu}{\Lambda_{QCD}} \to \frac{m_\mu}{\Lambda_{QCD}} \sim \frac{1}{3}$$



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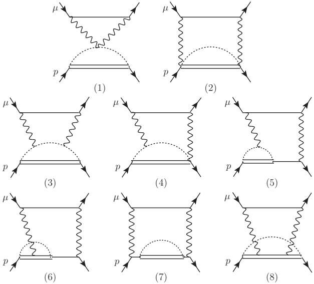

$$c_3^{\rm had} \sim \alpha^2 \frac{m_\mu}{m_\pi}\left[1 + \# \frac{m_\pi}{\Delta} + \cdots\right] + \mathcal{O}\left(\alpha^2 \frac{m_\mu}{\Lambda_{QCD}}\right) = \alpha^2 \frac{m_\mu}{m_\pi} \begin{cases} 47.2(23.6) & (\pi), \\ 56.7(20.6) & (\pi + \Delta), \end{cases}$$

$$\Delta E_{\rm TPE} = 28.59(\pi) + 5.86(\pi\&\Delta) = 34.4(12.5)\mu{\rm eV} \;\; ({\rm Peset\&AP})\,.$$

(Model dependent: $\Delta E_{\rm TPE} = 33(2)\mu{\rm eV}$ (Birse-McGovern))

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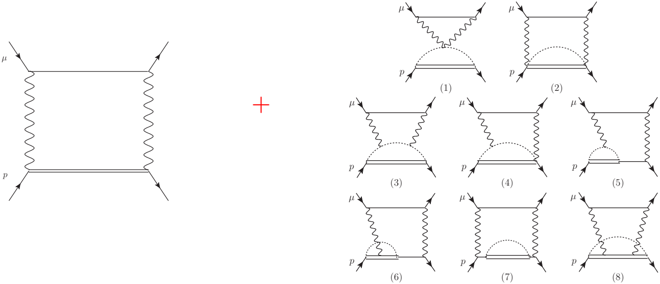

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$$\Delta E_{\rm TPE} \sim m_\mu \alpha^5 \times \frac{m_\mu^2}{(4\pi F_\pi)^2} \times \frac{m_\mu}{m_\pi} \sum_{n=0}^{\infty} c_n (N_c \sqrt{m_q})^n$$

$$\frac{\#}{\sqrt{m_q}} + ? + ?\sqrt{m_q} + \cdots$$

plus large N_c

$$\frac{\#}{\sqrt{m_q}} + \left[\#N_c + ? + \frac{?}{N_c} + \cdots\right] + \left[\#N_c^2 + ?N_c + ? + \cdots\right]\sqrt{m_q} + \cdots$$

? $\rightarrow$ Size of the counterterm in HBET

$c_3^{\rm had}$: Two-Photon-Exchange contribution= Born+polarizability

Born:

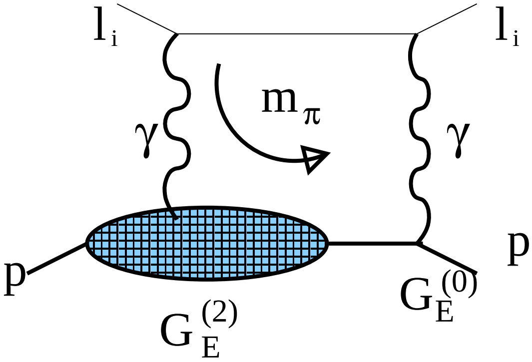

Figure: *Symbolic representation (plus permutations) of the Born $\langle r^3 \rangle$ correction.*

$$\Delta E_{\rm Born} = 0.010 \frac{\langle r^3 \rangle_{(2)}}{\rm fm^3}$$

$$\frac{\langle r^3 \rangle_{(2)}}{\rm fm^3} = \frac{48}{\pi} \int \frac{d^3k}{4\pi} \frac{1}{\mathbf{k}^6} \left(G_E^2 - 1 + \frac{1}{3} \langle r^2 \rangle \mathbf{k}^2 \right) = \frac{96}{\pi} \int \frac{d^{D-1}k}{4\pi} \frac{1}{\mathbf{k}^6} G_E^{(0)} G_E^{(2)}$$

$$\Delta E_{\rm Born} = 0.010\frac{\langle r^3\rangle_{(2)}}{\rm fm^3}$$

$$\frac{\langle r^3\rangle_{(2)}}{\rm fm^3} = \frac{96}{\pi}\int d^{D-1}k\frac{1}{\mathbf{k}^6}G_E^{(0)}G_E^{(2)}$$

$$\delta c_{3,\rm Born}^{pl_i} = \frac{\pi}{3}\alpha^2 m_p^2 m_\mu \langle r^3\rangle_{(2)} = 2(\pi\alpha)^2\left(\frac{m_p}{4\pi F_0}\right)^2\frac{m_{l_i}}{m_\pi}\left\{\frac{3}{4}g_A^2+\frac{1}{8}\right.$$
$$\left.+\frac{2}{\pi}g_{\pi N\Delta}^2\frac{m_\pi}{\Delta}\sum_{r=0}^{\infty}C_r\left(\frac{m_\pi}{\Delta}\right)^{2r}+g_{\pi N\Delta}^2\sum_{r=1}^{\infty}H_r\left(\frac{m_\pi}{\Delta}\right)^{2r}\right\},$$

where ($\Delta = M_\Delta - M_p \sim 300$ MeV)

$$C_r = \frac{(-1)^r\Gamma(-3/2)}{\Gamma(r+1)\Gamma(-3/2-r)}\left\{B_{6+2r}-\frac{2(r+2)}{3+2r}B_{4+2r}\right\}, \qquad r\geq 0,$$

$$B_n \equiv \int_0^\infty dt\frac{t^{2-n}}{\sqrt{1-t^2}}\ln\left[\frac{1}{t}+\sqrt{\frac{1}{t^2}-1}\right]$$

$$H_n \equiv \frac{n!(2n-1)!!\Gamma[-3/2]}{2(2n)!!\Gamma[1/2+n]}.$$

Including Pions and Δ particles

$$\Delta E_{\rm Born} = 0.010 \frac{\langle r^3 \rangle_{(2)}}{\rm fm^3}$$

$$\frac{\langle r^3 \rangle_{(2)}}{\rm fm^3} = \frac{96}{\pi} \int d^{D-1}k \frac{1}{\mathbf{k}^6} G_E^{(0)} G_E^{(2)}$$

$$\delta c_{3,{\rm Born}}^{pl_i} = \frac{\pi}{3}\alpha^2 m_p^2 m_\mu \langle r^3 \rangle_{(2)} = 2(\pi\alpha)^2 \left(\frac{m_p}{4\pi F_0}\right)^2 \frac{m_{l_i}}{m_\pi} \left\{ \frac{3}{4} g_A^2 + \frac{1}{8} + \frac{2}{\pi} g_{\pi N\Delta}^2 \frac{m_\pi}{\Delta} \sum_{r=0}^{\infty} C_r \left(\frac{m_\pi}{\Delta}\right)^{2r} + g_{\pi N\Delta}^2 \sum_{r=1}^{\infty} H_r \left(\frac{m_\pi}{\Delta}\right)^{2r} \right\},$$

where ($\Delta = M_\Delta - M_p \sim 300$ MeV)

$$C_r = \frac{(-1)^r \Gamma(-3/2)}{\Gamma(r+1)\Gamma(-3/2-r)} \left\{ B_{6+2r} - \frac{2(r+2)}{3+2r} B_{4+2r} \right\}, \qquad r \geq 0,$$

$$B_n \equiv \int_0^\infty dt \frac{t^{2-n}}{\sqrt{1-t^2}} \ln\left[\frac{1}{t} + \sqrt{\frac{1}{t^2} - 1}\right]$$

$$H_n \equiv \frac{n!(2n-1)!!\Gamma[-3/2]}{2(2n)!!\Gamma[1/2+n]}.$$

Including Pions and Δ particles

	$\langle r^3 \rangle$	$\langle r^5 \rangle$	$\langle r^6 \rangle$	$\langle r^3 \rangle_{(2)}$
π	0.4980	1.619	5.203	0.9960
$\pi\&\Delta$	0.4071	1.522	4.978	0.8142
Dipole	0.7706	1.775	3.325	2.023
Kelly	0.9838	3.209	7.440	2.526
Distler et al.	1.16(4)	8.0(1.2)(1.0)	29.8(7.6)(12.6)	2.85(8)

Table: *The first two rows give the prediction from the effective theory (Peset&AP). The third row corresponds to the standard dipole fit with $\langle r^2 \rangle = 0.6581$ fm^3. The fourth and fifth rows correspond to different parameterizations of experimental data. For completeness, we also quote $\langle r^3 \rangle_{(2)} = 2.71$ fm^3 from Friar.*

μeV	DR	*Pachucki*	*Carlson et al*	HBET	*Peset&AP*(π)	($\pi\&\Delta$)
$\Delta E_{\rm Born}$		23.2(1.0)	24.7(1.6)		10.1(5.1)	8.3(4.3)

Table: *Predictions for the Born contribution to the $n = 2$ Lamb shift. The first two entries correspond to dispersion relations. The last two entries are the predictions of HBET: The 3rd entry is the prediction of HBET at leading order (only pions) and the last entry is the prediction of HBET at leading and next-to-leading order (pions and Deltas).*

The proton radius in *ep* scattering from χPT
PRELIMINARY: Hessels, Horbatsch and Pineda, in preparation

$$G_E(Q^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} Q^{2n} \langle r^{2n} \rangle$$

- Extrapolation from $|\mathbf{q}| \sim 100$ MeV to $|\mathbf{q}| = 0$
  $|\mathbf{q}| \sim m_\mu \alpha \sim m_e \sim 0.5$ MeV (muonic hydrogen)
  $|\mathbf{q}| \sim m_e \alpha \sim 5.10^{-3}$ MeV (hydrogen)
- dependence on the fitting functions: normalization factors, full data set ...

Higher moments diverge in the chiral limit

$$\langle r^{2k} \rangle \sim m_\pi^{2-2k}$$

Extrapolation controlled by χPT (at low Q^2): $r_p \sim 0.84$.

Bigger values for the moments produce larger values of r_p.

c_3 Polarizability effects

(μeV)	[1]	[2]	[3]	[4]	$B\chi PT(\pi)$	HBET(π)	(π&Δ)
$\Delta E_{\rm pol}$	12(2)	11.5	7.4(2.4)	15.3(5.6)	$8.2(^{+1.2}_{-2.5})$	18.5(9.3)	26.2(10.0)

Table: *Polarizability contribution to the $n = 2$ Lamb shift. The first four entries use dispersion relations for the inelastic term and different modeling functions for the subtraction term. [1] Pachucki, [2] Martynenko, [3] Carlson&Vanderhaeghen, [4] Gorchtein et al.. The 5th entry is the prediction obtained using $B\chi PT$ (Alarcon et al.). The last two entries are the predictions of HBET (Nevado&AP and Peset&AP).*

Polarizability=Inelastic+subtraction

$$c_{3,\mathrm{sub}}^{pl_i} = -e^4 M_p m_{l_i} \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{k_E^4} \frac{1}{k_E^4 + 4m_{l_i}^2 k_{0,E}^2} (3k_{0,E}^2 + \mathbf{k}^2) S_1(0, -k_E^2)$$

$$c_{3,\mathrm{inel}}^{pl_i} = -e^4 M_p m_{l_i} \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{k_E^4} \frac{1}{k_E^4 + 4m_{l_i}^2 k_{0,E}^2} \times \left\{ (3k_{0,E}^2 + \mathbf{k}^2)(S_1(ik_{0,E}, -k_E^2) - S_1(0, -k_E^2)) - \mathbf{k}^2 S_2(ik_{0,E}, -k_E^2) \right\}$$

$$\Delta E^{(\mathrm{sub})}(\pi{-}\mathrm{loop}) = -1.62\ \mu\mathrm{eV}\ ; \qquad \Delta E^{(\mathrm{sub})}(\pi\Delta{-}\mathrm{loop}) = -1.23\ \mu\mathrm{eV}.$$

$$\delta c_{3,\mathrm{sub}}^{pl_i} \sim -\delta c_{3,\mathrm{inel}}^{pl_i} \simeq -\frac{4}{3}\alpha^2 \frac{m_{l_i}}{\Delta} b_{1,F}^2 \ln(\nu/m_{l_i}) \to \Delta E^{(\mathrm{sub})}|_{\nu=m_\rho} \sim -11.4\ \mu\mathrm{eV}$$

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Definition of the proton radius

$$\langle p', s|J^\mu|p, s\rangle = \bar{u}(p')\left[F_1(q^2)\gamma^\mu + iF_2(q^2)\frac{\sigma^{\mu\nu}q_\nu}{2m_p}\right]u(p)\,,$$

$$F_i(q^2) = F_i + \frac{q^2}{m_p^2}F_i' + \ldots$$

$$G_E(q^2) = F_1(q^2) + \frac{q^2}{4m_p^2}F_2(q^2), \qquad G_M(q^2) = F_1(q^2) + F_2(q^2).$$

$$r_p^2 = 6\frac{d}{dq^2}G_{E,p}(q^2)|_{q^2=0}$$

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$$\langle p', s | J^\mu | p, s \rangle = \bar{u}(p') \left[F_1(q^2)\gamma^\mu + iF_2(q^2)\frac{\sigma^{\mu\nu} q_\nu}{2m_p} \right] u(p) \,,$$

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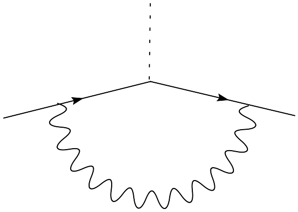
$$r_p^2(\nu) = 6\frac{d}{dq^2}G_{E,p}(q^2)|_{q^2=0}$$

Infrared divergent! $\rightarrow$ Wilson coefficient



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$$r_p^2(\nu) = 6\frac{d}{dq^2}G_{E,p}(q^2)|_{q^2=0} = \frac{3}{4}\frac{1}{m_p^2}\left(c_D^{(p)}(\nu) - 1\right)$$

$$c_D(\nu) = 1 + 2F_2 + 8F_1' = 1 + 8m_p^2\left.\frac{dG_{p,E}(q^2)}{d\,q^2}\right|_{q^2=0}\,,$$

Standard definition (corresponds to the experimental number):

$$r_p^2 = \frac{3}{4}\frac{1}{m_p^2}\left(c_D(\nu) - c_{D,point-like}(\nu)\right)$$

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	$\mathcal{O}(m_r\alpha^3)$	$V_{\rm VP}^{(0)}$	205. 00737
	$\mathcal{O}(m_r\alpha^4)$	$V_{\rm VP}^{(0)}$	1. 50795
	$\mathcal{O}(m_r\alpha^4)$	$V_{\rm VP}^{(0)}$	0. 15090
	$\mathcal{O}(m_r\alpha^5)$	$V_{\rm VP}^{(0)}$	0. 00752
	$\mathcal{O}(m_r\alpha^5)$	$V_{\rm LbL}^{(0)}$	$-0.$ 00089(2)
	$\mathcal{O}(m_r\alpha^4 \times \frac{m_\mu^2}{m_p^2})$	$V^{(2,1)} + V^{(3,0)}$	0. 05747
	$\mathcal{O}(m_r\alpha^5)$	$V_{\rm VP}^{(2,2)} + V^{(2,1)} \times V_{\rm VP}^{(0,2)}$	0. 01876
	$\mathcal{O}(m_r\alpha^5)$	$V_{\rm no-VP}^{(2,2)} + \text{ultrasoft}$	$-0.$ 71896
	$\mathcal{O}(m_r\alpha^6 \times \ln(\frac{m_\mu}{m_e}))$	$V^{(2,3)}; c_D^{(\mu)}$	$-0.$ 00127
	$\mathcal{O}(m_r\alpha^6 \times \ln\alpha)$	$V_{\rm VP}^{(2,3)}; c_D^{(\mu)}$	$-0.$ 00454
	$\mathcal{O}(m_r\alpha^4 \times m_r^2 r_p^2)$	$V^{(2,1)}; c_D^{(p)}$	$-5.\ 1975\frac{r_p^2}{\rm fm^2}$
	$\mathcal{O}(m_r\alpha^5 \times m_r^2 r_p^2)$	$V_{\rm VP}^{(2,2)} + V^{(2,1)} \times V_{\rm VP}^{(0,2)}; c_D^{(p)}$	$-0.\ 0282\frac{r_p^2}{\rm fm^2}$
	$\mathcal{O}(m_r\alpha^6 \ln\alpha \times m_r^2 r_p^2)$	$V^{(2,3)}; c_D^{(p)}$	$-0.\ 0014\frac{r_p^2}{\rm fm^2}$
	$\mathcal{O}(m_r\alpha^5 \times \frac{m_r^2}{m_\rho^2})$	$V_{\rm VP_{had}}^{(2)}; d_2^{\rm had}$	0. 0111(2)
	$\mathcal{O}(m_r\alpha^5 \times \frac{m_r^2}{m_\rho^2}\frac{m_\mu}{m_\pi})$	$V^{(2)}; c_3^{\rm had}$	0. 0344(125)

$$\Delta E_L^{\text{our work}} = \left[206.0243(30) - 5.2270(7)\frac{r_p^2}{\text{fm}^2} + 0.0455(125)\right] \text{meV}\,.$$

This expression includes the leading logarithmic $\mathcal{O}(m_\mu\alpha^6)$ terms, as well as the leading $\mathcal{O}\left(m_\mu\alpha^5\frac{m_\mu^2}{m_\rho^2}\right)$ hadronic effects. The accuracy of our result is limited by uncomputed terms of $\mathcal{O}(m_\mu\alpha^5\frac{m_\mu^3}{m_\rho^3}, m_\mu\alpha^6)$.

Using

$$\Delta E_L^{\text{exp}} \equiv E(2P_{3/2}) - E(2S_{1/2}) = 202.3706(23)\,\text{meV} \quad \text{Antognini et al.}$$

$$r_p = 0.8413(15)\,\text{fm}.$$

At 6.8σ variance with respect the CODATA value.

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Hadronic corrections: Spin-dependent

Pineda: hep-ph/0210210, hep-ph/0308193; Peset&Pineda, 1406.4524

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x},\mathbf{X},t)\left\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) + e\mathbf{x}\cdot\mathbf{E}(\mathbf{X},t)\right\} S(\mathbf{x},\mathbf{X},t) - \int d^3\mathbf{x}\frac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

$$V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

$$\frac{\delta V^{(2)}(r)}{m_\mu^2} \to \frac{1}{m_p^2} D_d^{had.}(\mathbf{S}_1+\mathbf{S}_2)^2\delta^3(\mathbf{r})$$

$$D_s^{had.} = 2c_4$$

c_4, matching coefficient of NRQED.

$$HBET(m_\pi/m_\mu) \to NRQED(m_\mu\alpha) \to pNRQED$$

$$\delta\mathcal{L} = \cdots - \frac{c_4}{m_p^2} N_p^\dagger \boldsymbol{\sigma} N_p \mu^\dagger \boldsymbol{\sigma}\mu$$

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Pineda: hep-ph/0210210, hep-ph/0308193; Peset&Pineda, 1406.4524

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x},\mathbf{X},t)\left\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) + e\mathbf{x}\cdot\mathbf{E}(\mathbf{X},t)\right\} S(\mathbf{x},\mathbf{X},t) - \int d^3\mathbf{x}\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,,$$

$$V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

$$\frac{\delta V^{(2)}(r)}{m_\mu^2} \to \frac{1}{m_p^2} D_d^{had.}(\mathbf{S}_1+\mathbf{S}_2)^2\delta^3(\mathbf{r})$$

$$D_s^{had.} = 2c_4$$

c_4, matching coefficient of NRQED.

$$HBET(m_\pi/m_\mu) \to NRQED(m_\mu\alpha) \to pNRQED$$

$$\delta\mathcal{L} = \dots - \frac{c_4}{m_p^2} N_p^\dagger \boldsymbol{\sigma} N_p \mu^\dagger \boldsymbol{\sigma}\mu$$

Hadronic corrections: Spin-dependent

Pineda: hep-ph/0210210, hep-ph/0308193; Peset&Pineda, 1406.4524

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x},\mathbf{X},t) \left\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) + e\mathbf{x}\cdot\mathbf{E}(\mathbf{X},t) \right\} S(\mathbf{x},\mathbf{X},t) - \int d^3\mathbf{x} \frac{1}{4} F_{\mu\nu}F^{\mu\nu} \,,$$

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c_4, Spin-dependent effects

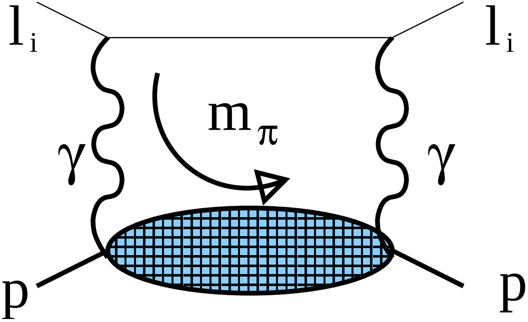


Figure: *Symbolic representation (plus permutations) of the spin-dependent correction.*

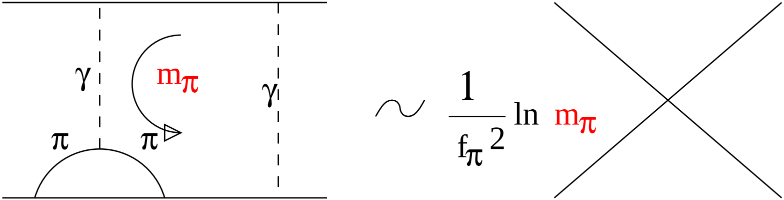
$$c_4^{pl} = -\frac{ig^4}{3}\int \frac{d^Dk}{(2\pi)^D}\frac{1}{k^2}\frac{1}{k^4-4m_l^2k_0^2}\left\{A_1(k_0,k^2)(k_0^2+2k^2)+3k^2\frac{k_0}{m_p}A_2(k_0,k^2)\right\}$$

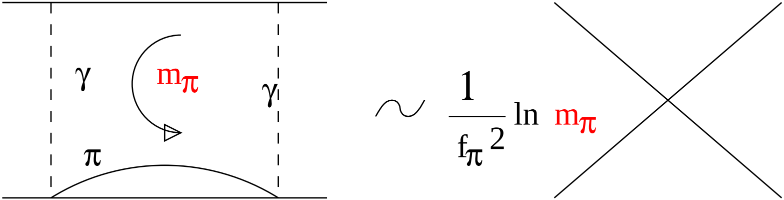
$$T^{\mu\nu} = i\int d^4x\, e^{iq\cdot x}\langle p,s|TJ^{\mu}(x)J^{\nu}(0)|p,s\rangle\,,$$

which has the following structure ($\rho = q\cdot p/m$):

$$\begin{aligned} T^{\mu\nu} &= \left(-g^{\mu\nu}+\frac{q^{\mu}q^{\nu}}{q^2}\right)S_1(\rho,q^2) \\ &+\frac{1}{m_p^2}\left(p^{\mu}-\frac{m_p\rho}{q^2}q^{\mu}\right)\left(p^{\nu}-\frac{m_p\rho}{q^2}q^{\nu}\right)S_2(\rho,q^2) \\ &-\frac{i}{m_p}\,\epsilon^{\mu\nu\rho\sigma}q_{\rho}s_{\sigma}A_1(\rho,q^2) \\ &-\frac{i}{m_p^3}\,\epsilon^{\mu\nu\rho\sigma}q_{\rho}\left((m_p\rho)s_{\sigma}-(q\cdot s)p_{\sigma}\right)A_2(\rho,q^2) \end{aligned}$$

Leading chiral logs to the hyperfine splitting





$$\delta V = 2\frac{c_4}{m_p^2}\mathbf{S}^2\delta^{(3)}(\mathbf{r}) \,.$$

$$\delta E_{HF} \sim \mathcal{O}(m_\mu \alpha^5 \times \frac{m_\mu^2}{\Lambda_\chi^2} \times \ln m_\pi)$$

The leading chiral logs are determined for Hydrogen and muonic hydrogen hyperfine splitting. Model independent result!

$$\begin{aligned} c_4^{pl_i} &\simeq \left(1-\frac{\mu_p^2}{4}\right)\alpha^2 \ln\frac{m_{l_i}^2}{\nu^2} + \frac{b_{1,F}^2}{18}\alpha^2 \ln\frac{\Delta^2}{\nu^2} \\ &+ \frac{m_p^2}{(4\pi F_0)^2}\alpha^2 \frac{2}{3}\left(\frac{2}{3}+\frac{7}{2\pi^2}\right)\pi^2 g_A^2 \ln\frac{m_\pi^2}{\nu^2} \\ &+ \frac{m_p^2}{(4\pi F_0)^2}\alpha^2 \frac{8}{27}\left(\frac{5}{3}-\frac{7}{\pi^2}\right)\pi^2 g_{\pi N\Delta}^2 \ln\frac{\Delta^2}{\nu^2} \\ &\overset{(N_c\to\infty)}{\simeq} \alpha^2 \ln\frac{m_l^2}{\nu^2} + \frac{m_p^2}{(4\pi F_0)^2}\alpha^2\pi^2 g_A^2 \ln\frac{m_\pi^2}{\nu^2}\,. \end{aligned}$$

$$E_{\rm HF} = 4\frac{c_4^{pl_i}}{m_p^2}\frac{1}{\pi}(\mu_{l_ip}\alpha)^3 \sim m_{l_i}\alpha^5 \frac{m_{l_i}^2}{m_p^2} \times (\ln m_q, \ln\Delta, \ln m_{l_i})\,.$$

$$c_4^{pl_i} = c_{4,\mathrm{R}}^{pl_i} + c_{4,\mathrm{point-like}}^{pl_i} + c_{4,\mathrm{Born}}^{pl_i} + c_{4,\mathrm{pol}}^{pl_i} + \mathcal{O}(\alpha^3)\,.$$

$$\begin{aligned}
c_{4,\mathrm{point-like}}^{pl_i} &= \left(1-\frac{\kappa_p^2}{4}\right)\alpha^2 \ln\frac{m_{l_i}^2}{\nu^2}\,,\\
c_{4,\mathrm{Born}}^{pl_i} &\simeq (4\pi\alpha)^2 M_p \frac{2}{3}\int \frac{d^{D-1}k}{(2\pi)^{D-1}}\frac{1}{\mathbf{k}^4} G_E^{(0)} G_M^{(1)}\\
&\simeq \frac{M_p^2}{(4\pi F_0)^2}\alpha^2 \frac{2}{3}\pi^2\left[g_A^2 \ln\frac{m_\pi^2}{\nu^2} + \frac{4}{9} g_{\pi N\Delta}^2 \ln\frac{\Delta^2}{\nu^2}\right],\\
c_{4,\mathrm{pol}}^{pl_i} &= \frac{M_p^2}{(4\pi F_0)^2}\frac{\alpha^2}{\pi}\frac{8}{3}\left(\frac{7\pi}{8}-\frac{\pi^3}{12}\right)\left[g_A^2 \ln\frac{m_\pi^2}{\nu^2} - \frac{8}{9} g_{\pi N\Delta}^2 \ln\frac{\Delta^2}{\nu^2}\right]\\
&\quad + \frac{b_{1,F}^2}{18}\alpha^2 \ln\frac{\Delta^2}{\nu^2}\,.
\end{aligned}$$

$$c_{4,\mathrm{point-like}}^{pl_i} + c_{4,pol}^{pl_i} \overset{(N_c\to\infty)}{\simeq} \alpha^2 \ln\frac{m_{l_i}^2}{\nu^2}$$

Polarizability contribution small and vanishes in the large N_c limit!!

$$c_4^{pl_i} = c_{4,\mathrm{R}}^{pl_i} + c_{4,\mathrm{point-like}}^{pl_i} + c_{4,\mathrm{Born}}^{pl_i} + c_{4,\mathrm{pol}}^{pl_i} + \mathcal{O}(\alpha^3)\,.$$

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c_{4,\mathrm{pol}}^{pl_i} &= \frac{M_p^2}{(4\pi F_0)^2}\frac{\alpha^2}{\pi}\frac{8}{3}\left(\frac{7\pi}{8}-\frac{\pi^3}{12}\right)\left[g_A^2\ln\frac{m_\pi^2}{\nu^2}-\frac{8}{9}g_{\pi N\Delta}^2\ln\frac{\Delta^2}{\nu^2}\right]\\
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Polarizability contribution small and vanishes in the large N_c limit!!

c_4, Spin-dependent effects (Born): $\mathcal{O}(m_\mu \alpha^5 \times \frac{m_\mu^2}{\Lambda_\chi^2} \times \ln m_\pi)$

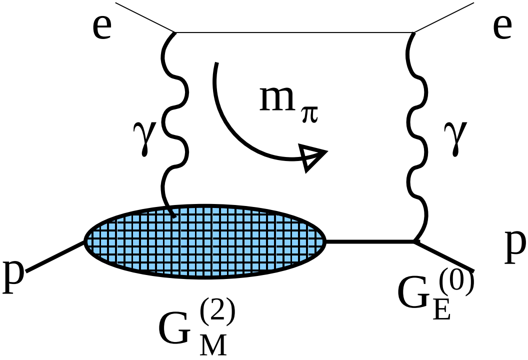

Figure: *Symbolic representation (plus permutations) of the Born correction.*

$$c^{pl}_{4,\rm Born} = (4\pi\alpha)^2 M_p \frac{2}{3} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{1}{\mathbf{k}^4} G_E^{(0)} G_M^{(2)} .$$

Zemach magnetic radius

Chiral logs can be determined and constitute the leading contribution!

$$\langle r_Z \rangle = -\frac{4}{\pi}\int_0^\infty \frac{dQ}{Q^2}\left[G_E(Q^2)G_M(Q^2)-1\right] = -\frac{4}{\pi}\int_0^\infty q^{D-4}\frac{dQ}{Q^2}G_E^{(0)}(Q^2)G_M^{(2)}(Q^2)\,.$$

$$\langle r_Z \rangle = -\frac{3}{4\pi}\frac{1}{\alpha^2 M_p}c_{4,\mathrm{Born}}^{pl_i} \simeq -\frac{\pi}{2}\frac{M_p}{(4\pi F_0)^2}\left[g_A^2 \ln\frac{m_\pi^2}{\nu^2}+\frac{4}{9}g_{\pi N\Delta}^2\ln\frac{\Delta^2}{\nu^2}\right] \stackrel{(\nu=m_\rho)}{=} 1.35\ \mathrm{fm}\,.$$

"Experiment" $\sim 1.04 - 1.08$ fm.

Hydrogen. Fixing $\nu = m_\rho$ we obtain:

$$E_{\rm HF,logarithms}(m_\rho) = -0.031 \ {\rm MHz}\,.$$

It accounts for 2/3 of the difference between theory (QED) and experiment.

$$E_{\rm HF}(QED) - E_{\rm HF}(exp) = -0.046 \ {\rm MHz}.$$

What is left gives the size of the counterterm. Experimentally what we have is $c_4^{ple} = -48\alpha^2$ and $c_{4,R}^{pl}(m_\rho) \simeq c_{4,R}^{p}(m_\rho) \simeq -16\alpha^2$.

Muonic hydrogen.

$$c_4^{pl_\mu} = c_4^{ple} + \left[c_{4,\rm point-like}^{pl_\mu} - c_{4,\rm point-like}^{ple}\right] + \left[c_{4,\rm pol}^{pl_\mu} - c_{4,\rm pol}^{ple}\right] + \mathcal{O}(\alpha^3, \alpha^2 m_\mu/\Lambda_{QCD})\,.$$

$$c_{4,\rm point-like}^{pl_\mu} - c_{4,\rm point-like}^{ple} = \left(1 - \frac{\kappa_p^2}{4}\right)\alpha^2 \ln\frac{m_\mu^2}{m_e^2} \simeq 2.09\alpha^2$$

$$c_{4,\rm pol}^{pl_\mu} - c_{4,\rm pol}^{ple} = 0.17\alpha^2(\pi) + 0.07\alpha^2(\Delta) + 0.008\alpha^2(\pi\&\Delta) = 0.24\alpha^2\,.$$

Overall we obtain $c_4^{pl_\mu} \simeq -46\alpha^2$. The bulk of this contribution comes from the Born term, which in turn is related to the Zemach magnetic radius

$$\Delta E_{HF} \simeq -\frac{1.1746}{n^3}\bigg|_{n=2} = -0.147(50) meV \ (Pachucki: -0.145)$$

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Recent work:
Hagelstein et al.: 1511.04301&1512.03765

These works should check if the leading chiral contribution is reproduced (which is a model independent prediction).

CONCLUSIONS

Effective Field Theories provide with a model independent, efficient and systematic (Power counting) approach to the dynamics of NR systems and a unified framework to determine the nonperturbative effects.

Possible to obtain a rigorous connection between Quantum Field Theories and a NR Quantum-mechanical formulation of the NR systems.

The proton radius is a matching coefficient of the effective theory. In general it is an scheme/scale dependent object.

The two-photon exchange energy shift (and the associated error) is (and can only be) computed in a model independent way with χPT. Overall number consistent with determinations from a combined use of dispersion relations and models, but individual contributions are quite different.

χPT predicts the chiral logs of the hyperfine splitting and the difference between hydrogen and muonic hydrogen.

Analytic understanding of the QCD dynamics: m_q and N_c dependence.

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$$\Delta E_L^{\text{our work}} = \left[206.0243(30) - 5.2270(7)\frac{r_p^2}{\text{fm}^2} + 0.0455(125)\right] \text{meV}\,.$$

Using

$$\Delta E_L^{\text{exp}} \equiv E(2P_{3/2}) - E(2S_{1/2}) = 202.3706(23)\,\text{meV} \quad \text{Antognini et al.}$$

$$r_p = 0.8413(15)\,\text{fm}.$$

At 6.8σ variance with respect the CODATA value.

Definition of the neutron radius

$$\langle p', s | J^\mu | p, s \rangle = \bar{u}(p') \left[F_1(q^2)\gamma^\mu + iF_2(q^2)\frac{\sigma^{\mu\nu} q_\nu}{2m_p} \right] u(p) \,,$$

$$F_i(q^2) = F_i + \frac{q^2}{m_p^2} F_i' + \ldots$$

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$$F_i(q^2) = 0 + \frac{q^2}{m_p^2}F_i' + \ldots$$

$$G_E(q^2) = F_1(q^2) + \frac{q^2}{4m_p^2}F_2(q^2), \qquad G_M(q^2) = F_1(q^2) + F_2(q^2).$$

$$r_n^2 = 6\frac{d}{dq^2}G_{n,E}(q^2)|_{q^2=0} = \frac{3}{4}\frac{1}{m_p^2}c_D^{(n)}$$

$$c_D = 0 + 2F_2 + 8F_1' = 0 + 8m_n^2\left.\frac{dG_{n,E}(q^2)}{d\,q^2}\right|_{q^2=0}$$

Standard definition (corresponds to the experimental number):

$$r_n^2 = \frac{3}{4}\frac{1}{m_n^2}c_D$$

Neutron-lepton scattering length = REAL low energy constant

$$b_{nl} = \frac{1}{4m_n}\left(\alpha c_D - \frac{2}{\pi}c_{3,NR}^{nl}\right) \sim D_d^{(n)had}$$

It is not proportional to the radius