



# Quantum Geometry

Daniel Krefl

based on

0: arXiv: 1105.0630  
(with Aganagic, Cheng, Dijkgraaf & Vafa)

I: arXiv: 1311.0584  
II: arXiv: 1410.7116  
III: arXiv: 1605.00182 ←



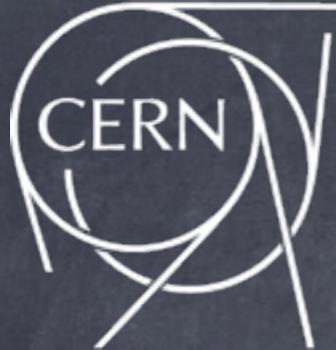
# Classical Geometry



# Quantum Geometry

Classical geometry:

$$\Sigma : f(x, p) = 0$$



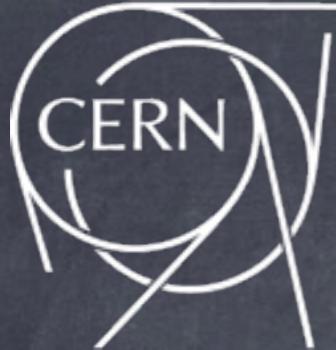
# Quantum Geometry

Classical geometry:

$$\Sigma : f(x, p) = 0$$



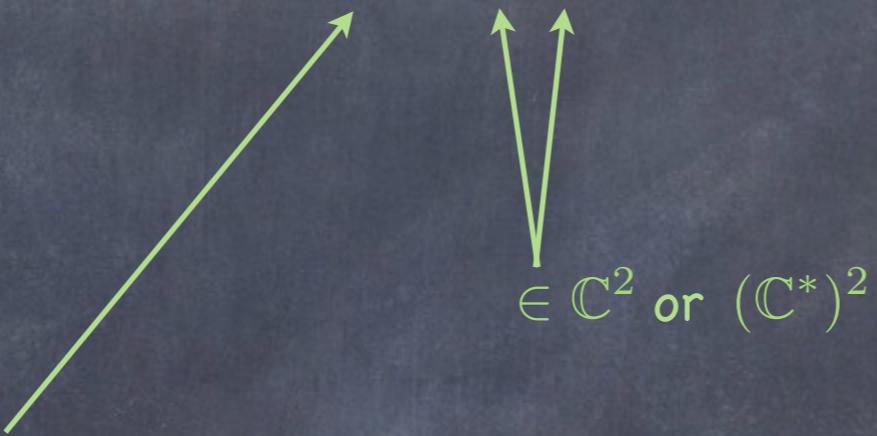
$\in \mathbb{C}^2$  or  $(\mathbb{C}^*)^2$



# Quantum Geometry

Classical geometry:

$$\Sigma : f(x, p) = 0$$



Algebraic curve (not necessarily polynomial)



# Quantum Geometry

Classical geometry:

In full generality, depends on auxiliary parameters  $z_i$

$$\Sigma : f(x, p) = 0$$

$\in \mathbb{C}^2$  or  $(\mathbb{C}^*)^2$

Algebraic curve (not necessarily polynomial)



# Quantum Geometry

Classical geometry:

$$\begin{array}{c} \Sigma : f(x, p) = 0 \\ \downarrow \\ \mathcal{M} \end{array}$$

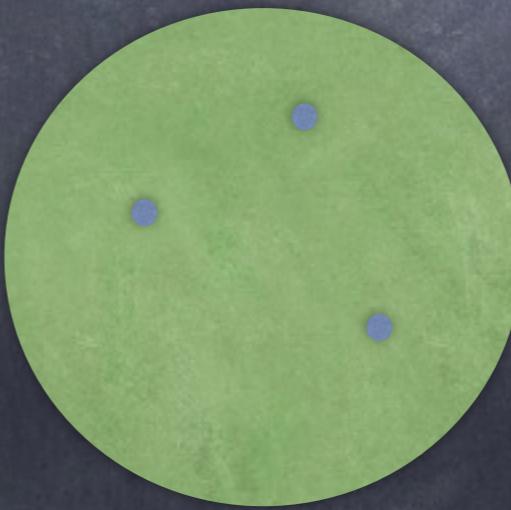


# Quantum Geometry

Classical geometry:

$$\Sigma : f(x, p) = 0$$
$$\downarrow$$
$$\mathcal{M}$$

Moduli space of the curve

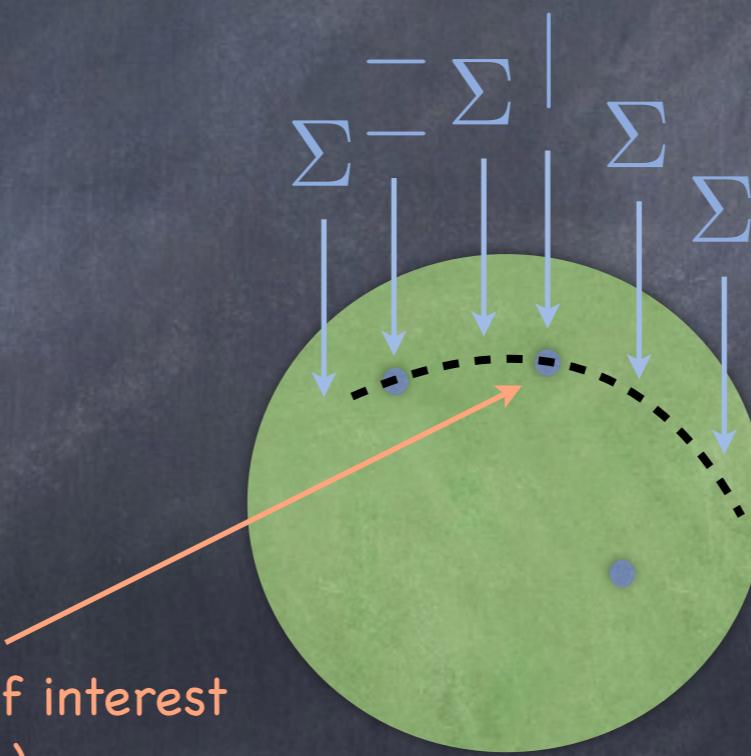




# Quantum Geometry

Classical geometry:

$$\Sigma : f(x, p) = 0$$
$$\downarrow$$
$$\mathcal{M}$$



Special points in moduli space of interest  
(for instance, degenerating cycle)



# Quantum Geometry

Classical geometry:

$$\Sigma : f(x, p) = 0$$



The first homology group  $H_1(\Sigma, \mathbb{Z})$  possesses a canonical basis  $A_g, B_g$  (not unique)



# Quantum Geometry

Classical geometry:

$$\Sigma : f(x, p) = 0$$

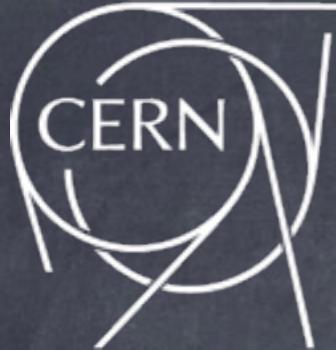


The first homology group  $H_1(\Sigma, \mathbb{Z})$  possesses a canonical basis  $A_g, B_g$  (not unique)

Under introducing a (meromorphic) 1-form  $d\lambda$  we obtain periods

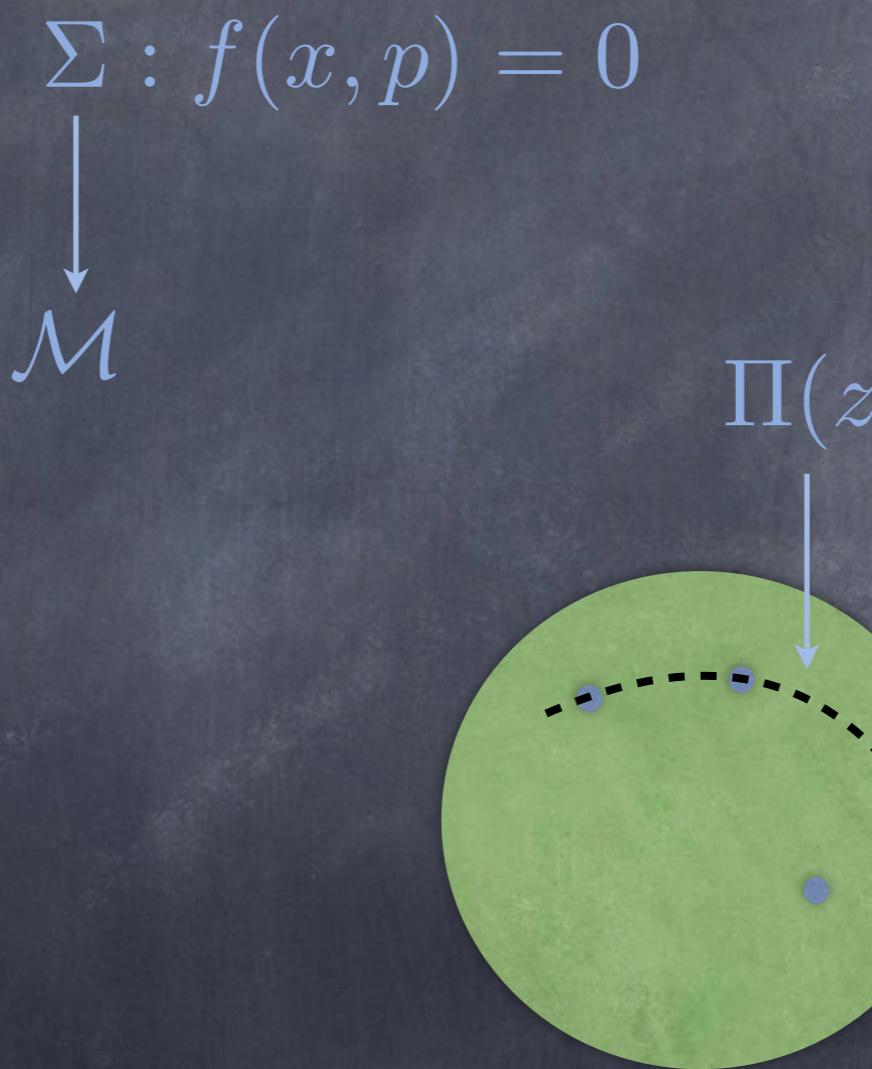
$$\begin{pmatrix} \Pi_A \\ \Pi_B \end{pmatrix}$$

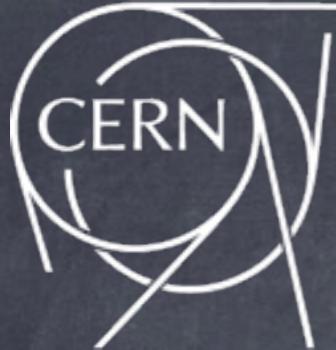
$$\Pi(z) = \oint d\lambda$$



# Quantum Geometry

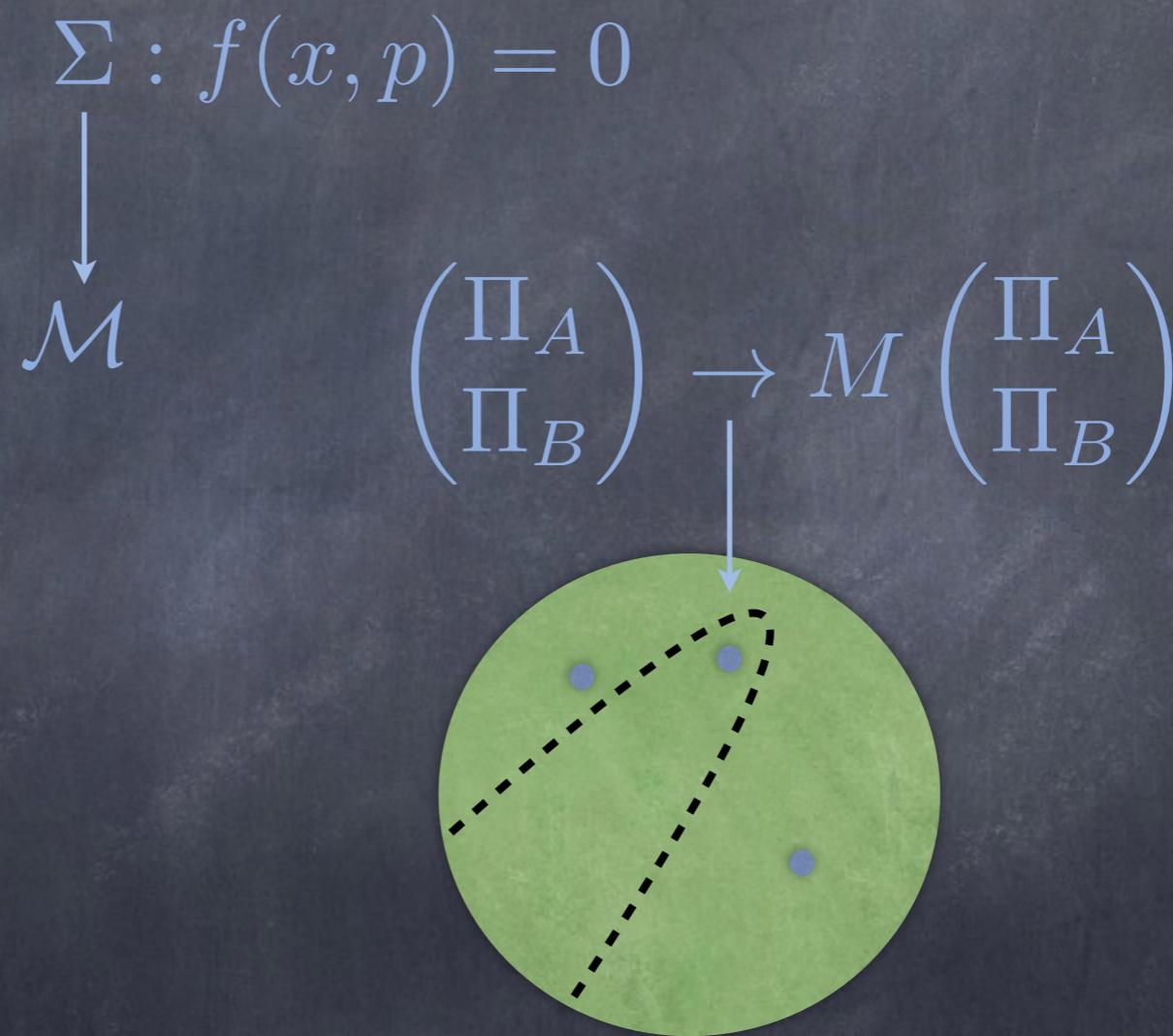
Classical geometry:





# Quantum Geometry

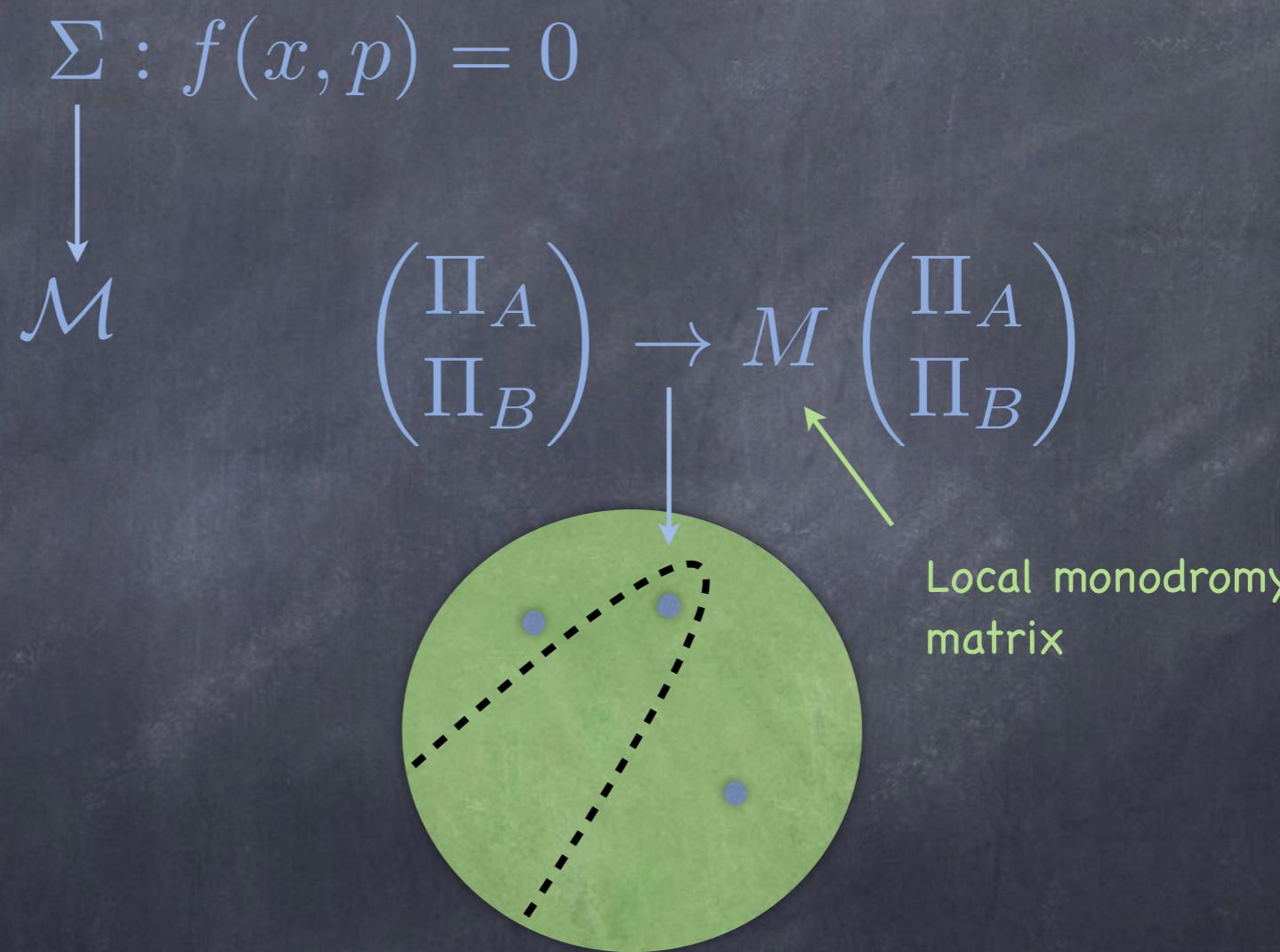
Classical geometry:





# Quantum Geometry

Classical geometry:





# Quantum Geometry

Classical geometry:

$$\Sigma : f(x, p) = 0$$

Highly relevant for :

- \* Seiberg-Witten solution of  $\mathcal{N} = 2$  supersymmetric gauge theories in 4d



# Quantum Geometry

Classical geometry:

$$\Sigma : f(x, p) = 0$$

Highly relevant for :

- ★ Seiberg-Witten solution of  $\mathcal{N} = 2$  supersymmetric gauge theories in 4d
- ★ Topological strings on toric Calabi-Yaus



# Quantum Geometry

Classical geometry:

$$\Sigma : f(x, p) = 0$$

Highly relevant for :

- ★ Seiberg-Witten solution of  $\mathcal{N} = 2$  supersymmetric gauge theories in 4d
- ★ Topological strings on toric Calabi-Yaus
- ★ Matrix Models



# Quantum Geometry

Classical geometry:

$$\Sigma : f(x, p) = 0$$

Highly relevant for :

- ★ Seiberg-Witten solution of  $\mathcal{N} = 2$  supersymmetric gauge theories in 4d
- ★ Topological strings on toric Calabi-Yaus
- ★ Matrix Models
- ★ ...



# Quantum Geometry

Classical geometry:

$$\Sigma : f(x, p) = 0$$

Highly relevant for :

- ★ Seiberg-Witten solution of  $\mathcal{N} = 2$  supersymmetric gauge theories in 4d
  - ★ Topological strings on toric Calabi-Yaus
  - ★ Matrix Models
  - ★ ...
- “Geometrification” of physics



# Quantum Geometry

Classical geometry:

$$\Sigma : f(x, p) = 0$$

Usual physical quantity of interest is the partition function  $Z$  or, equivalently, free energy  $\mathcal{F} = \log Z$



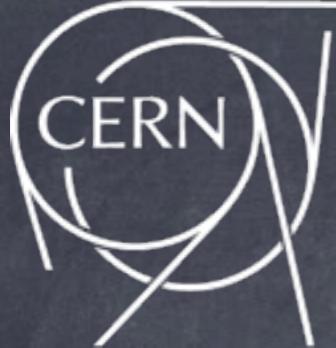
# Quantum Geometry

Classical geometry:

$$\Sigma : f(x, p) = 0$$

Usual physical quantity of interest is the partition function  $Z$  or, equivalently, free energy  $\mathcal{F} = \log Z$

→ “Geometrize” the tree-level part



# Quantum Geometry

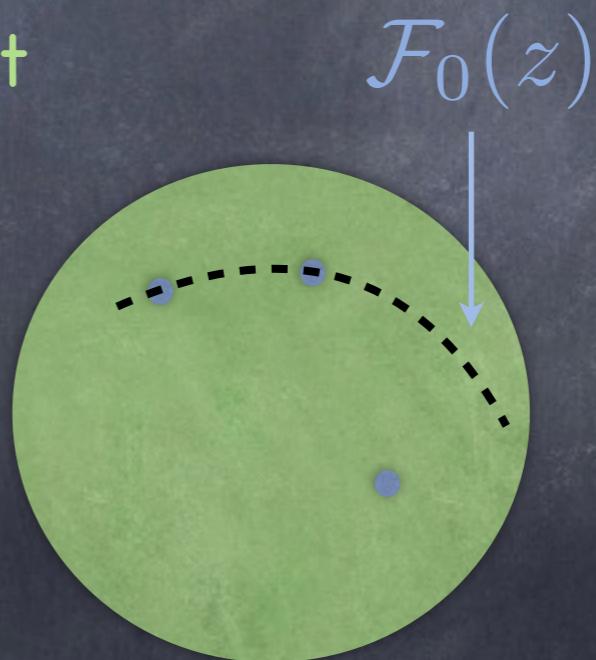
Classical geometry:

$$\Sigma : f(x, p) = 0$$

Usual physical quantity of interest is the partition function  $Z$  or, equivalently, free energy  $\mathcal{F} = \log Z$

→ “Geometrize” the tree-level part

i.e. find  $(\Sigma, \lambda)$  and fix a basis  $\begin{pmatrix} \Pi_A \\ \Pi_B \end{pmatrix}$   
such that we can obtain  $\mathcal{F}_0$  from  $\Pi$

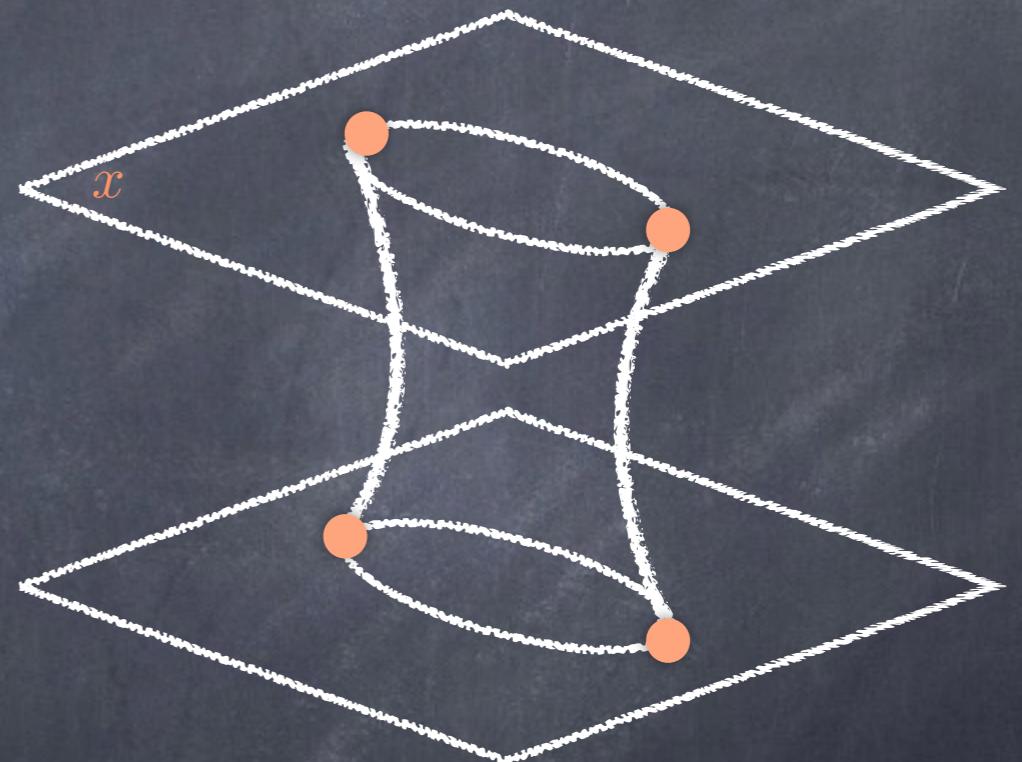




# Quantum Geometry

Example:

$$p^2 + x^2 = E$$



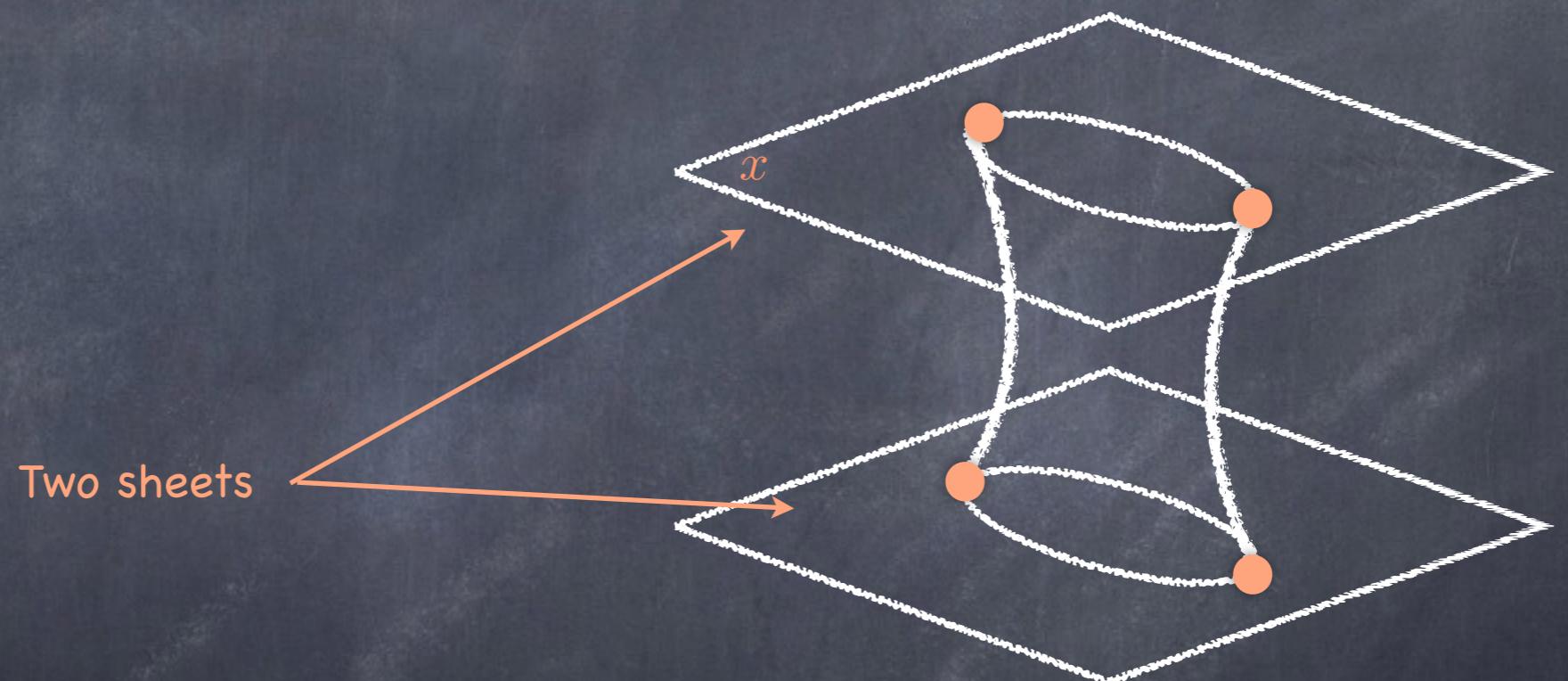
(deformed conifold)



# Quantum Geometry

Example:

$$p^2 + x^2 = E$$

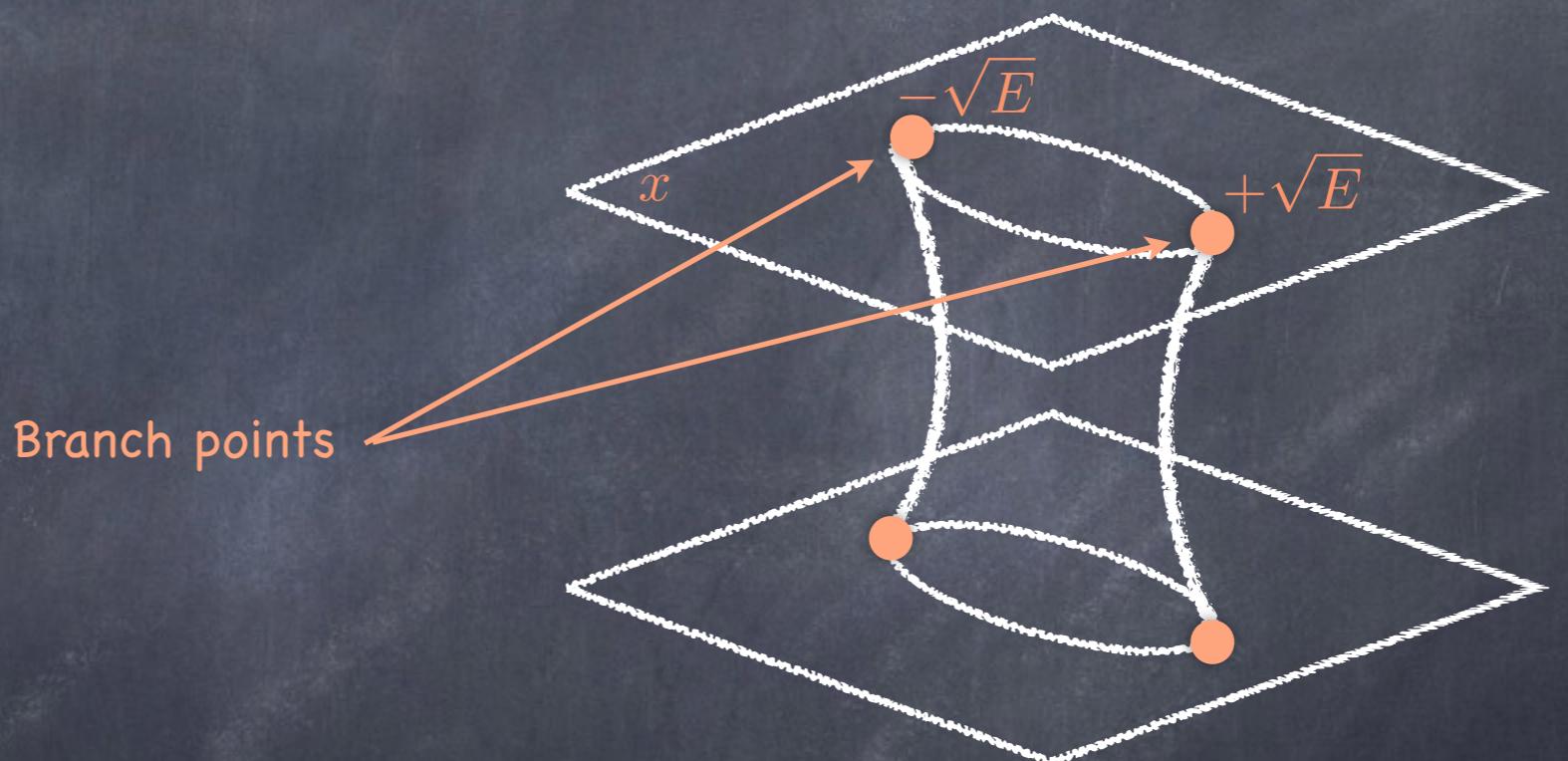




# Quantum Geometry

Example:

$$p^2 + x^2 = E$$

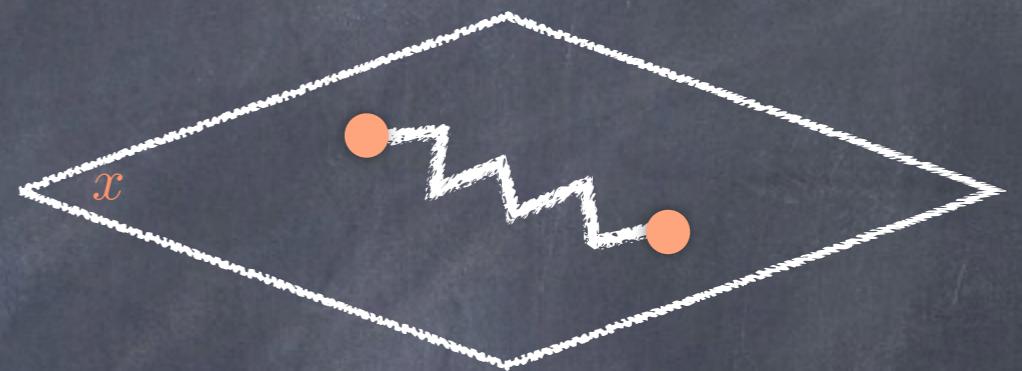




# Quantum Geometry

Example:

$$p^2 + x^2 = E$$





# Quantum Geometry

Example:

$$p^2 + x^2 = E$$

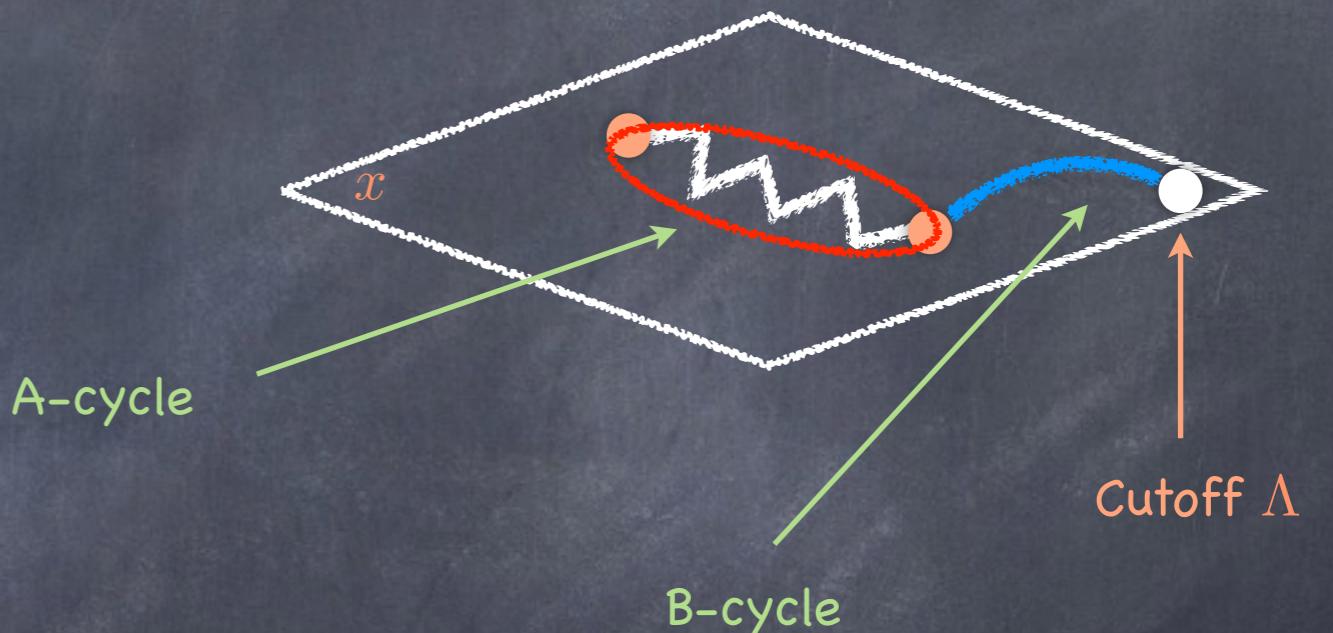




# Quantum Geometry

Example:

$$p^2 + x^2 = E$$





# Quantum Geometry

Example:

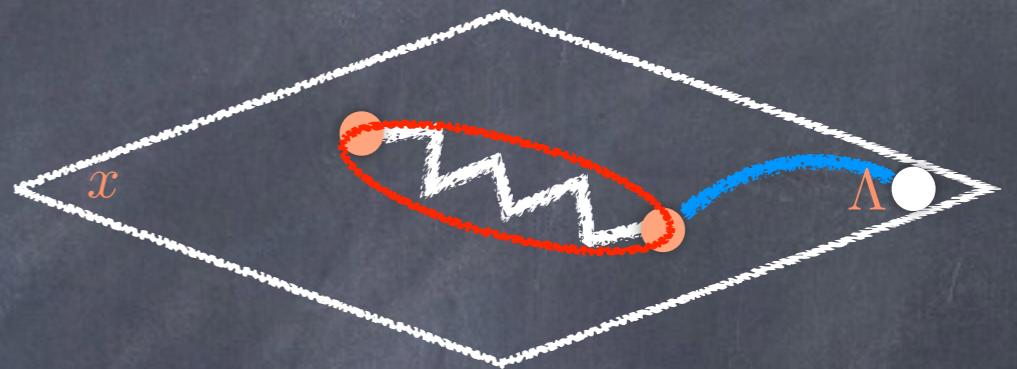
$$p^2 + x^2 = E$$

Take as differential:

$$d\lambda \sim \sqrt{E - x^2}$$

→  $\Pi_A \sim \int_{-\sqrt{E}}^{\sqrt{E}} d\lambda \sim E$

$$\Pi_B \sim \int_{\sqrt{E}}^{\Lambda} d\lambda \sim E - E \log \left( \frac{E}{4\Lambda^2} \right) - \Lambda^2$$





# Quantum Geometry

Example:

$$p^2 + x^2 = E$$

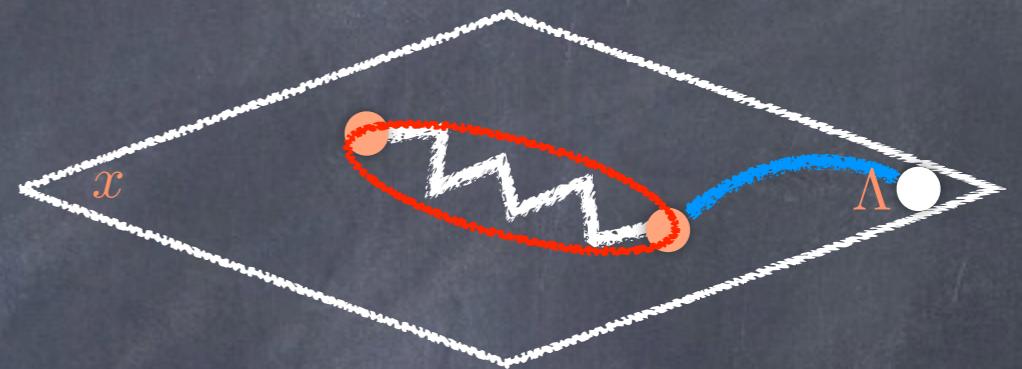
Take as differential:

$$d\lambda \sim \sqrt{E - x^2}$$

$$\rightarrow \Pi_A \sim \int_{-\sqrt{E}}^{\sqrt{E}} d\lambda \sim E$$

Modulus is already the flat coordinate

$$\Pi_B \sim \int_{\sqrt{E}}^{\Lambda} d\lambda \sim E - E \log \left( \frac{E}{4\Lambda^2} \right) - \Lambda^2$$





# Quantum Geometry

Example:

$$p^2 + x^2 = E$$

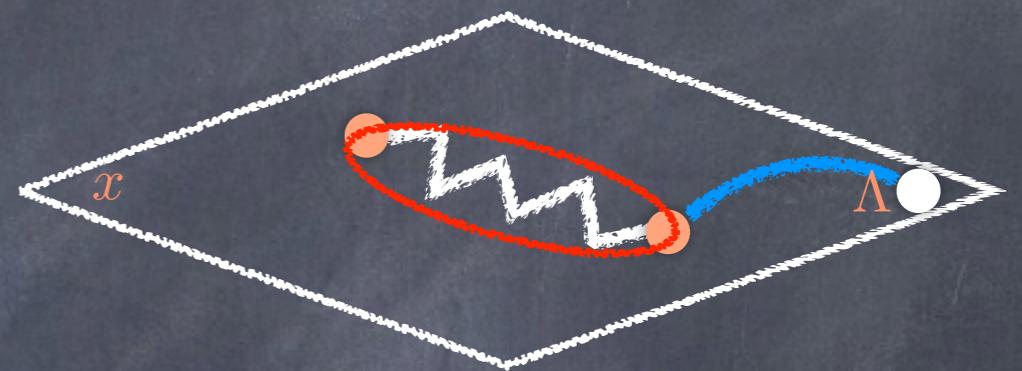
Take as differential:

$$d\lambda \sim \sqrt{E - x^2}$$

$$\rightarrow \Pi_A \sim \int_{-\sqrt{E}}^{\sqrt{E}} d\lambda \sim E$$

Modulus is already the flat coordinate

$$\Pi_B \sim \int_{\sqrt{E}}^{\Lambda} d\lambda \sim E - E \log \left( \frac{E}{4\Lambda^2} \right) - \Lambda^2$$



The constant will be important later !



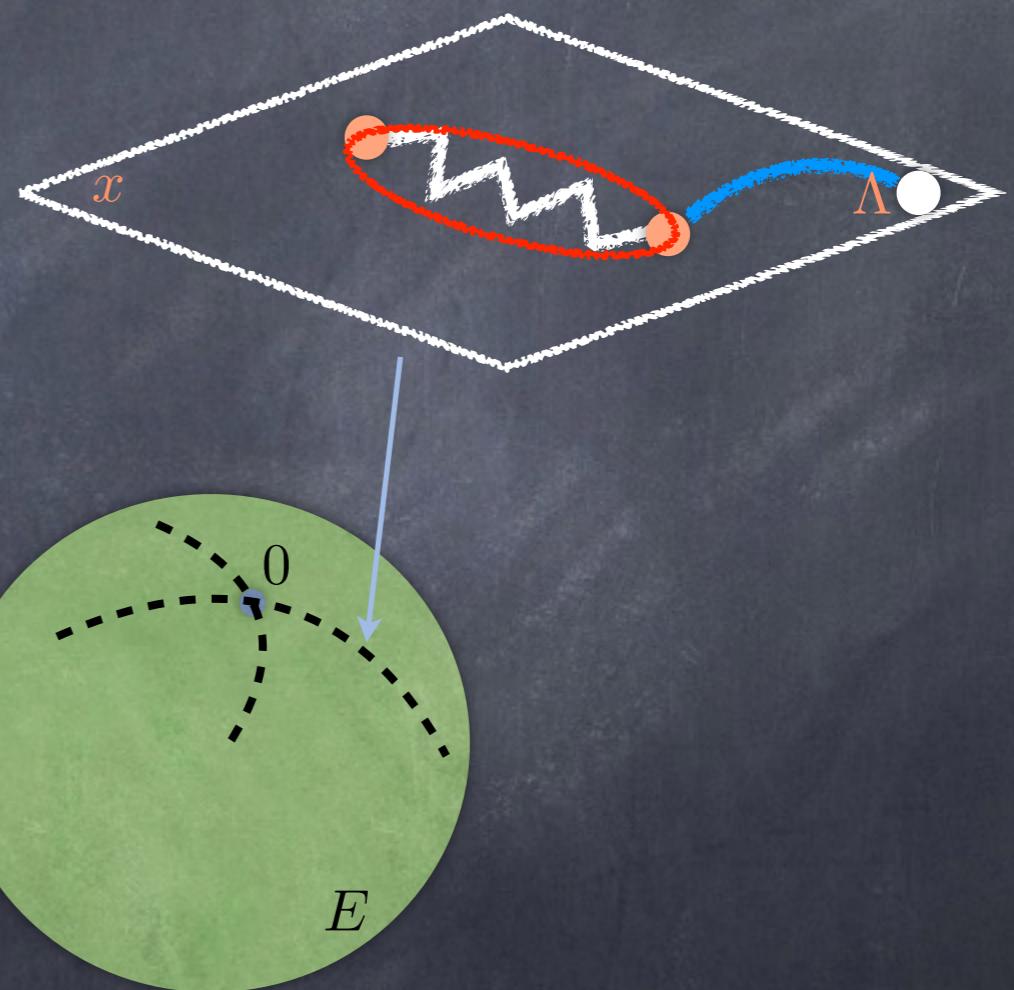
# Quantum Geometry

Example:

$$p^2 + x^2 = E$$



$$\mathcal{M} = \mathbb{C}_{\Lambda^2}$$





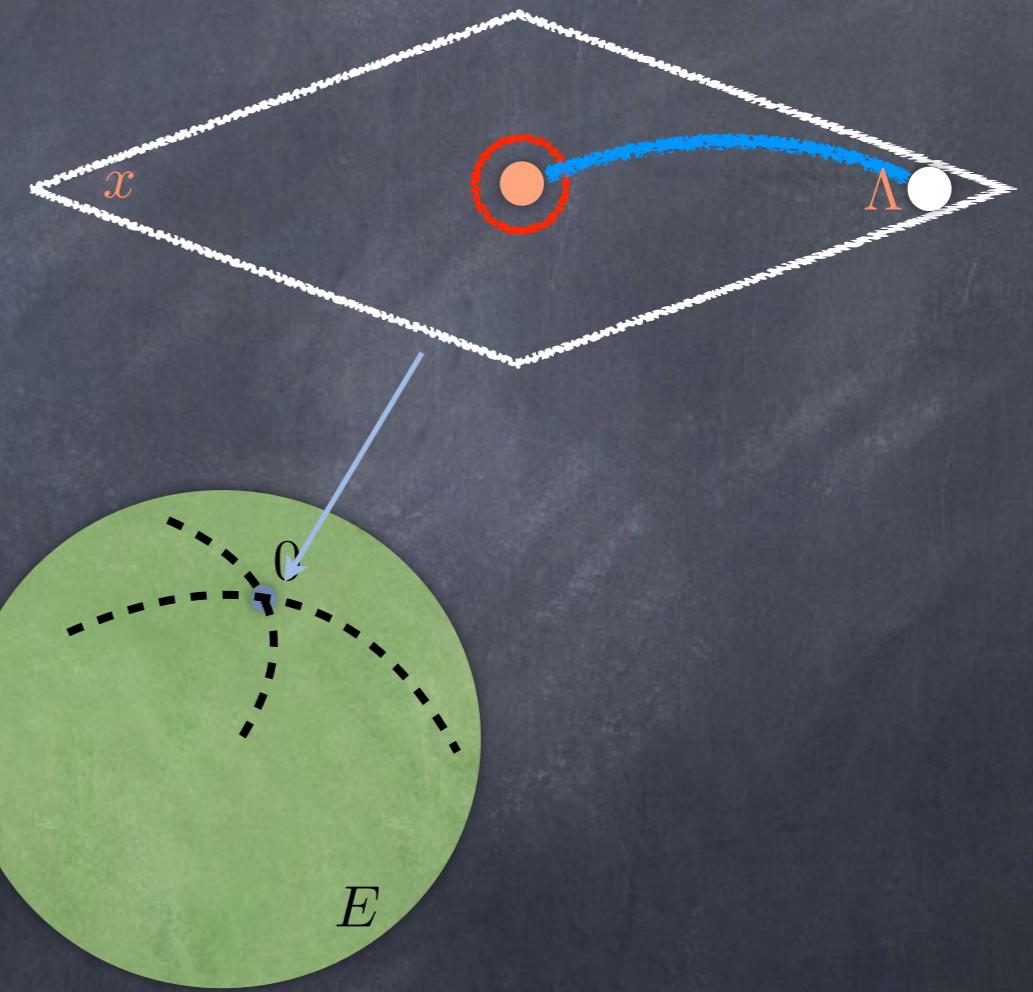
# Quantum Geometry

Example:

$$p^2 + x^2 = E$$



$$\mathcal{M} = \mathbb{C}_{\Lambda^2}$$





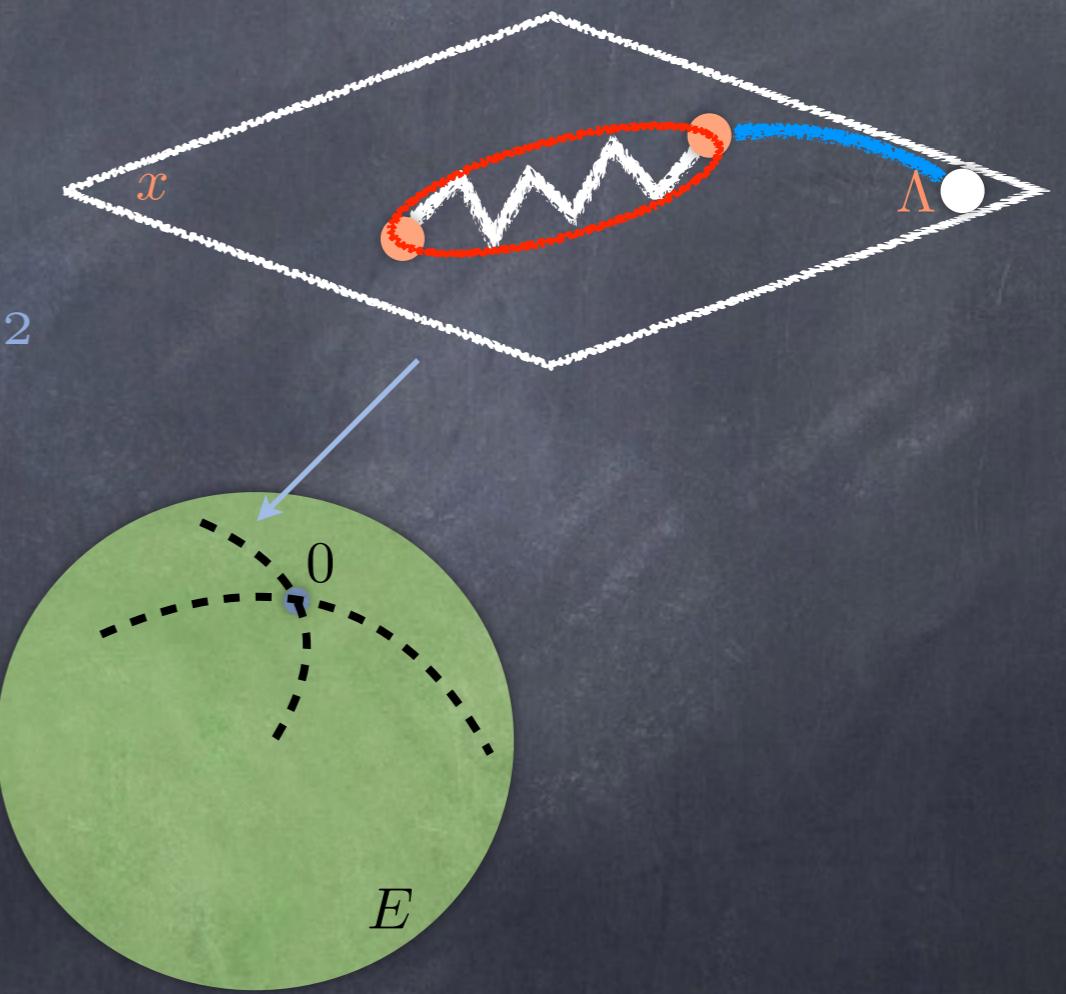
# Quantum Geometry

Example:

$$p^2 + x^2 = E$$



$$\mathcal{M} = \mathbb{C}_{\Lambda^2}$$





# Quantum Geometry

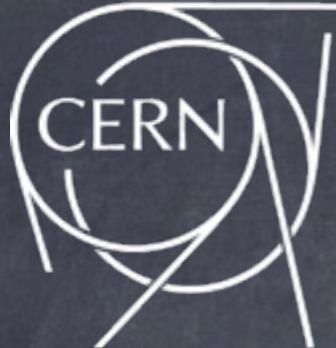


# Quantum Geometry

Quantum Geometry:

$$\Sigma : f(x, p) = 0$$

Perform canonical quantization, i.e.,  $[x, p] = i|\hbar|e^{i\theta} \in \mathbb{C}$



# Quantum Geometry

Quantum Geometry:

$$\Sigma : f(x, p) = 0$$

Perform canonical quantization, i.e.,  $[x, p] = i|\hbar|e^{i\theta} \in \mathbb{C}$

→  $\Sigma \rightarrow \hat{\Sigma}$

$$\hat{\Sigma} : \hat{f} \Psi(x) = 0$$



# Quantum Geometry

Quantum Geometry:

$$\Sigma : f(x, p) = 0$$

Perform canonical quantization, i.e.,  $[x, p] = i|\hbar|e^{i\theta} \in \mathbb{C}$

→  $\Sigma \rightarrow \hat{\Sigma}$

$$\hat{\Sigma} : \hat{f} \Psi(x) = 0$$

(As for quantum integrable systems, we take a right-ordering prescription of operators)



# Quantum Geometry

Quantum Geometry:

$$\Sigma : f(x, p) = 0$$

Perform canonical quantization, i.e.,  $[x, p] = i|\hbar|e^{i\theta} \in \mathbb{C}$

→  $\Sigma \rightarrow \hat{\Sigma}$

We consider here only curves which lead to a second order operator

$$\hat{\Sigma} : \hat{f} \Psi(x) = 0$$



# Quantum Geometry

Quantum Geometry:

$$\Sigma : f(x, p) = 0$$

Perform canonical quantization, i.e.,  $[x, p] = i|\hbar|e^{i\theta} \in \mathbb{C}$

→  $\Sigma \rightarrow \widehat{\Sigma}$

We consider here only curves which lead to a second order operator

$$\widehat{\Sigma} : \widehat{f} \Psi(x) = 0$$

General solution:  $\Psi(x) = \sum_i c_i \Psi^{(i)}(x)$



# Quantum Geometry

Quantum Geometry:

$$\Sigma : f(x, p) = 0$$

Perform canonical quantization, i.e.,  $[x, p] = i|\hbar|e^{i\theta} \in \mathbb{C}$

→  $\Sigma \rightarrow \hat{\Sigma}$

$$\hat{\Sigma} : \hat{f} \Psi(x) = 0$$

$\Psi \rightarrow \Sigma$  ← Normalizability:  $\int_{\mathcal{C}} dx \overline{\Psi(x)} \Psi(x) < \infty$



# Quantum Geometry

Quantum Geometry:

$$\Sigma : f(x, p) = 0$$

Perform canonical quantization, i.e.,  $[x, p] = i|\hbar|e^{i\theta} \in \mathbb{C}$

$$\rightarrow \Sigma \rightarrow \hat{\Sigma}$$

Integration path connecting two infinities of  $\Sigma$

$$\hat{\Sigma} : \hat{f} \Psi(x) = 0$$

$$\Psi \rightarrow \Sigma$$



Normalizability:

$$\int_{\mathcal{C}} dx \overline{\Psi(x)} \Psi(x) < \infty$$



# Quantum Geometry

Quantum Geometry:

$$\Sigma : f(x, p) = 0$$

Perform canonical quantization, i.e.,  $[x, p] = i|\hbar|e^{i\theta} \in \mathbb{C}$

$$\rightarrow \Sigma \rightarrow \hat{\Sigma}$$

Note: Non-holomorphic, hence path dependent !

$$\hat{\Sigma} : \hat{f} \Psi(x) = 0$$

$$\Psi \rightarrow \Sigma$$



Normalizability:

$$\int_{\mathcal{C}} dx \overline{\Psi(x)} \Psi(x) < \infty$$



# Quantum Geometry

Quantum Geometry:

$$\Psi \rightarrow \Sigma$$

Constraints possible solutions

Normalizability:  $\int_{\mathcal{C}} dx \overline{\Psi(x)} \Psi(x) < \infty$



# Quantum Geometry

Quantum Geometry:

$$\Psi \rightarrow \Sigma$$



$$\mathcal{M}_\hbar \subset \mathcal{M}$$

Equivalently, constraints for fixed solution the moduli space !

Normalizability:  $\int_{\mathcal{C}} dx \overline{\Psi(x)} \Psi(x) < \infty$



# Quantum Geometry

Quantum Geometry:

$$\Psi \rightarrow \Sigma$$



Use to define a quantum differential:

$$dS \sim \partial_x \log \Psi$$

Note: A priori no unique differential



# Quantum Geometry

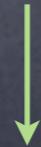
Quantum Geometry:

$$\Psi \rightarrow \Sigma$$



Use to define a quantum differential:

$$dS \sim \partial_x \log \Psi$$



Quantum periods    $\Pi = \oint dS$



# Quantum Geometry

Example:

$$p^2 + \omega^2 x^2 = E$$



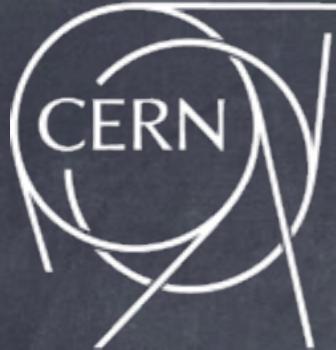
# Quantum Geometry

Example:

$$p^2 + \omega^2 x^2 = E$$



Convenient to introduce additional parameter



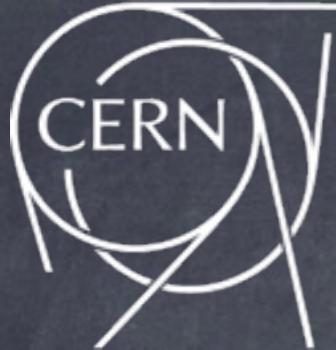
# Quantum Geometry

Example:

$$p^2 + \omega^2 x^2 = E$$



$$\hat{f} : \frac{\partial^2}{\partial x^2} - \kappa^2 x^2 + \frac{E}{\hbar^2}$$
$$\kappa := \frac{\omega}{\hbar}$$



# Quantum Geometry

Example:

$$p^2 + \omega^2 x^2 = E$$



$$\hat{f} : \frac{\partial^2}{\partial x^2} - \kappa^2 x^2 + \frac{E}{\hbar^2}$$

Note:

The interpretation in terms of a (real) 1d quantum mechanical problem is ambiguous and depends on how we take a real slice and where we sit in moduli space !

Harmonic oscillator



$$\kappa \rightarrow i\kappa$$

A double-headed green arrow indicating a transformation between the two states.

Parabolic barrier





# Quantum Geometry

Example:

$$p^2 + \omega^2 x^2 = E$$



$$\hat{f} : \frac{\partial^2}{\partial x^2} - \kappa^2 x^2 + \frac{E}{\hbar^2}$$

Two independent solutions  $\Psi^\pm$  can be found in terms of parabolic cylinder functions  $D(N, x)$ :

$$\Psi^+ = D\left(\frac{1}{2}\left(\frac{E}{\hbar\omega} - 1\right), \sqrt{2\kappa}x\right)$$

$$\Psi^- = D\left(-\frac{1}{2}\left(\frac{E}{\hbar\omega} + 1\right), i\sqrt{2\kappa}x\right)$$



# Quantum Geometry

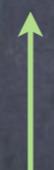
Example:

$$p^2 + \omega^2 x^2 = E$$



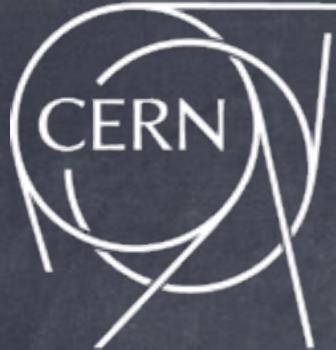
$$\hat{f} : \frac{\partial^2}{\partial x^2} - \kappa^2 x^2 + \frac{E}{\hbar^2}$$

Two independent solutions  $\Psi^\pm$  can be found in terms of parabolic cylinder functions  $D(N, x)$ :



For  $N \in \mathbb{N}$  reduce to Hermite polynomials

$$D(N, x) \sim e^{-\frac{x^2}{4}} H_N \left( \frac{x}{\sqrt{2}} \right)$$



# Quantum Geometry

Example:

$$p^2 + \omega^2 x^2 = E$$



$$\hat{f} : \frac{\partial^2}{\partial x^2} - \kappa^2 x^2 + \frac{E}{\hbar^2}$$

Two independent solutions  $\Psi^\pm$  can be found in terms of parabolic cylinder functions  $D(N, x)$ :

Normalizability depends on where we go to infinity in the complex plane !

↑ For  $N \in \mathbb{N}$  reduce to Hermite polynomials

$$D(N, x) \sim e^{-\frac{x^2}{4}} H_N \left( \frac{x}{\sqrt{2}} \right)$$



# Quantum Geometry

Example:

$$p^2 + \omega^2 x^2 = E$$



$$\hat{f} : \frac{\partial^2}{\partial x^2} - \kappa^2 x^2 + \frac{E}{\hbar^2}$$

Two independent solutions  $\Psi^\pm$  can be found in terms of parabolic cylinder functions  $D(N, x)$ :

$$\Psi^+ \sim e^{-\frac{\kappa x^2}{2}}$$

$$\xleftarrow{N \in \mathbb{N}}$$

$$\Psi^- \sim e^{\frac{\kappa x^2}{2}}$$



For  $N \in \mathbb{N}$  reduce to Hermite polynomials

$$D(N, x) \sim e^{-\frac{x^2}{4}} H_N \left( \frac{x}{\sqrt{2}} \right)$$



# Quantum Geometry

Example:

$$p^2 + \omega^2 x^2 = E$$



$$\hat{f} : \frac{\partial^2}{\partial x^2} - \kappa^2 x^2 + \frac{E}{\hbar^2}$$

Two independent solutions  $\Psi^\pm$  can be found in terms of parabolic cylinder functions  $D(N, x)$ :

$$\begin{array}{c} \Psi^+ \sim e^{-\frac{\kappa x^2}{2}} \\ \downarrow \kappa \rightarrow -\kappa \\ \Psi^- \sim e^{\frac{\kappa x^2}{2}} \end{array}$$

$$\xleftarrow{N \in \mathbb{N}}$$

$$D(N, x) \sim e^{-\frac{x^2}{4}} H_N \left( \frac{x}{\sqrt{2}} \right)$$



For  $N \in \mathbb{N}$  reduce to Hermite polynomials



# Quantum Geometry

Example:

$$p^2 + \omega^2 x^2 = E$$



$$\hat{f} : \frac{\partial^2}{\partial x^2} - \kappa^2 x^2 + \frac{E}{\hbar^2}$$

Two independent solutions  $\Psi^\pm$  can be found in terms of parabolic cylinder functions  $D(N, x)$ :

$$\Psi^+ : E = +\hbar^2 \kappa(2N + 1)$$

$$\xleftarrow{N \in \mathbb{N}} D(N, x) \sim e^{-\frac{x^2}{4}} H_N \left( \frac{x}{\sqrt{2}} \right)$$

$$\Psi^- : E = -\hbar^2 \kappa(2N + 1)$$

↑ For  $N \in \mathbb{N}$  reduce to Hermite polynomials



# Quantum Geometry

Example:

$$p^2 + \omega^2 x^2 = E$$



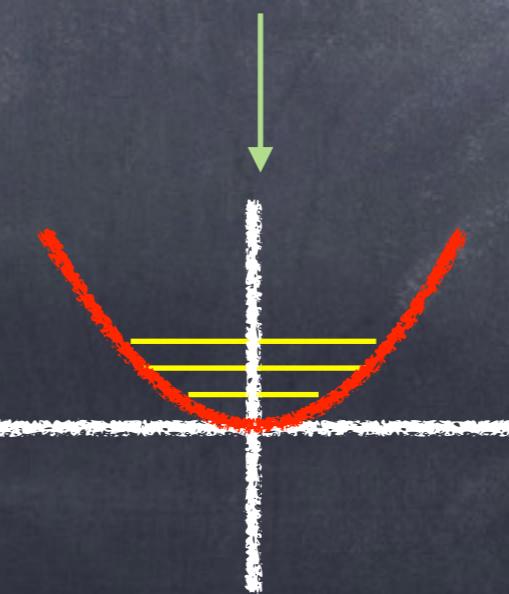
$$\hat{f} : \frac{\partial^2}{\partial x^2} - \kappa^2 x^2 + \frac{E}{\hbar^2}$$

Two independent solutions  $\Psi^\pm$  can be found in terms of parabolic cylinder functions  $D(N, x)$ :

$$\Psi^+ : E = +\hbar^2 \kappa(2N + 1)$$

$$\xrightarrow{\text{Im } \kappa = 0}$$

$$\Psi^- : E = -\hbar^2 \kappa(2N + 1)$$





# Quantum Geometry

Example:

$$p^2 + \omega^2 x^2 = E$$



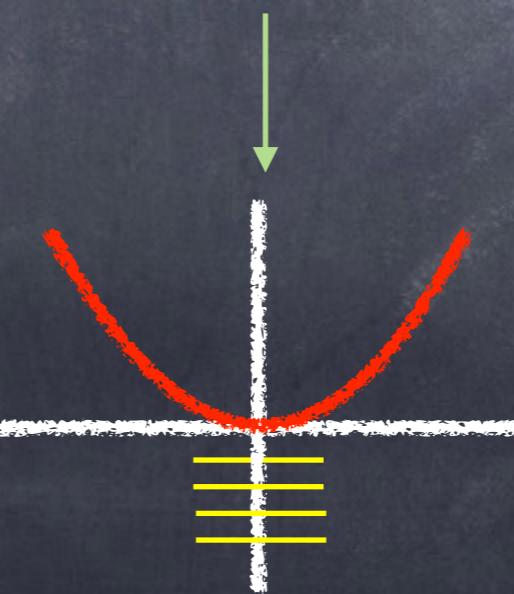
$$\hat{f} : \frac{\partial^2}{\partial x^2} - \kappa^2 x^2 + \frac{E}{\hbar^2}$$

Two independent solutions  $\Psi^\pm$  can be found in terms of parabolic cylinder functions  $D(N, x)$ :

$$\Psi^+ : E = +\hbar^2 \kappa(2N + 1)$$

$$\xrightarrow{\text{Im } \kappa = 0}$$

$$\boxed{\Psi^- : E = -\hbar^2 \kappa(2N + 1)}$$





# Quantum Geometry

Example:

$$p^2 + \omega^2 x^2 = E$$



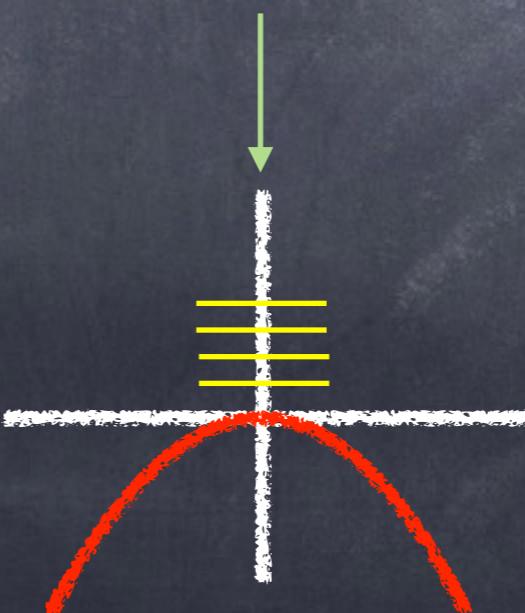
$$\hat{f} : \frac{\partial^2}{\partial x^2} - \kappa^2 x^2 + \frac{E}{\hbar^2}$$

Two independent solutions  $\Psi^\pm$  can be found in terms of parabolic cylinder functions  $D(N, x)$ :

$$\Psi^+ : E = +\hbar^2 \kappa(2N + 1)$$

$$\xrightarrow{\text{Re } \kappa = 0}$$

$$\Psi^- : E = -\hbar^2 \kappa(2N + 1)$$





# Quantum Geometry

Example:

$$p^2 + \omega^2 x^2 = E$$



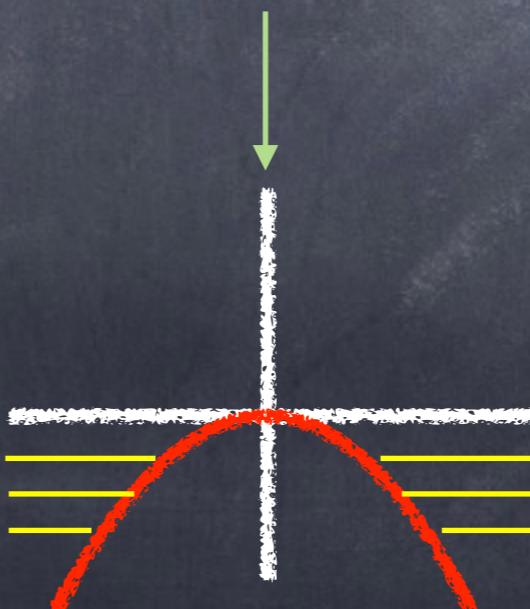
$$\hat{f} : \frac{\partial^2}{\partial x^2} - \kappa^2 x^2 + \frac{E}{\hbar^2}$$

Two independent solutions  $\Psi^\pm$  can be found in terms of parabolic cylinder functions  $D(N, x)$ :

$$\Psi^+ : E = +\hbar^2 \kappa(2N + 1)$$

$$\xrightarrow{\text{Re } \kappa = 0}$$

$$\Psi^- : E = -\hbar^2 \kappa(2N + 1)$$





# Quantum Geometry

Example:

$$p^2 + \omega^2 x^2 = E$$

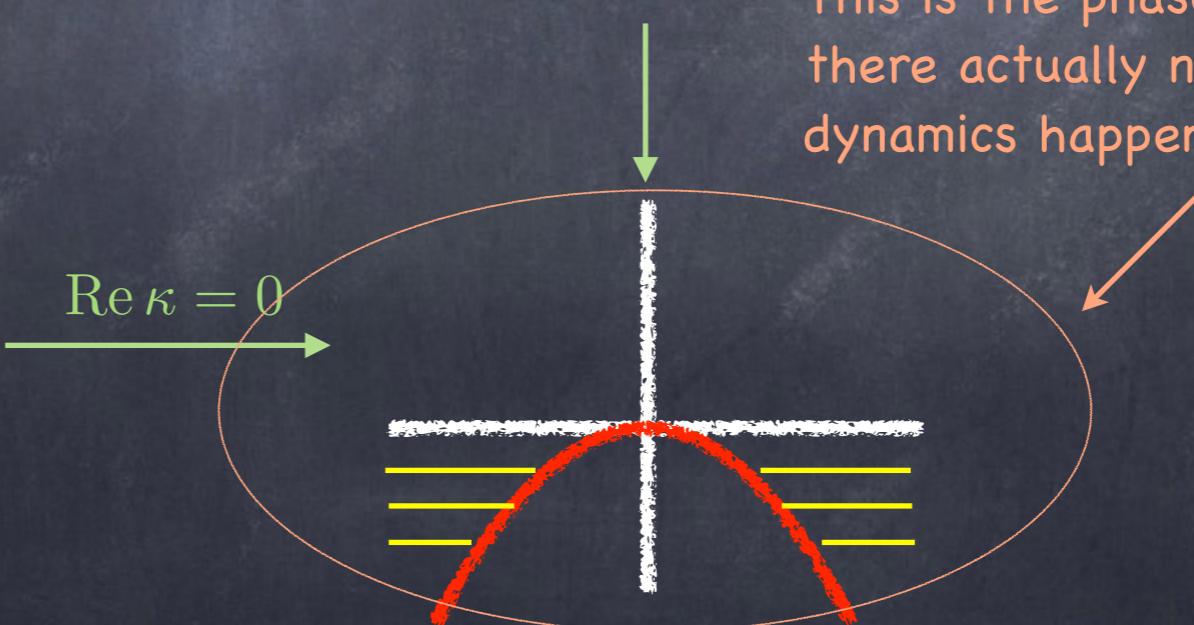


$$\hat{f} : \frac{\partial^2}{\partial x^2} - \kappa^2 x^2 + \frac{E}{\hbar^2}$$

Two independent solutions  $\Psi^\pm$  can be found in terms of parabolic cylinder functions  $D(N, x)$ :

$$\Psi^+ : E = +\hbar^2 \kappa(2N + 1)$$

$$\Psi^- : E = -\hbar^2 \kappa(2N + 1)$$



This is the phase in moduli space  
there actually non-perturbative  
dynamics happens !



# Quantum Geometry

Example:

$$p^2 + \omega^2 x^2 = E$$



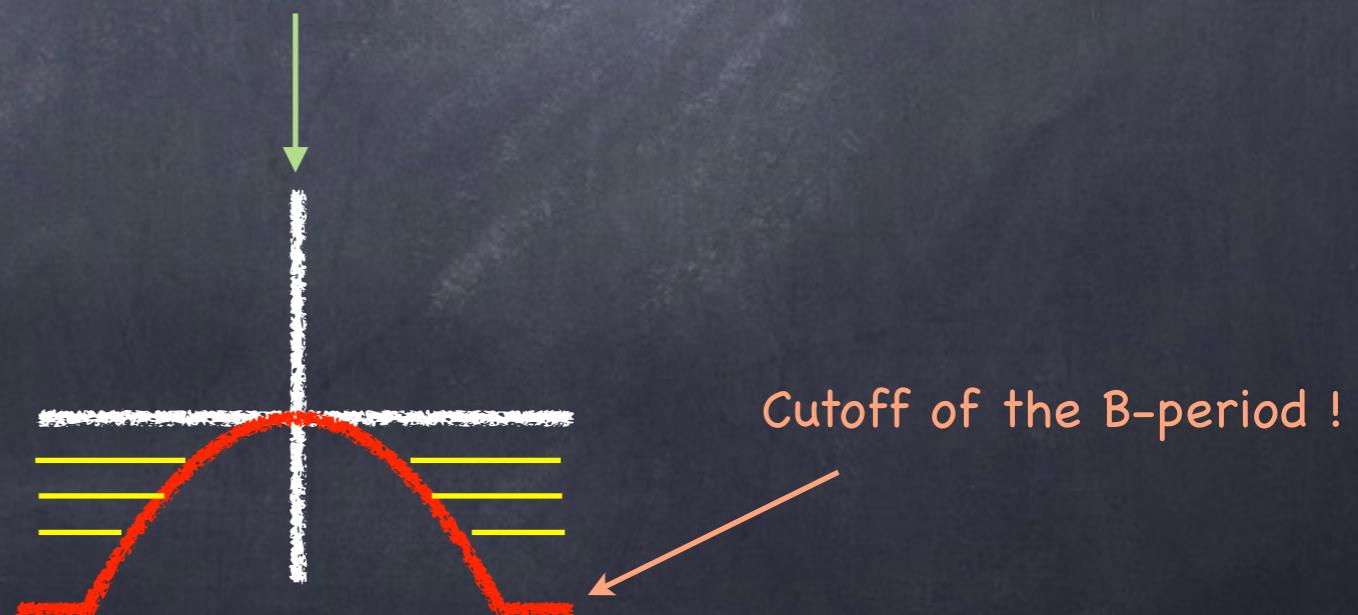
$$\hat{f} : \frac{\partial^2}{\partial x^2} - \kappa^2 x^2 + \frac{E}{\hbar^2}$$

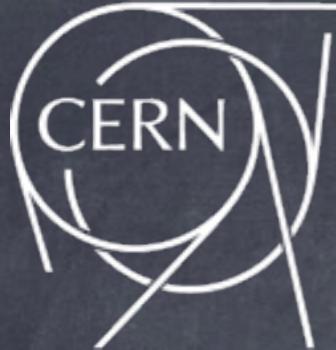
Two independent solutions  $\Psi^\pm$  can be found in terms of parabolic cylinder functions  $D(N, x)$ :

$$\Psi^+ : E = +\hbar^2 \kappa(2N + 1)$$

$$\xrightarrow{\text{Re } \kappa = 0}$$

$$\boxed{\Psi^- : E = -\hbar^2 \kappa(2N + 1)}$$





# Quantum Geometry

Example:

$$p^2 + \omega^2 x^2 = E$$



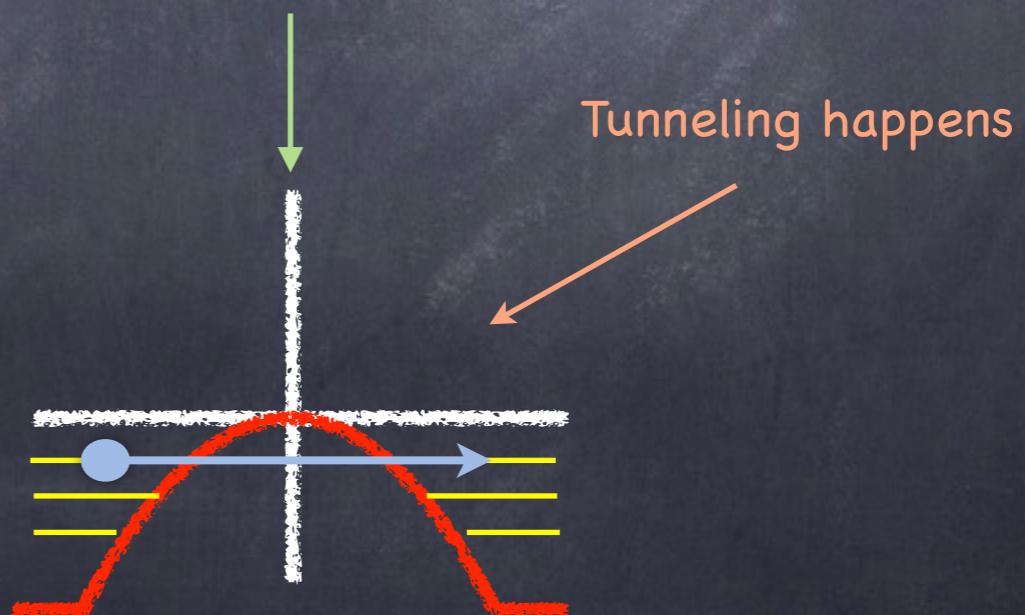
$$\hat{f} : \frac{\partial^2}{\partial x^2} - \kappa^2 x^2 + \frac{E}{\hbar^2}$$

Two independent solutions  $\Psi^\pm$  can be found in terms of parabolic cylinder functions  $D(N, x)$ :

$$\Psi^+ : E = +\hbar^2 \kappa(2N + 1)$$

$$\xrightarrow{\text{Re } \kappa = 0}$$

$$\boxed{\Psi^- : E = -\hbar^2 \kappa(2N + 1)}$$





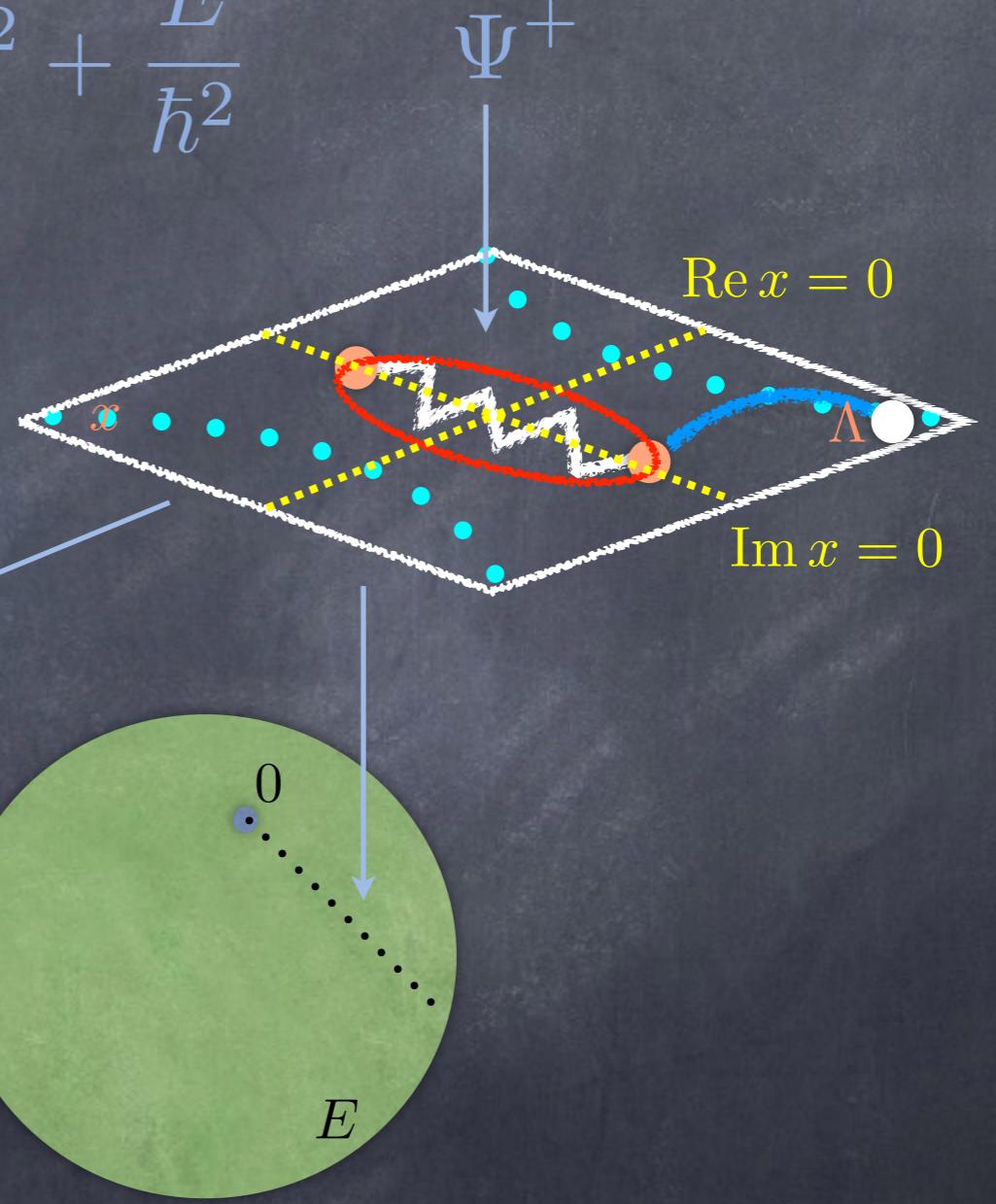
# Quantum Geometry

Example:

$$\Psi^+ \sim e^{-\frac{\kappa x^2}{2}}$$

$$\Psi^- \sim e^{\frac{\kappa x^2}{2}}$$

$$\hat{f} : \frac{\partial^2}{\partial x^2} - \kappa^2 x^2 + \frac{E}{\hbar^2}$$





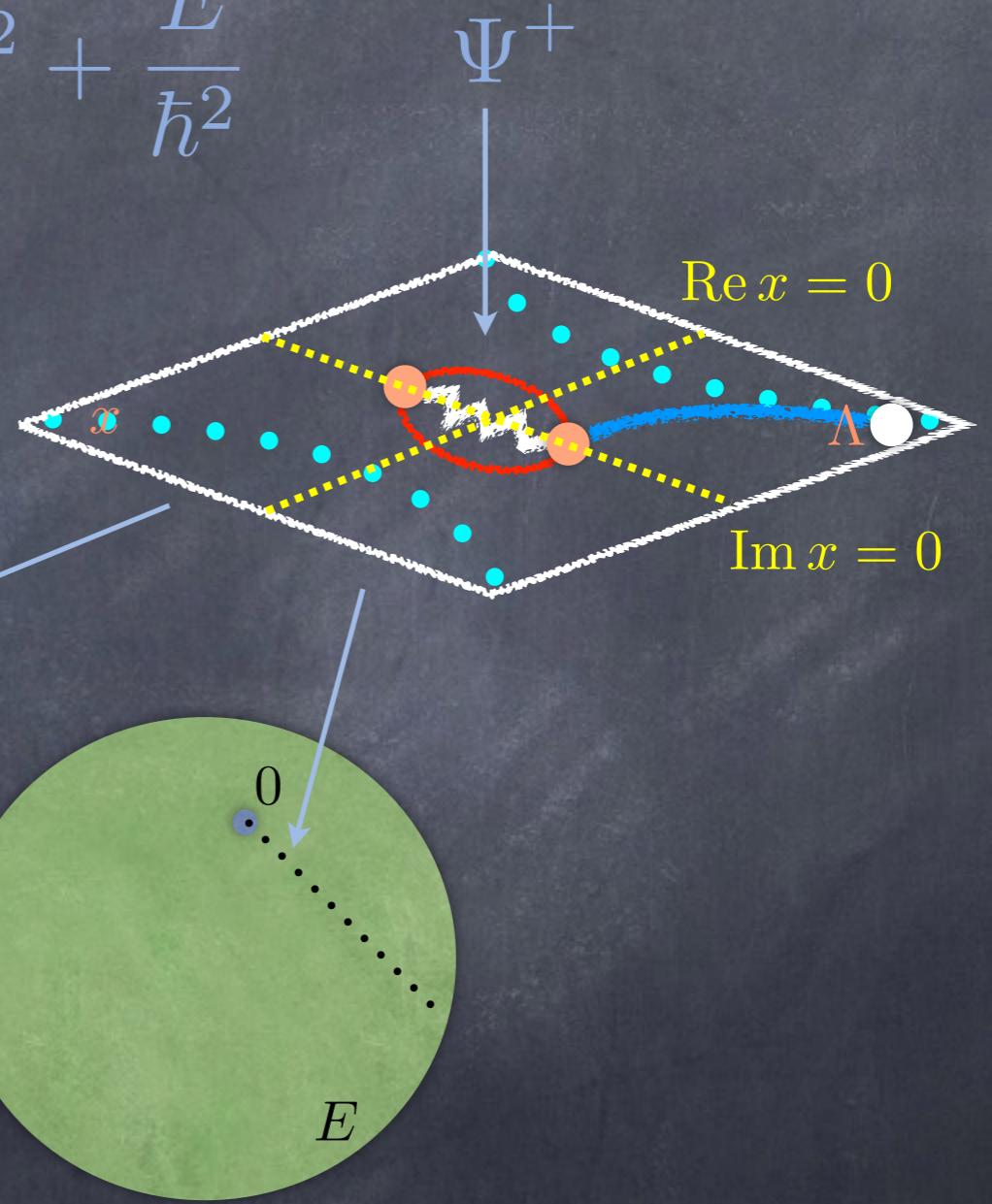
# Quantum Geometry

Example:

$$\Psi^+ \sim e^{-\frac{\kappa x^2}{2}}$$

$$\Psi^- \sim e^{\frac{\kappa x^2}{2}}$$

$$\hat{f} : \frac{\partial^2}{\partial x^2} - \kappa^2 x^2 + \frac{E}{\hbar^2}$$





# Quantum Geometry

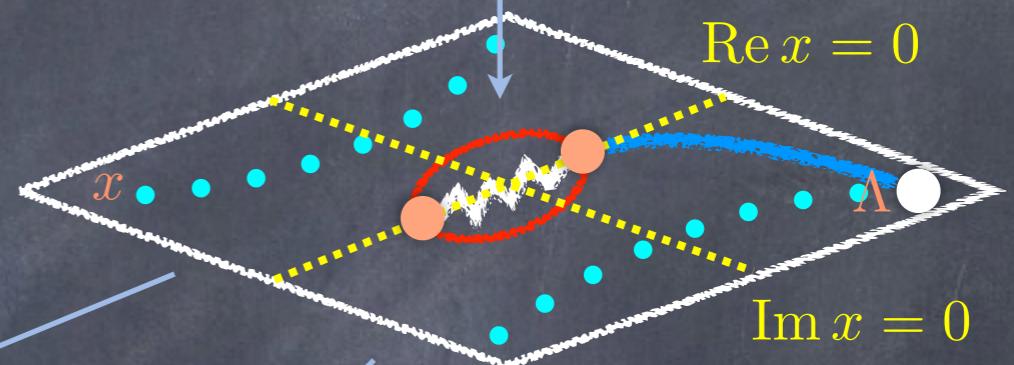
Example:

$$\Psi^+ \sim e^{-\frac{\kappa x^2}{2}}$$

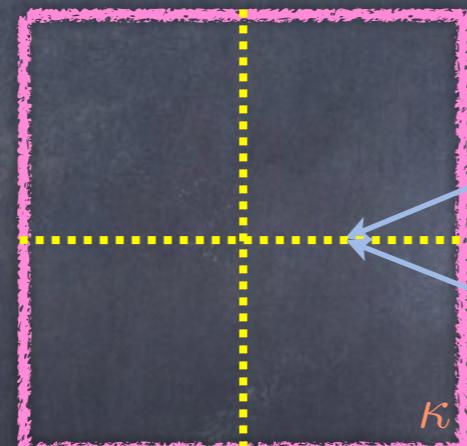
$$\Psi^- \sim e^{\frac{\kappa x^2}{2}}$$

$$\hat{f} : \frac{\partial^2}{\partial x^2} - \kappa^2 x^2 + \frac{E}{\hbar^2}$$

$\Psi^-$

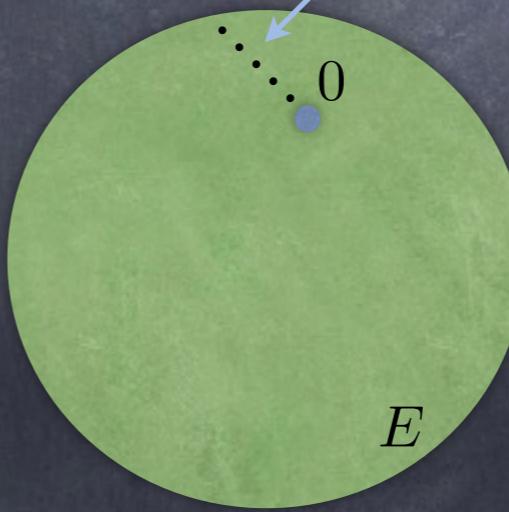


Phase transition !



Re  $\kappa = 0$

Im  $\kappa = 0$



$E$



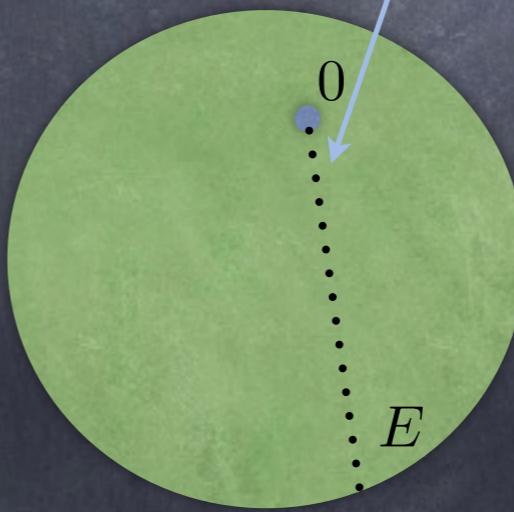
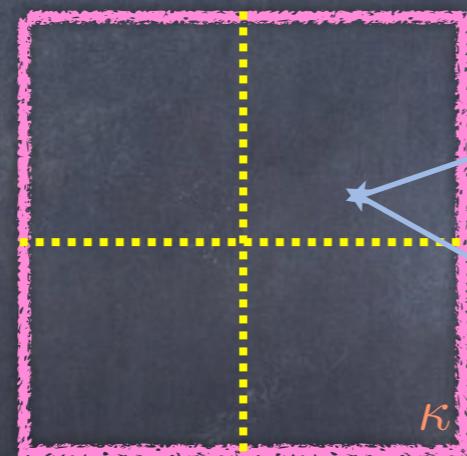
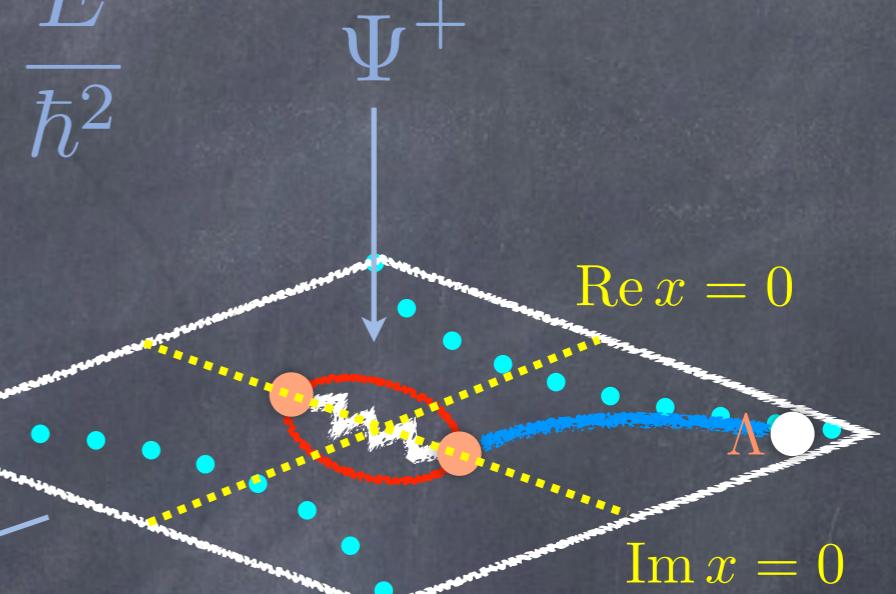
# Quantum Geometry

Example:

$$\Psi^+ \sim e^{-\frac{\kappa x^2}{2}}$$

$$\Psi^- \sim e^{\frac{\kappa x^2}{2}}$$

$$\hat{f} : \frac{\partial^2}{\partial x^2} - \kappa^2 x^2 + \frac{E}{\hbar^2}$$





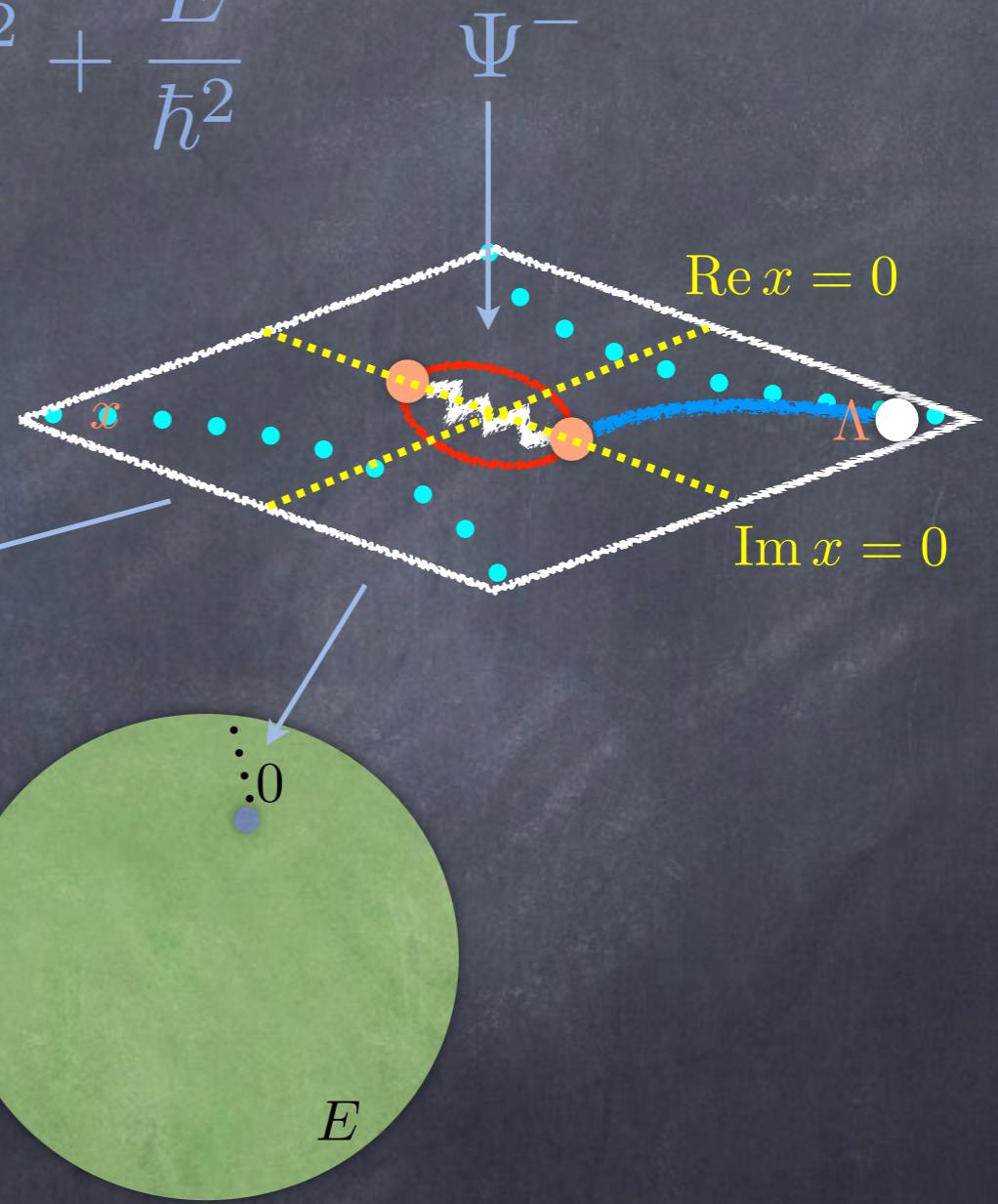
# Quantum Geometry

Example:

$$\Psi^+ \sim e^{-\frac{\kappa x^2}{2}}$$

$$\Psi^- \sim e^{\frac{\kappa x^2}{2}}$$

$$\hat{f} : \frac{\partial^2}{\partial x^2} - \kappa^2 x^2 + \frac{E}{\hbar^2}$$





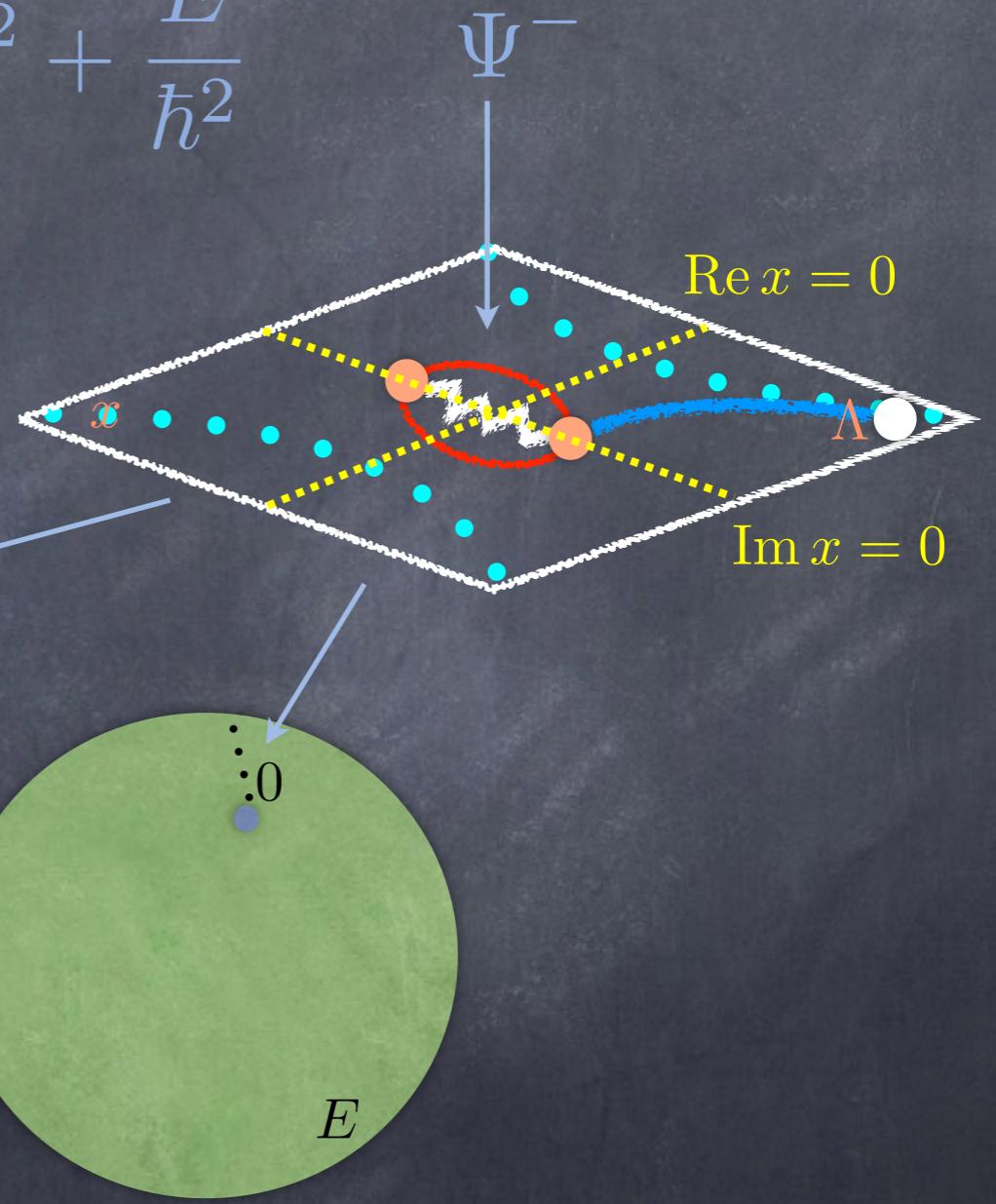
# Quantum Geometry

Example:

$$\Psi^+ \sim e^{-\frac{\kappa x^2}{2}}$$

$$\Psi^- \sim e^{\frac{\kappa x^2}{2}}$$

$$\hat{f} : \frac{\partial^2}{\partial x^2} - \kappa^2 x^2 + \frac{E}{\hbar^2}$$

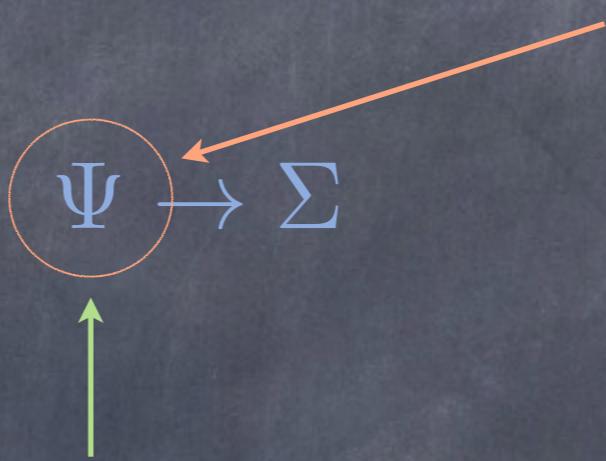


Non-trivial phase structure over the extended moduli space !



# Quantum Geometry

Quantum Geometry:



In general we do not know exact solutions

Use to define a quantum differential:

$$dS \sim \partial_x \log \Psi$$

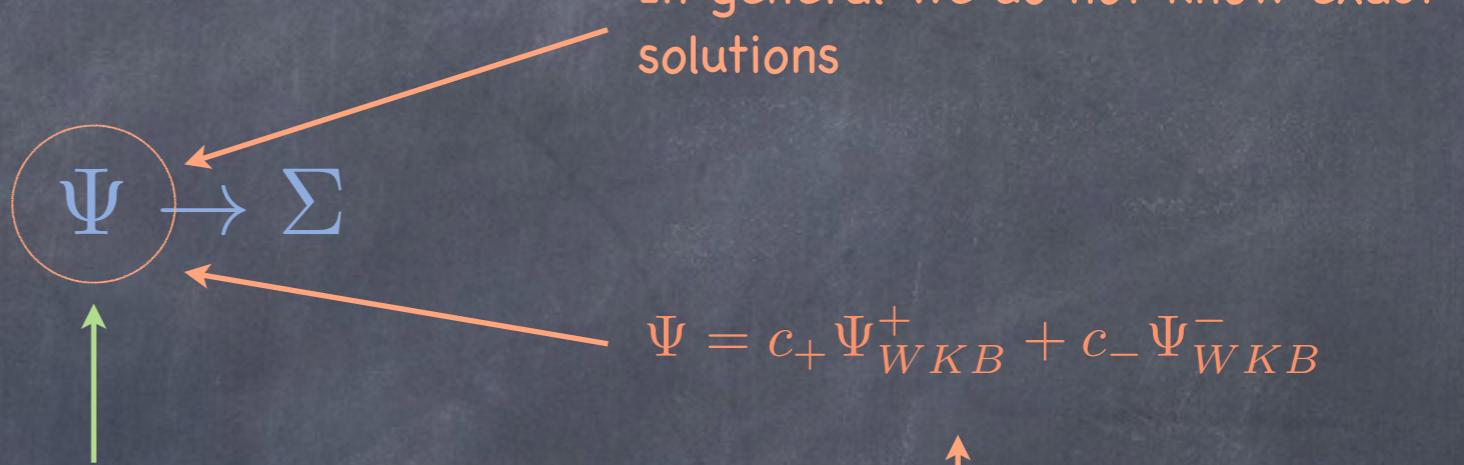


Quantum periods  $\Pi = \oint dS$



# Quantum Geometry

Quantum Geometry:



In general we do not know exact solutions

$$\Psi = c_+ \Psi_{WKB}^+ + c_- \Psi_{WKB}^-$$

Use to define a quantum differential:

$$dS \sim \partial_x \log \Psi$$

$$\Psi_{WKB}^\pm(x) \sim e^{\pm \frac{1}{\hbar} \int^x dS}$$

↓

Quantum periods  $\Pi = \oint dS$

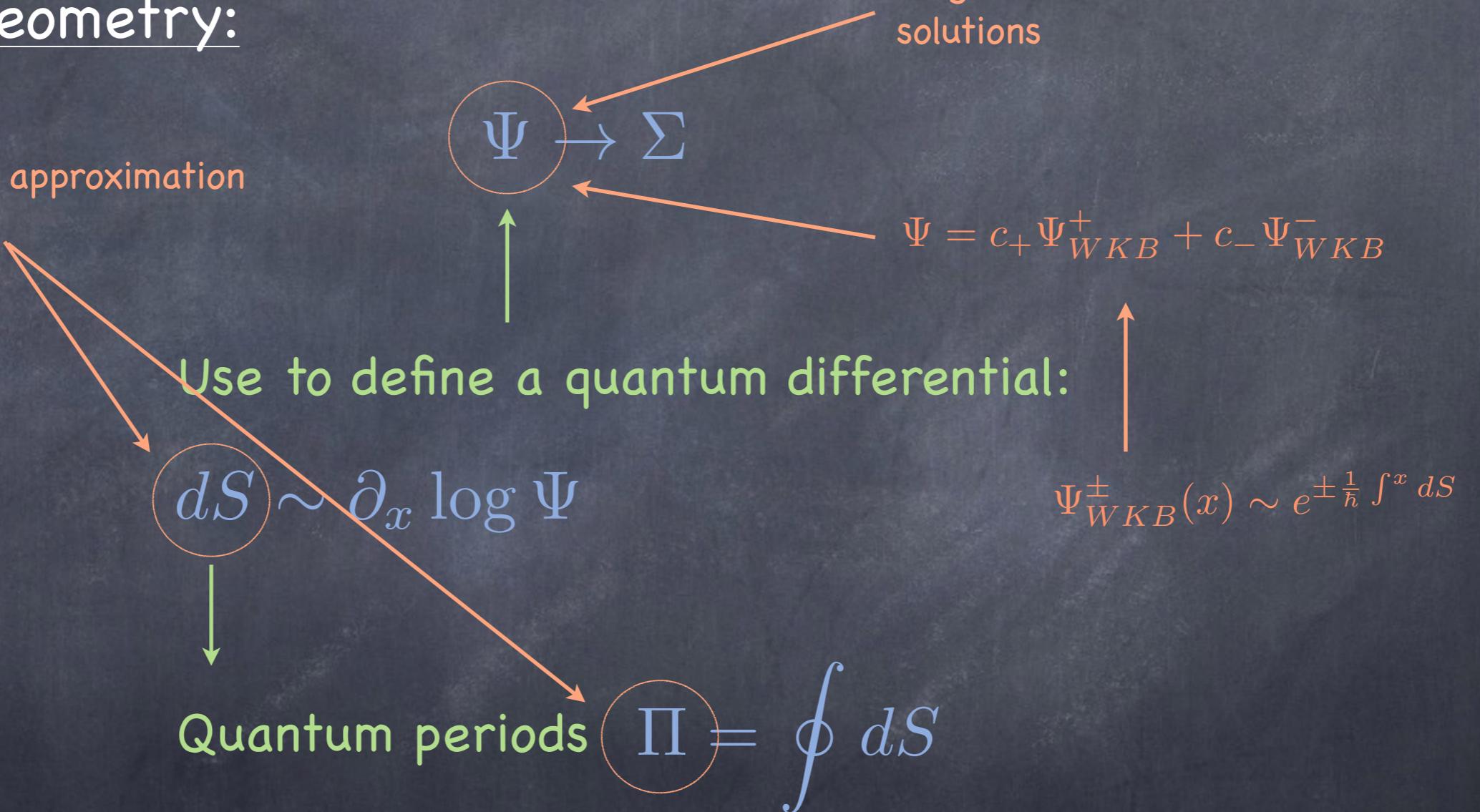


# Quantum Geometry

## Quantum Geometry:

“Semi-classical” approximation

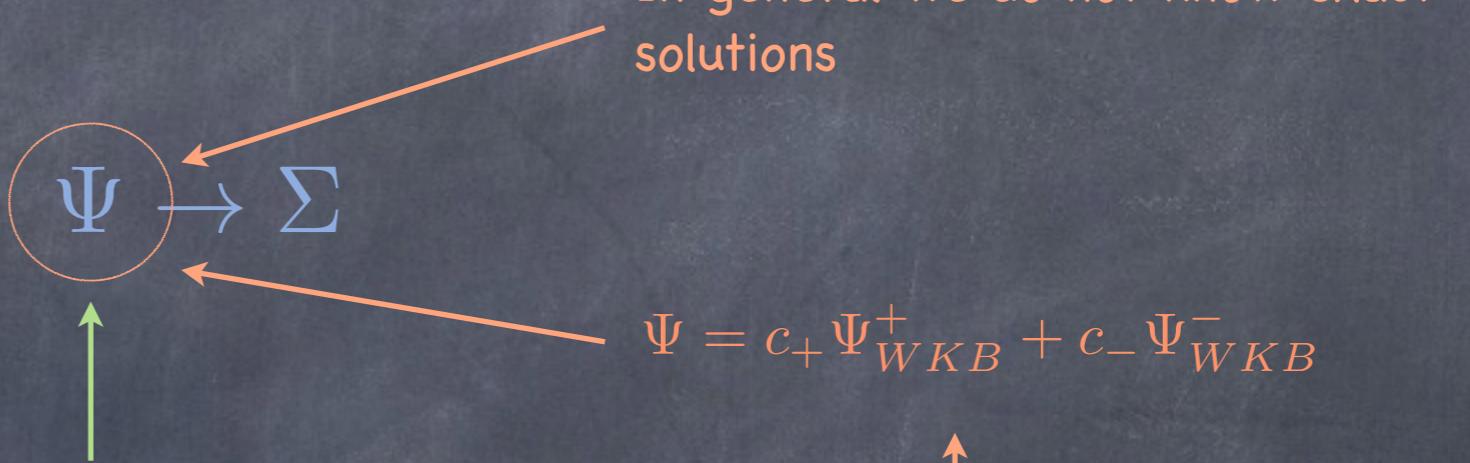
In general we do not know exact solutions





# Quantum Geometry

Quantum Geometry:



In general we do not know exact solutions

$$\Psi = c_+ \Psi_{WKB}^+ + c_- \Psi_{WKB}^-$$

Use to define a quantum differential:

$$dS \sim \partial_x \log \Psi$$

$$\Psi_{WKB}^\pm(x) \sim e^{\pm \frac{1}{\hbar} \int^x dS}$$

↓  
Quantum periods

$$\Pi = \oint dS$$



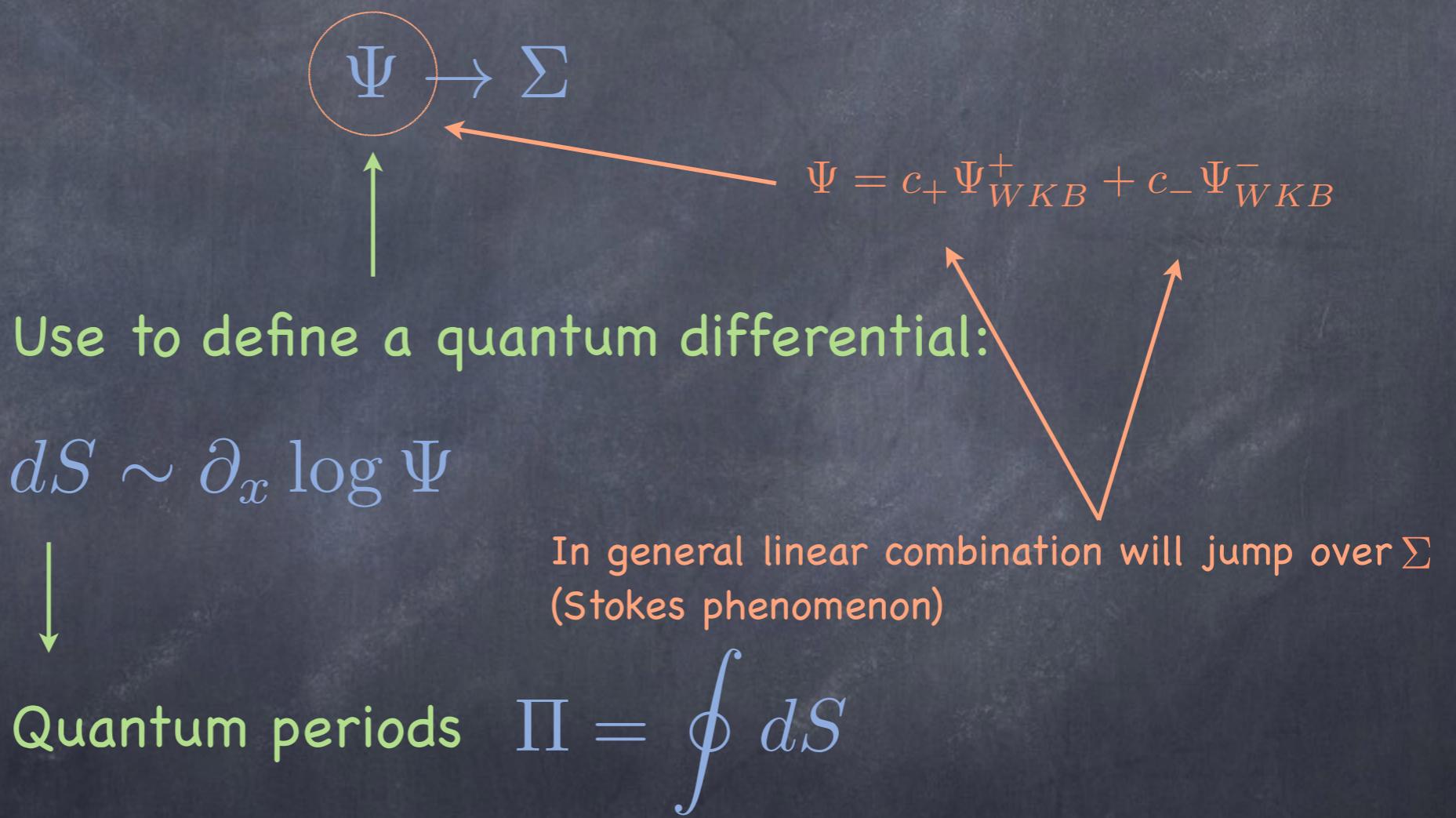
Encodes the NS limit  
of the corresponding  
physical theory

[Mironov+Morozov '09]  
[Part 0: '11]



# Quantum Geometry

Quantum Geometry:





# Quantum Geometry

Quantum Geometry:

$$\Psi \rightarrow \Sigma$$

$$\Psi = c_+ \Psi_{WKB}^+ + c_- \Psi_{WKB}^-$$

Exact quantization condition



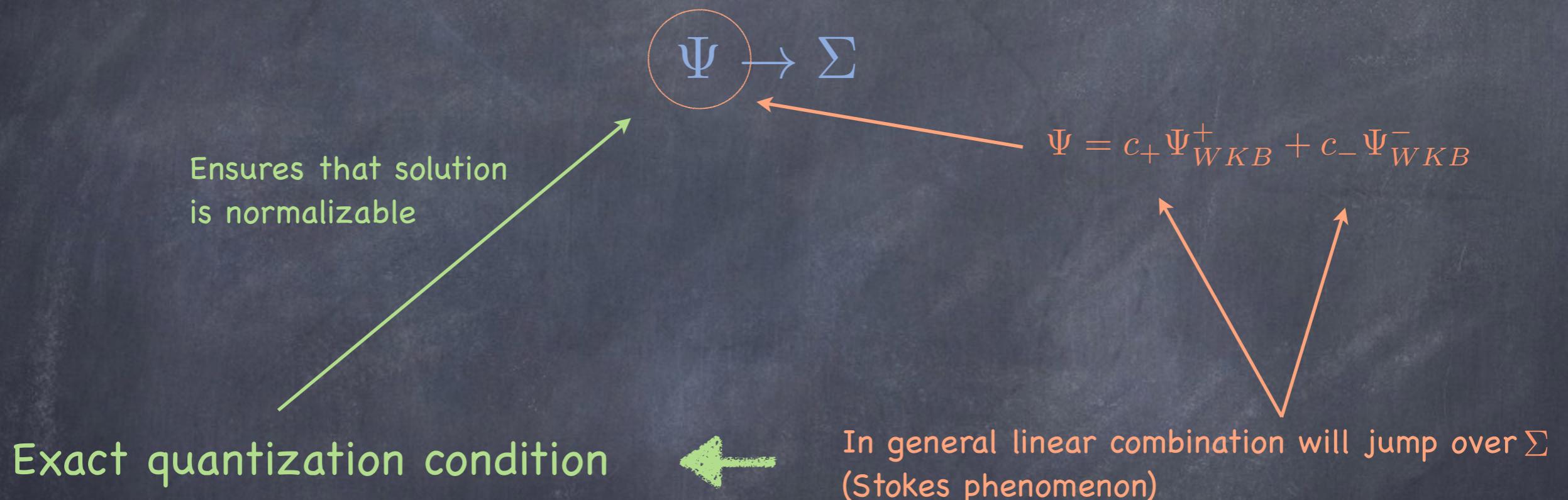
In general linear combination will jump over  $\Sigma$   
(Stokes phenomenon)





# Quantum Geometry

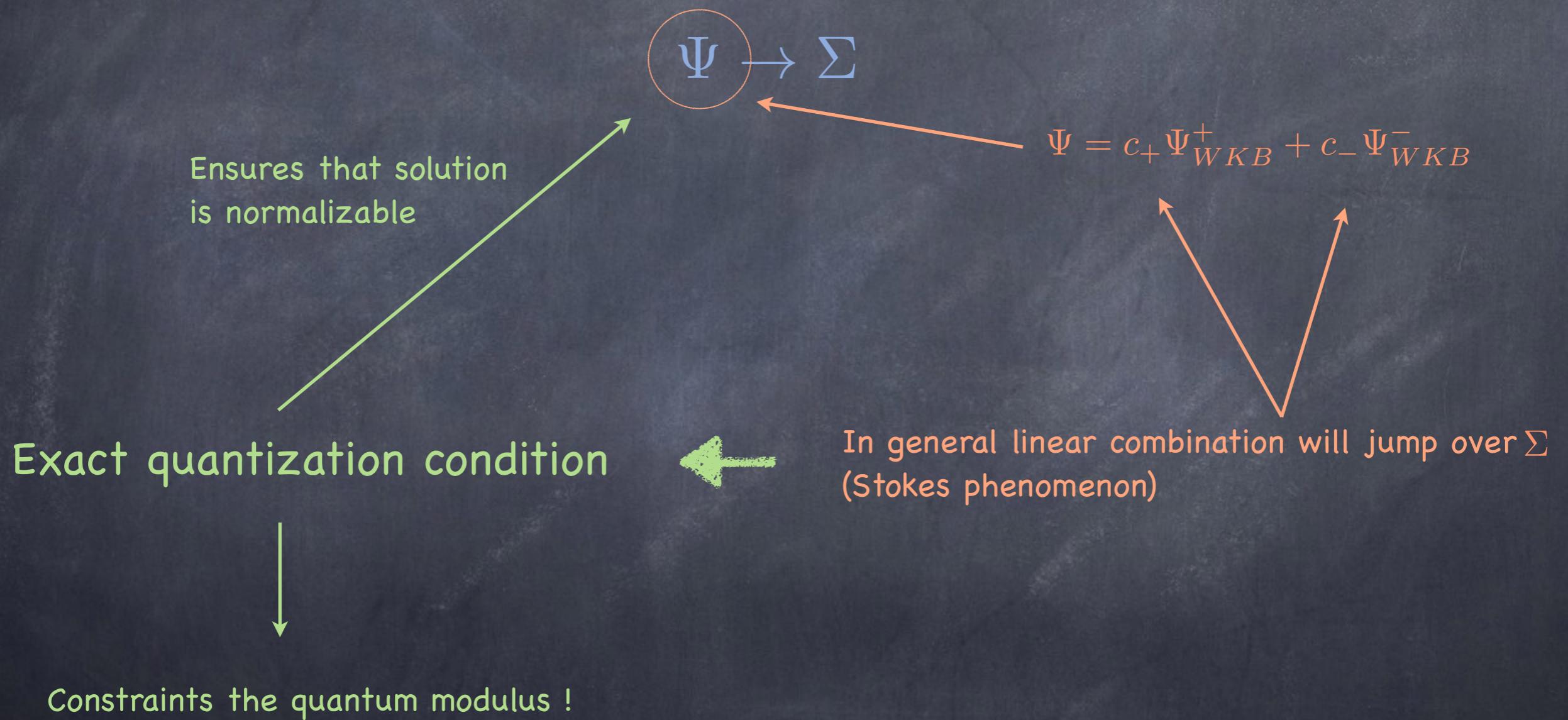
## Quantum Geometry:





# Quantum Geometry

## Quantum Geometry:

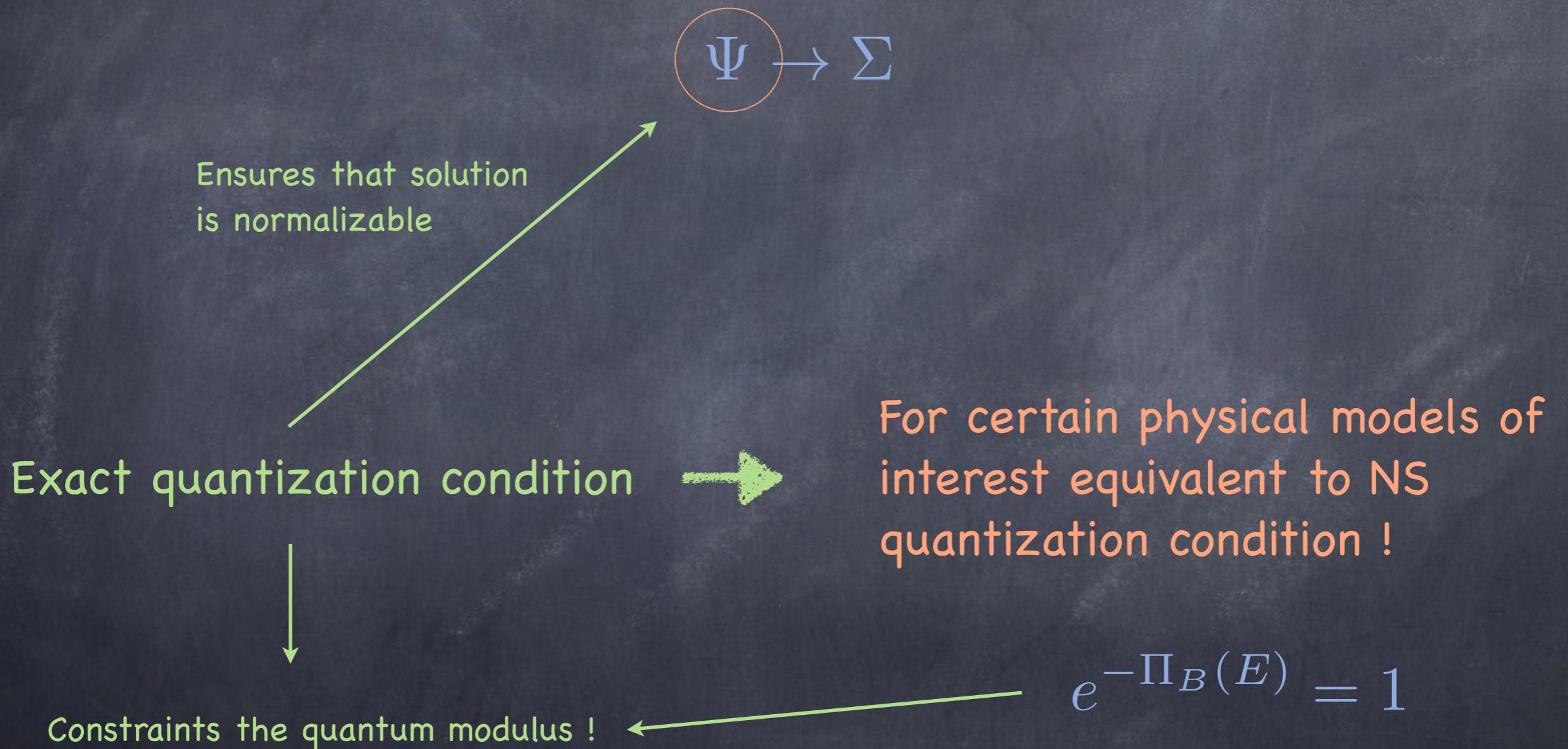




# Quantum Geometry

[Part I,II: D.K. '13,'14]

## Quantum Geometry:

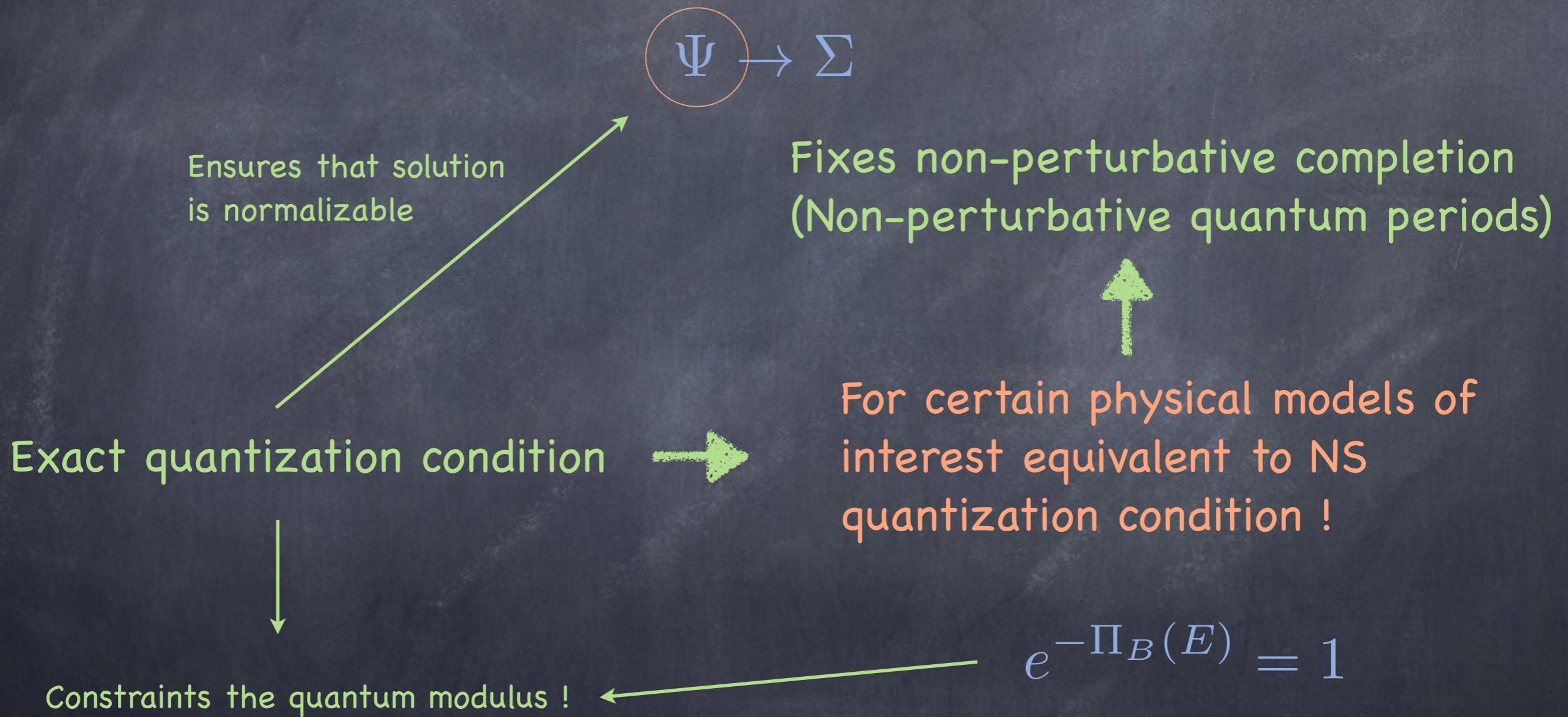




# Quantum Geometry

[Part I,II: D.K. '13,'14]

## Quantum Geometry:

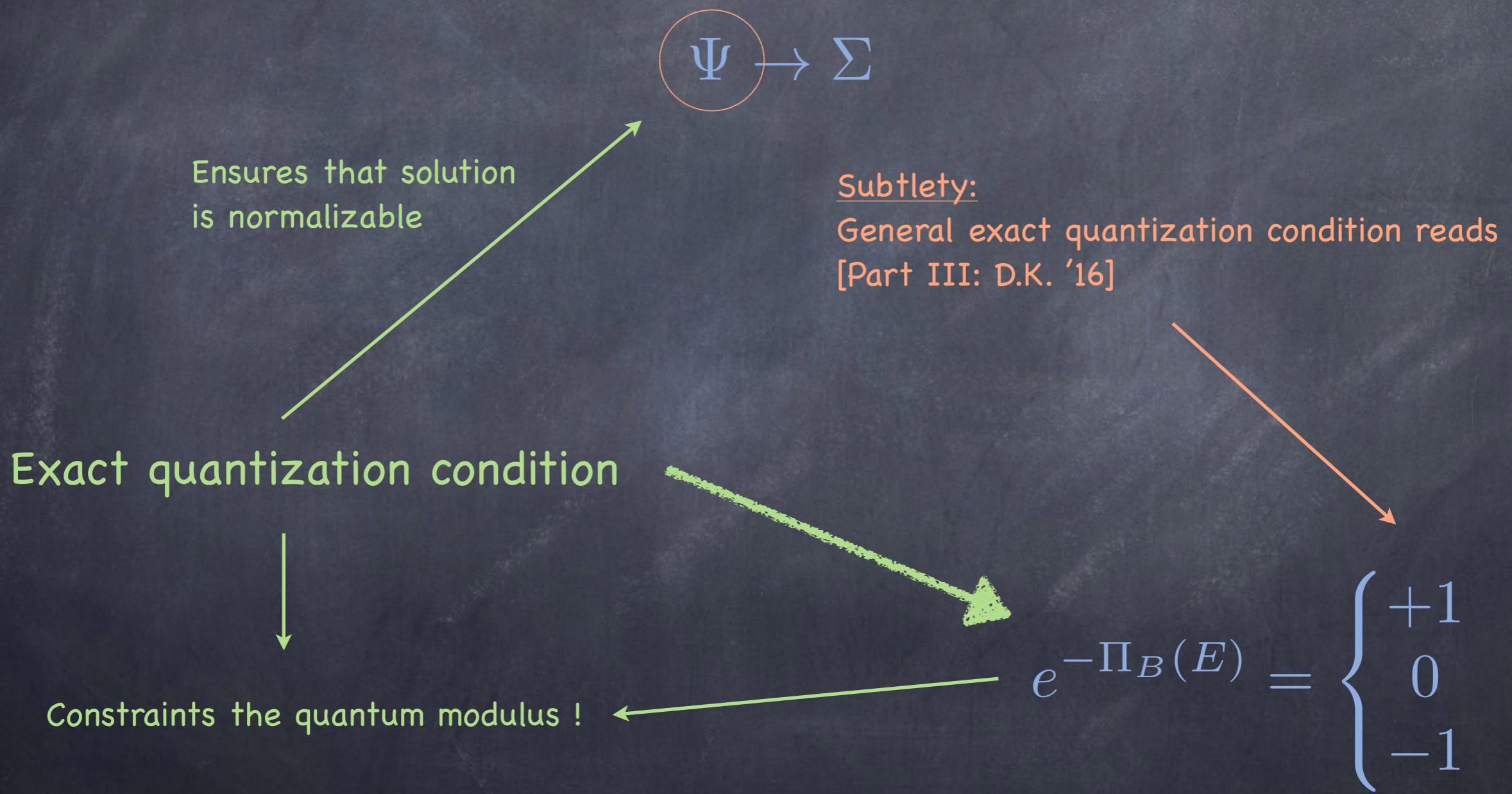




# Quantum Geometry

[Part I,II: D.K. '13,'14]

## Quantum Geometry:





# Quantum Geometry

Example:

$$\hat{f} : \frac{\partial^2}{\partial x^2} - \kappa^2 x^2 + \frac{E}{\hbar^2}$$



# Quantum Geometry

Example:

$$\hat{f} : \frac{\partial^2}{\partial x^2} - \kappa^2 x^2 + \frac{E}{\hbar^2}$$

WKB solution:

$$\Pi_A = \frac{E}{2}$$

$$\Pi_B = -\Lambda^2 + \frac{E}{2} \left( 1 - \log \left( \frac{E}{4\Lambda^2} \right) \right) - \frac{1}{12} \frac{\hbar^2}{E} - \frac{7}{360} \frac{\hbar^4}{E^3} + \mathcal{O}(\hbar^6)$$



# Quantum Geometry

Example:

$$\hat{f} : \frac{\partial^2}{\partial x^2} - \kappa^2 x^2 + \frac{E}{\hbar^2}$$

WKB solution:

$$\Pi_A = \frac{E}{2}$$

$$\Pi_B = -\Lambda^2 + \frac{E}{2} \left( 1 - \log \left( \frac{E}{4\Lambda^2} \right) \right) - \frac{1}{12} \frac{\hbar^2}{E} - \frac{7}{360} \frac{\hbar^4}{E^3} + \mathcal{O}(\hbar^6)$$



$$\frac{1}{i\hbar} \Pi_B(E) = \log \Gamma \left( \frac{1}{2} + \frac{iE}{2\hbar} \right) + \frac{E}{\hbar} \log \left( \frac{\Lambda}{\hbar} \right) - \frac{\Lambda^2}{i\hbar} - \frac{1}{2} \log 2\pi$$



# Quantum Geometry

Example:

$$\hat{f} : \frac{\partial^2}{\partial x^2} - \kappa^2 x^2 + \frac{E}{\hbar^2}$$

WKB solution:

$$\Pi_A = \frac{E}{2}$$

$$\Pi_B = -\Lambda^2 + \frac{E}{2} \left( 1 - \log \left( \frac{E}{4\Lambda^2} \right) \right) - \frac{1}{12} \frac{\hbar^2}{E} - \frac{7}{360} \frac{\hbar^4}{E^3} + \mathcal{O}(\hbar^6)$$



$$\frac{1}{i\hbar} \Pi_B(E) = \boxed{\log \Gamma \left( \frac{1}{2} + \frac{iE}{2\hbar} \right) + \frac{E}{\hbar} \log \left( \frac{\Lambda}{\hbar} \right) - \frac{\Lambda^2}{i\hbar} - \frac{1}{2} \log 2\pi}$$



As apparent from Euler's reflection formula,  
there is a Stokes phenomenon present !



# Quantum Geometry

Example:

$$\hat{f} : \frac{\partial^2}{\partial x^2} - \kappa^2 x^2 + \frac{E}{\hbar^2}$$

WKB solution:

$$\Pi_A = \frac{E}{2}$$

$$\Pi_B = -\Lambda^2 + \frac{E}{2} \left( 1 - \log \left( \frac{E}{4\Lambda^2} \right) \right) - \frac{1}{12} \frac{\hbar^2}{E} - \frac{7}{360} \frac{\hbar^4}{E^3} + \mathcal{O}(\hbar^6)$$



$$\frac{1}{i\hbar} \Pi_B(E) = \boxed{\log \Gamma \left( \frac{1}{2} + \frac{iE}{2\hbar} \right)} + \frac{E}{\hbar} \log \left( \frac{\Lambda}{\hbar} \right) - \frac{\Lambda^2}{i\hbar} - \frac{1}{2} \log 2\pi$$

$$\Pi_B(-E) = \Pi_B(E) - i\hbar \sum_{k=1}^{\infty} \frac{e^{-\frac{k\pi E}{\hbar}}}{k} + \dots$$



# Quantum Geometry

Example:

$$\hat{f} : \frac{\partial^2}{\partial x^2} - \kappa^2 x^2 + \frac{E}{\hbar^2}$$

WKB solution:

$$\Pi_A = \frac{E}{2}$$

$$\Pi_B = -\Lambda^2 + \frac{E}{2} \left( 1 - \log \left( \frac{E}{4\Lambda^2} \right) \right) - \frac{1}{12} \frac{\hbar^2}{E} - \frac{7}{360} \frac{\hbar^4}{E^3} + \mathcal{O}(\hbar^6)$$



$$\frac{1}{i\hbar} \Pi_B(E) = \boxed{\log \Gamma \left( \frac{1}{2} + \frac{iE}{2\hbar} \right)} + \frac{E}{\hbar} \log \left( \frac{\Lambda}{\hbar} \right) - \frac{\Lambda^2}{i\hbar} - \frac{1}{2} \log 2\pi$$

$$\Pi_B(-E) = \Pi_B(E) - i\hbar \sum_{k=1}^{\infty} \frac{e^{-\frac{k\pi E}{\hbar}}}{k} + \dots$$

Non-Perturbative corrections:  $\zeta^k$



# Quantum Geometry

Example:

$$\hat{f} : \frac{\partial^2}{\partial x^2} - \kappa^2 x^2 + \frac{E}{\hbar^2}$$

$$\frac{1}{i\hbar} \Pi_B(E) = \log \Gamma \left( \frac{1}{2} + \frac{iE}{2\hbar} \right) + \frac{E}{\hbar} \log \left( \frac{\Lambda}{\hbar} \right) - \frac{\Lambda^2}{i\hbar} - \frac{1}{2} \log 2\pi$$



Impose exact quantization condition

$$E \rightarrow \mathcal{E}^\pm = E + \sum_{n=1}^{\infty} E_{\pm, np}^{(n)} \xi^n$$



Non-perturbative flat coordinate !



# Quantum Geometry

Example:

$$\hat{f} : \frac{\partial^2}{\partial x^2} - \kappa^2 x^2 + \frac{E}{\hbar^2}$$

$$\frac{1}{i\hbar} \Pi_B(E) = \log \Gamma \left( \frac{1}{2} + \frac{iE}{2\hbar} \right) + \frac{E}{\hbar} \log \left( \frac{\Lambda}{\hbar} \right) - \frac{\Lambda^2}{i\hbar} - \frac{1}{2} \log 2\pi$$



Impose exact quantization condition

$$E \rightarrow \mathcal{E}^\pm = E + \sum_{n=1}^{\infty} E_{\pm, np}^{(n)} \xi^n$$

Bohr-Sommerfeld quantized energy



# Quantum Geometry

Example:

$$\hat{f} : \frac{\partial^2}{\partial x^2} - \kappa^2 x^2 + \frac{E}{\hbar^2}$$

$$\frac{1}{i\hbar} \Pi_B(E) = \log \Gamma \left( \frac{1}{2} + \frac{iE}{2\hbar} \right) + \frac{E}{\hbar} \log \left( \frac{\Lambda}{\hbar} \right) - \frac{\Lambda^2}{i\hbar} - \frac{1}{2} \log 2\pi$$



Impose exact quantization condition

$$E \rightarrow \mathcal{E}^\pm = E + \sum_{n=1}^{\infty} E_{\pm, np}^{(n)} \xi^n$$



Can be solved for analytically



# Quantum Geometry

Example:

$$\hat{f} : \frac{\partial^2}{\partial x^2} - \kappa^2 x^2 + \frac{E}{\hbar^2}$$

$$\frac{1}{i\hbar} \Pi_B(E) = \log \Gamma \left( \frac{1}{2} + \frac{iE}{2\hbar} \right) + \frac{E}{\hbar} \log \left( \frac{\Lambda}{\hbar} \right) - \frac{\Lambda^2}{i\hbar} - \frac{1}{2} \log 2\pi$$



Impose exact quantization condition

$$E \rightarrow \mathcal{E}^\pm = E + \sum_{n=1}^{\infty} E_{\pm, np}^{(n)} \xi^n$$

Instanton counting parameter



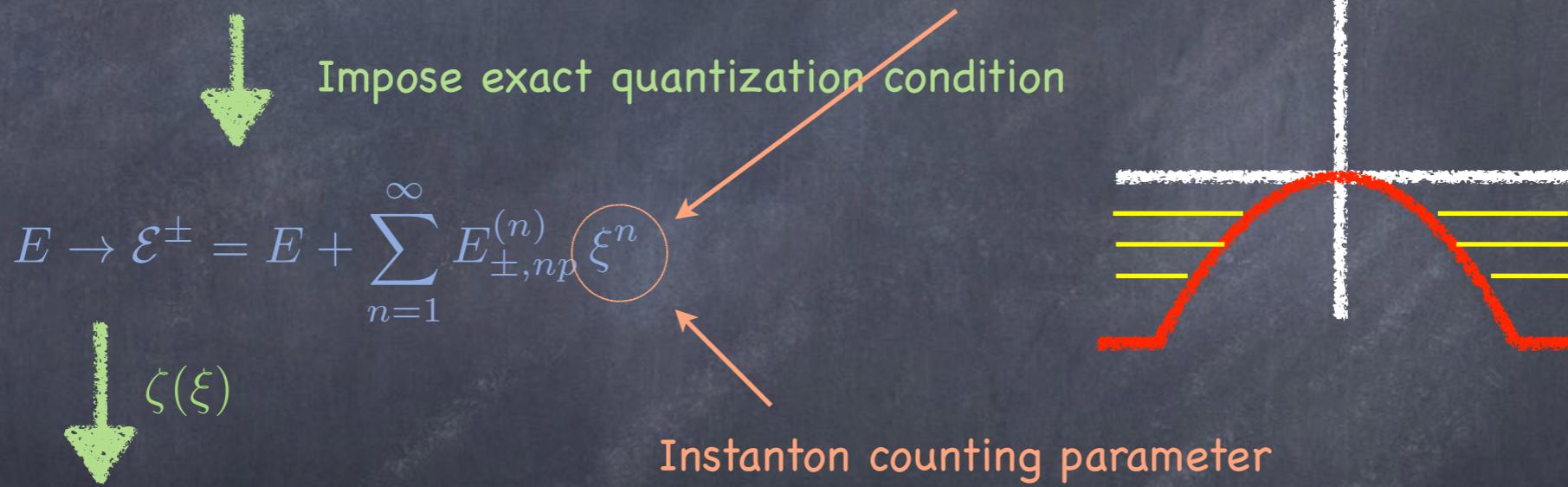
# Quantum Geometry

[Part III: D.K. '16]

Example:

$$\hat{f} : \frac{\partial^2}{\partial x^2} - \kappa^2 x^2 + \frac{E}{\hbar^2}$$

$$\frac{1}{i\hbar} \Pi_B(E) = \log \Gamma \left( \frac{1}{2} + \frac{iE}{2\hbar} \right) + \frac{E}{\hbar} \log \left( \frac{\Lambda}{\hbar} \right) - \frac{\Lambda^2}{i\hbar} - \frac{1}{2} \log 2\pi$$



Trans-series expansion of both periods



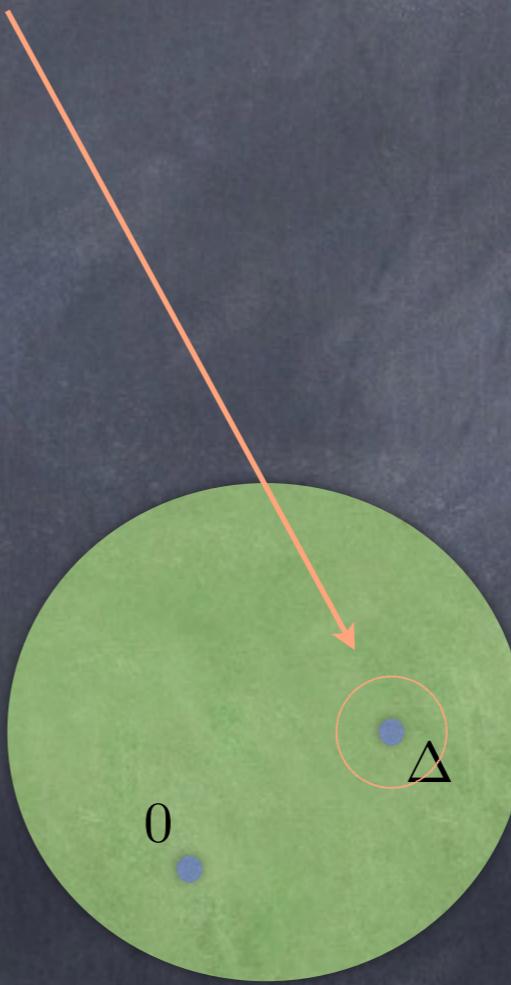
## Toric Calabi-Yau example



# Quantum Geometry

Generalization:

Consider a toric Calabi-Yau with a conifold point in moduli space



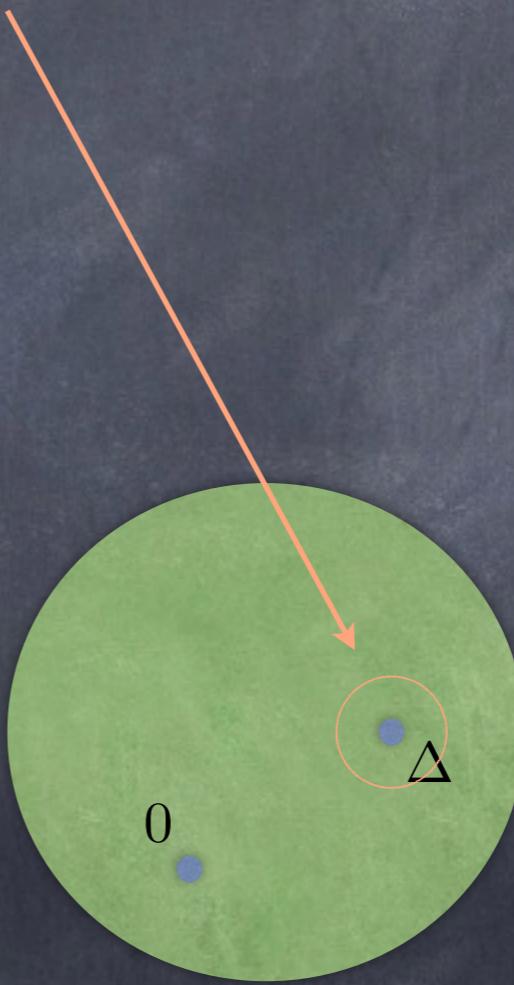


# Quantum Geometry

Generalization:

Consider a toric Calabi-Yau with a conifold point in moduli space

→ Massless hypermultiplet  
[Vafa '95; Ghoshal-Vafa '95]





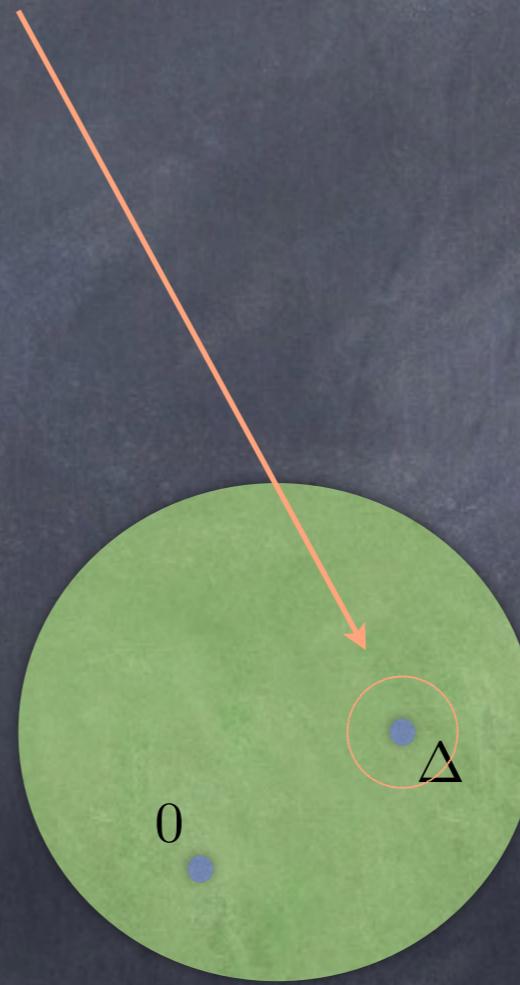
# Quantum Geometry

## Generalization:

Consider a toric Calabi-Yau with a conifold point in moduli space

- Massless hypermultiplet  
[Vafa '95; Ghoshal-Vafa '95]
- Leading singular contribution  
to refined top. string given  
by Schwinger integral  
[D.K. + Walcher '10]

$$\mathcal{F}_{sing} \sim \int_{\delta}^{\infty} \frac{dx}{x} \frac{e^{-tx}}{\sinh\left(\frac{\epsilon_1 x}{2}\right) \sinh\left(\frac{\epsilon_2 x}{2}\right)}$$





# Quantum Geometry

## Generalization:

Consider a toric Calabi-Yau with a conifold point in moduli space

→ Massless hypermultiplet  
[Vafa '95; Ghoshal-Vafa '95]

→ Leading singular contribution  
to refined top. string given  
by Schwinger integral  
[D.K. + Walcher '10]

NS limit easily taken

$$\mathcal{F}_{sing} \sim \int_{\delta}^{\infty} \frac{dx}{x} \frac{e^{-tx}}{\sinh\left(\frac{\epsilon_1 x}{2}\right) \sinh\left(\frac{\epsilon_2 x}{2}\right)}$$



# Quantum Geometry

## Generalization:

Consider a toric Calabi-Yau with a conifold point in moduli space

→ Massless hypermultiplet  
[Vafa '95; Ghoshal-Vafa '95]

→ Leading singular contribution  
to refined top. string given  
by Schwinger integral  
[D.K. + Walcher '10]

NS limit easily taken

$$e^{-\Pi_B^{sing}} \sim \left(\frac{1}{\hbar}\right)^{\frac{t}{\hbar}} \Gamma\left(\frac{1}{2} + \frac{t}{\hbar}\right)$$

$$\mathcal{F}_{sing} \sim \int_{\delta}^{\infty} \frac{dx}{x} \frac{e^{-tx}}{\sinh\left(\frac{\epsilon_1 x}{2}\right) \sinh\left(\frac{\epsilon_2 x}{2}\right)}$$



# Quantum Geometry

## Generalization:

Consider a toric Calabi-Yau with a conifold point in moduli space

→ Massless hypermultiplet  
[Vafa '95; Ghoshal-Vafa '95]

→ Leading singular contribution  
to refined top. string given  
by Schwinger integral  
[D.K. + Walcher '10]

$$\mathcal{F}_{sing} \sim \int_{\delta}^{\infty} \frac{dx}{x} \frac{e^{-tx}}{\sinh\left(\frac{\epsilon_1 x}{2}\right) \sinh\left(\frac{\epsilon_2 x}{2}\right)}$$

NS limit easily taken

$$e^{-\Pi_B^{sing}} \sim \left(\frac{1}{\hbar}\right)^{\frac{t}{\hbar}} \Gamma\left(\frac{1}{2} + \frac{t}{\hbar}\right)$$

$$e^{-\Pi_B} \sim \left(\frac{1}{\hbar}\right)^{\frac{t}{\hbar}} \Gamma\left(\frac{1}{2} + \frac{t}{\hbar}\right) e^{-A_p(t)} \xi$$

NS quantization condition

$$\Pi_B^{reg} = \frac{c_X}{\hbar} + A_p(t)$$



# Quantum Geometry

## Generalization:

Consider a toric Calabi-Yau with a conifold point in moduli space

→ Massless hypermultiplet  
[Vafa '95; Ghoshal-Vafa '95]

→ Leading singular contribution  
to refined top. string given  
by Schwinger integral  
[D.K. + Walcher '10]

$$\mathcal{F}_{sing} \sim \int_{\delta}^{\infty} \frac{dx}{x} \frac{e^{-tx}}{\sinh\left(\frac{\epsilon_1 x}{2}\right) \sinh\left(\frac{\epsilon_2 x}{2}\right)}$$

NS limit easily taken

$$e^{-\Pi_B^{sing}} \sim \left(\frac{1}{\hbar}\right)^{\frac{t}{\hbar}} \Gamma\left(\frac{1}{2} + \frac{t}{\hbar}\right)$$

$$e^{-\Pi_B} \sim \left(\frac{1}{\hbar}\right)^{\frac{t}{\hbar}} \Gamma\left(\frac{1}{2} + \frac{t}{\hbar}\right) e^{-A_p(t)} \xi$$

NS quantization condition

$$\Pi_B^{reg} = \frac{c_X}{\hbar} + A_p(t)$$



# Quantum Geometry

## Generalization:

Consider a toric Calabi-Yau with a conifold point in moduli space

→ Massless hypermultiplet  
[Vafa '95; Ghoshal-Vafa '95]

→ Leading singular contribution  
to refined top. string given  
by Schwinger integral  
[D.K. + Walcher '10]

$$\mathcal{F}_{sing} \sim \int_{\delta}^{\infty} \frac{dx}{x} \frac{e^{-tx}}{\sinh\left(\frac{\epsilon_1 x}{2}\right) \sinh\left(\frac{\epsilon_2 x}{2}\right)}$$

NS limit easily taken

$$e^{-\Pi_B^{sing}} \sim \left(\frac{1}{\hbar}\right)^{\frac{t}{\hbar}} \Gamma\left(\frac{1}{2} + \frac{t}{\hbar}\right)$$

$$e^{-\Pi_B} \sim \left(\frac{1}{\hbar}\right)^{\frac{t}{\hbar}} \Gamma\left(\frac{1}{2} + \frac{t}{\hbar}\right) e^{-A_p(t)} \xi$$

The pure deformed conifold  
discussion from before  
applies 1:1 !

$\sim \Lambda^2$

NS quantization condition

$$\Pi_B^{reg} = \frac{c_X}{\hbar} + A_p(t)$$



# Quantum Geometry

## Generalization:

Consider a toric Calabi-Yau with a conifold point in moduli space

→ Massless hypermultiplet  
[Vafa '95; Ghoshal-Vafa '95]

→ Leading singular contribution  
to refined top. string given  
by Schwinger integral  
[D.K. + Walcher '10]

$$\mathcal{F}_{sing} \sim \int_{\delta}^{\infty} \frac{dx}{x} \frac{e^{-tx}}{\sinh\left(\frac{\epsilon_1 x}{2}\right) \sinh\left(\frac{\epsilon_2 x}{2}\right)}$$

Non-perturbative completion  
of NS limit on toric CY, as  
predicted in [Part I, '13]

The pure deformed conifold  
discussion from before  
applies 1:1 !

NS limit easily taken

$$e^{-\Pi_B^{sing}} \sim \left(\frac{1}{\hbar}\right)^{\frac{t}{\hbar}} \Gamma\left(\frac{1}{2} + \frac{t}{\hbar}\right)$$

NS quantization condition

$$e^{-\Pi_B} \sim \left(\frac{1}{\hbar}\right)^{\frac{t}{\hbar}} \Gamma\left(\frac{1}{2} + \frac{t}{\hbar}\right) e^{-A_p(t)} \xi$$

Add regular part of period  
 $\Pi_B^{reg} = \frac{c_X}{\hbar} + A_p(t)$

$\sim \Lambda^2$



# Quantum Geometry

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$



# Quantum Geometry

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

$$\Sigma : -1 + e^x + e^p + z_1 e^{-x} + z_2 e^{-p} = 0$$



# Quantum Geometry

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

$$\Sigma : -1 + e^x + e^p + z_1 e^{-x} + z_2 e^{-p} = 0$$



Two moduli.

Large volume regime in moduli space:  $z_1, z_2 \ll 1$



# Quantum Geometry

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

$$\Sigma : -1 + e^x + e^p + z_1 e^{-x} + z_2 e^{-p} = 0$$

Change parameterization:

$$x \rightarrow ix + \frac{1}{2} \log z_1, \quad p \rightarrow p + \frac{1}{2} \log z_2$$

$$\lambda := i\sqrt{\frac{z_1}{z_2}}, \quad E := \frac{1}{\sqrt{z_2}}$$

$$2\lambda \cos(x) + e^p + e^{-p} = E$$



# Quantum Geometry

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

$$\Sigma : -1 + e^x + e^p + z_1 e^{-x} + z_2 e^{-p} = 0$$

Change parameterization:

$$x \rightarrow ix + \frac{1}{2} \log z_1, \quad p \rightarrow p + \frac{1}{2} \log z_2$$

$$\lambda := i\sqrt{\frac{z_1}{z_2}}, \quad E := \frac{1}{\sqrt{z_2}}$$

Important:

$E \ll 1$  is NOT at large volume !!!

$$2\lambda \cos(x) + e^p + e^{-p} = E$$



# Quantum Geometry

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

$$2\lambda \cos(x) + e^p + e^{-p} = E$$

Quantization:

$$[x, p] = i\hbar$$

$$e^p + e^{-p} \rightarrow \mathcal{D} = e^{i\hbar\partial_x} + e^{-i\hbar\partial_x}$$

$$(\mathcal{D} + 2\lambda \cos(x)) \Psi(x) = E\Psi(x)$$

"Quantum Mathieu" equation



# Quantum Geometry

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

$$2\lambda \cos(x) + e^p + e^{-p} = E$$

Quantization:

$$[x, p] = i\hbar$$

$$e^p + e^{-p} \rightarrow \mathcal{D} = e^{i\hbar\partial_x} + e^{-i\hbar\partial_x}$$

$$(\mathcal{D} + 2\lambda \cos(x)) \Psi(x) = E\Psi(x)$$

"Quantum Mathieu" equation

$$\downarrow \quad \hbar \ll 1$$

$$-\Psi''(x) + \frac{2\lambda}{\hbar^2} \cos(x)\Psi(x) + \mathcal{O}(\hbar^2) = \frac{E - 2}{\hbar^2} \Psi(x)$$

At leading order a Mathieu equation



# Quantum Geometry

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

$$2\lambda \cos(x) + e^p + e^{-p} = E$$

Quantization:

$$[x, p] = i\hbar$$

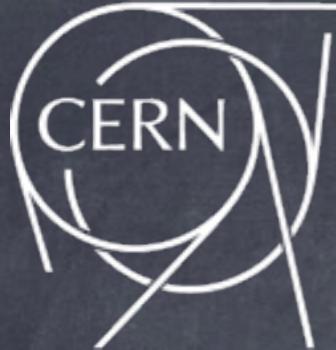
$$e^p + e^{-p} \rightarrow \mathcal{D} = e^{i\hbar\partial_x} + e^{-i\hbar\partial_x}$$

$$(\mathcal{D} + 2\lambda \cos(x)) \Psi(x) = E\Psi(x)$$

"Quantum Mathieu" equation

All my results about the non-perturbative  
aspects of the quantum SU(2) gauge theory  
MUST be reproduced as limiting case of  
local  $\mathbb{P}^1 \times \mathbb{P}^1$  !





# Quantum Geometry

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

[Part I,II: D.K. '13,'14]

$$2\lambda \cos(x) + e^p + e^{-p} = E$$

Quantization:

$$[x, p] = i\hbar$$

$$e^p + e^{-p} \rightarrow \mathcal{D} = e^{i\hbar\partial_x} + e^{-i\hbar\partial_x}$$

$$(\mathcal{D} + 2\lambda \cos(x)) \Psi(x) = E\Psi(x)$$

"Quantum Mathieu" equation

All my results about the non-perturbative aspects of the quantum SU(2) gauge theory MUST be reproduced as limiting case of local  $\mathbb{P}^1 \times \mathbb{P}^1$  !





# Quantum Geometry

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

[Part I,II: D.K. '13,'14]

$$2\lambda \cos(x) + e^p + e^{-p} = E$$

Quantization:

$$[x, p] = i\hbar$$

$$e^p + e^{-p} \rightarrow \mathcal{D} = e^{i\hbar\partial_x} + e^{-i\hbar\partial_x}$$

Can be verified  
numerically  
[Part III: D.K. '16]

$$(\mathcal{D} + 2\lambda \cos(x)) \Psi(x) = E\Psi(x)$$

"Quantum Mathieu" equation

All my results about the non-perturbative  
aspects of the quantum SU(2) gauge theory  
MUST be reproduced as limiting case of  
local  $\mathbb{P}^1 \times \mathbb{P}^1$  !

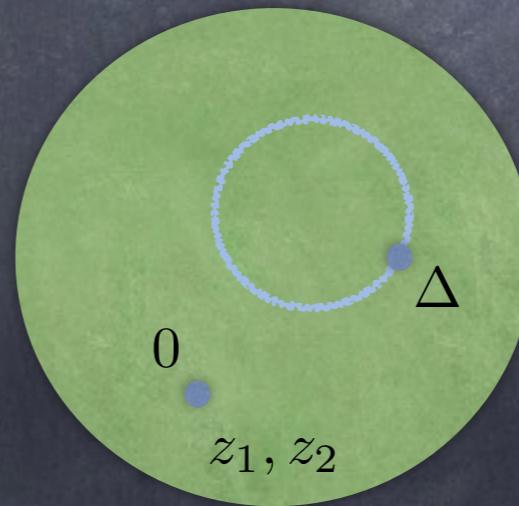




# Quantum Geometry

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

$$\Sigma : -1 + e^x + e^p + z_1 e^{-x} + z_2 e^{-p} = 0$$

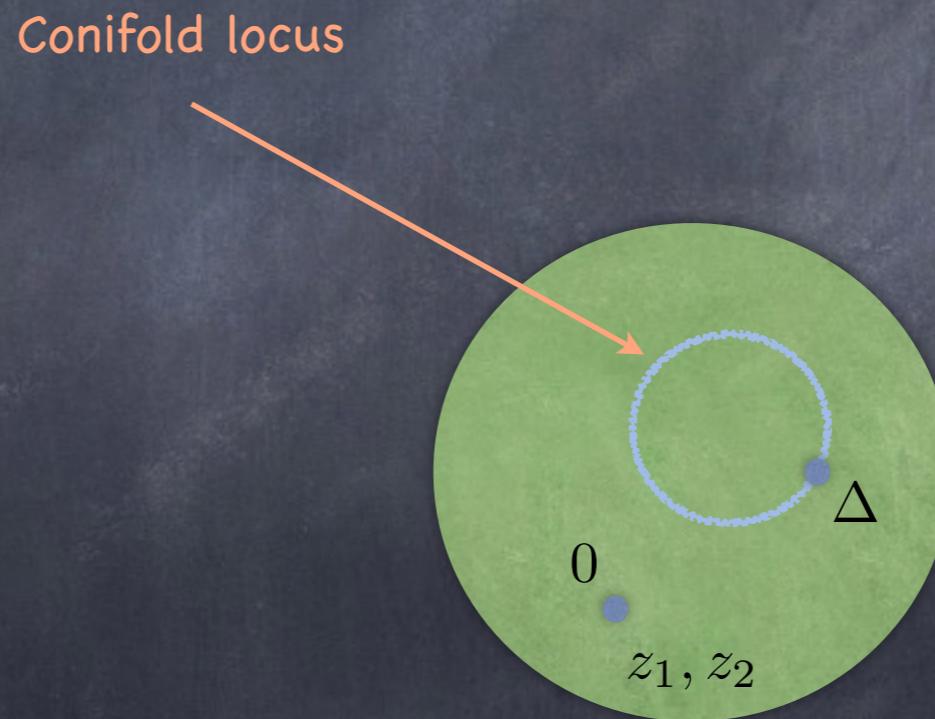




# Quantum Geometry

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

$$\Sigma : -1 + e^x + e^p + z_1 e^{-x} + z_2 e^{-p} = 0$$





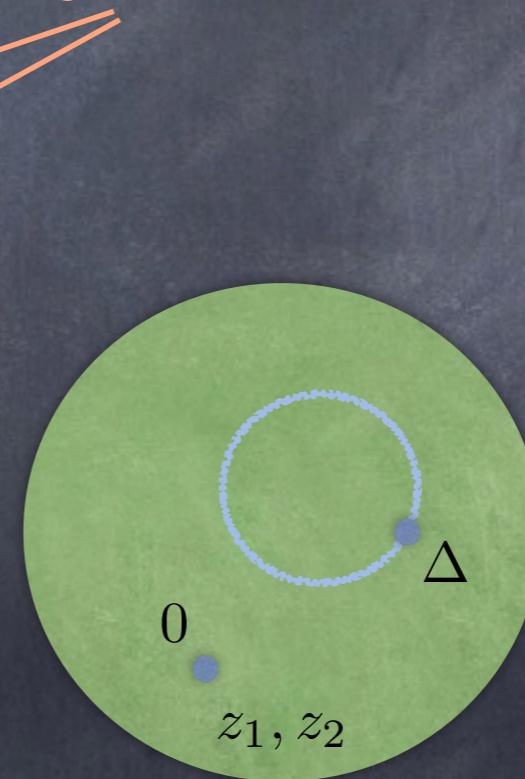
# Quantum Geometry

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

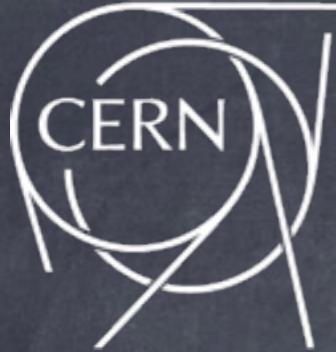
$$\Sigma : -1 + e^x + e^p + z_1 e^{-x} + z_2 e^{-p} = 0$$

Local coordinates at a conifold point  
(cf. Haghigat, Kleemann, Rauch '08)

$$z_1 = \frac{1}{8} - \frac{1}{8(2 + \Delta_1(\Delta_2 - 1) - \Delta_2)}$$
$$z_2 = \frac{\Delta_2 - 1}{8(2 + \Delta_1(\Delta_2 - 1) - \Delta_2)}$$



For simplicity, we set  $\lambda = 1 \rightarrow \Delta_1 = 0$



# Quantum Geometry

[Part III: D.K. '16]

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

$$\Sigma : -1 + e^x + e^p + z_1 e^{-x} + z_2 e^{-p} = 0$$

$$\downarrow [x, p] = -\hbar$$

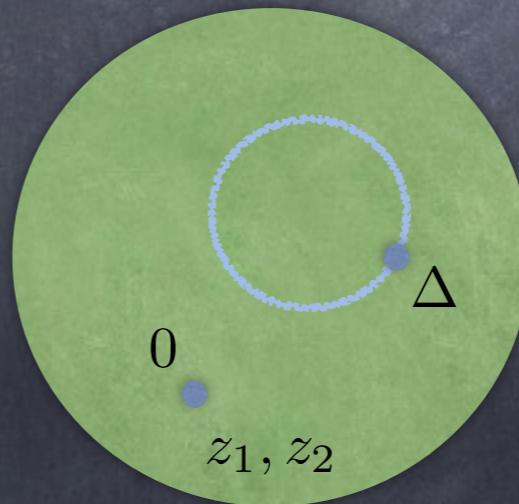
$\widehat{\Sigma}$  in terms of the conifold coordinate  $\Delta$

$$\uparrow$$

$$z_1 = \frac{1}{8} - \frac{1}{8(2 + \Delta_1(\Delta_2 - 1) - \Delta_2)}$$

$$z_2 = \frac{\Delta_2 - 1}{8(2 + \Delta_1(\Delta_2 - 1) - \Delta_2)}$$

For simplicity, we set  $\lambda = 1 \rightarrow \Delta_1 = 0$





# Quantum Geometry

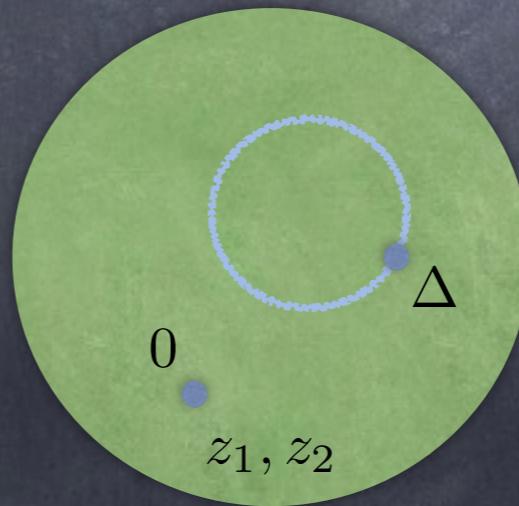
[Part III: D.K. '16]

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

$$\Sigma : -1 + e^x + e^p + z_1 e^{-x} + z_2 e^{-p} = 0$$

$$\downarrow [x, p] = -\hbar$$

$$(2 - \Delta + \frac{1}{8}(\Delta - 1)e^{-x} + (\Delta - 2)e^x)\Psi(x) + (\Delta - 2)\Psi(x + \hbar) + \frac{1}{8}(\Delta - 1)\Psi(x - \hbar) = 0$$





# Quantum Geometry

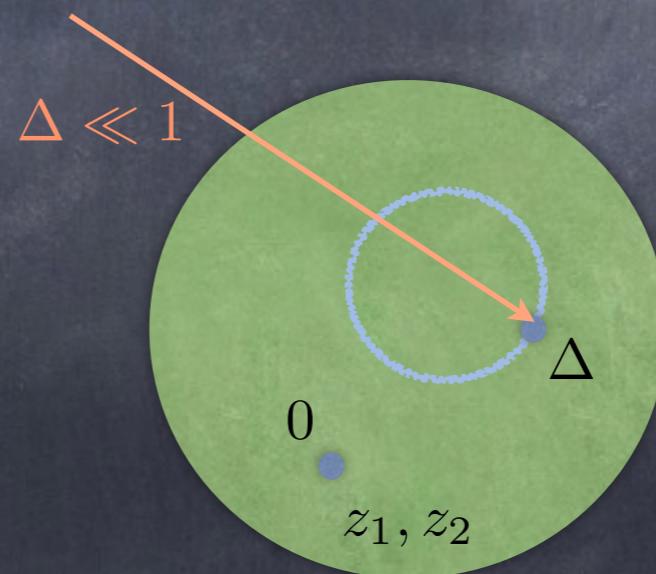
[Part III: D.K. '16]

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

$$\Sigma : -1 + e^x + e^p + z_1 e^{-x} + z_2 e^{-p} = 0$$

$$\downarrow [x, p] = -\hbar$$

$$(2 - \Delta + \frac{1}{8}(\Delta - 1)e^{-x} + (\Delta - 2)e^x)\Psi(x) + (\Delta - 2)\Psi(x + \hbar) + \frac{1}{8}(\Delta - 1)\Psi(x - \hbar) = 0$$





# Quantum Geometry

[Part III: D.K. '16]

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

$$(2 - \Delta + \frac{1}{8}(\Delta - 1)e^{-x} + (\Delta - 2)e^x)\Psi(x) + (\Delta - 2)\Psi(x + \hbar) + \frac{1}{8}(\Delta - 1)\Psi(x - \hbar) = 0$$



Can be easily solved for via a WKB Ansatz



Quantum periods at the conifold point order by order in  $\hbar$

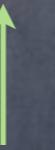


# Quantum Geometry

[Part III: D.K. '16]

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

$$(2 - \Delta + \frac{1}{8}(\Delta - 1)e^{-x} + (\Delta - 2)e^x)\Psi(x) + (\Delta - 2)\Psi(x + \hbar) + \frac{1}{8}(\Delta - 1)\Psi(x - \hbar) = 0$$



Can be easily solved for via a WKB Ansatz



Quantum periods at the conifold point order by order in  $\hbar$



Order  $\hbar^1$  is actually non-vanishing (sort of Maslov index) !



# Quantum Geometry

[Part III: D.K. '16]

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

$$(2 - \Delta + \frac{1}{8}(\Delta - 1)e^{-x} + (\Delta - 2)e^x)\Psi(x) + (\Delta - 2)\Psi(x + \hbar) + \frac{1}{8}(\Delta - 1)\Psi(x - \hbar) = 0$$

$$\rightarrow \Delta(N) = (2N + 1)\hbar - \frac{3 + 13N + 13N^2}{4}\hbar^2 + \mathcal{O}(\hbar^3)$$


Quantum level due to Bohr-Sommerfeld quantization  $\Pi_A = \hbar N$



# Quantum Geometry

[Part III: D.K. '16]

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

$$(2 - \Delta + \frac{1}{8}(\Delta - 1)e^{-x} + (\Delta - 2)e^x)\Psi(x) + (\Delta - 2)\Psi(x + \hbar) + \frac{1}{8}(\Delta - 1)\Psi(x - \hbar) = 0$$



$$\Delta(N) = (2N + 1)\hbar - \frac{3 + 13N + 13N^2}{4}\hbar^2 + \mathcal{O}(\hbar^3)$$



Map to  $E$  coordinate

$$E = 2\sqrt{\frac{2(\Delta - 2)}{\Delta - 1}} \rightarrow E(\Delta(N))$$



# Quantum Geometry

[Part III: D.K. '16]

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

$$(2 - \Delta + \frac{1}{8}(\Delta - 1)e^{-x} + (\Delta - 2)e^x)\Psi(x) + (\Delta - 2)\Psi(x + \hbar) + \frac{1}{8}(\Delta - 1)\Psi(x - \hbar) = 0$$



$$\Delta(N) = (2N + 1)\hbar - \frac{3 + 13N + 13N^2}{4}\hbar^2 + \mathcal{O}(\hbar^3)$$



Map to  $E$  coordinate

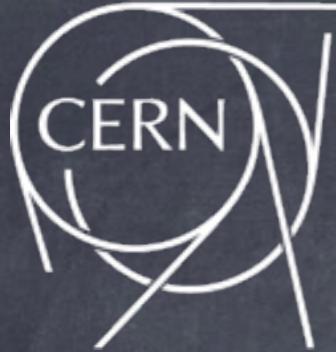
$$E = 2\sqrt{\frac{2(\Delta - 2)}{\Delta - 1}}$$



WKB approximation



$$E(\Delta(N))$$

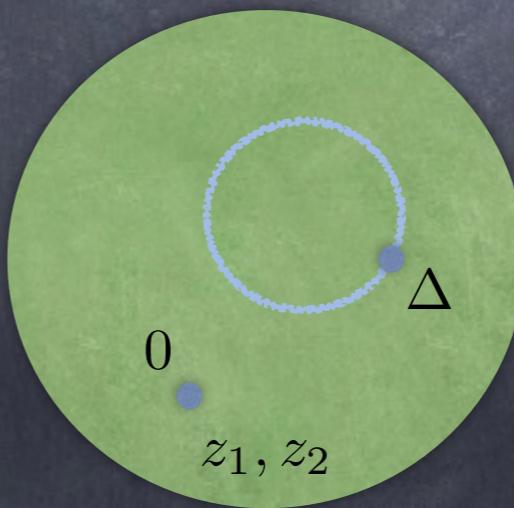


# Quantum Geometry

[Part III: D.K. '16]

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

$$E(\Delta(N))$$





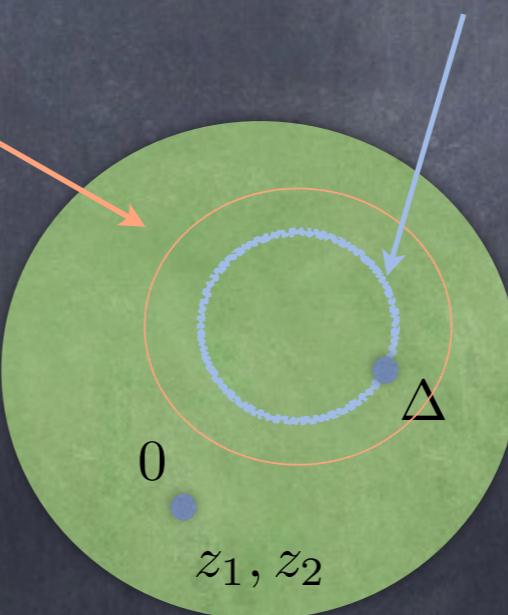
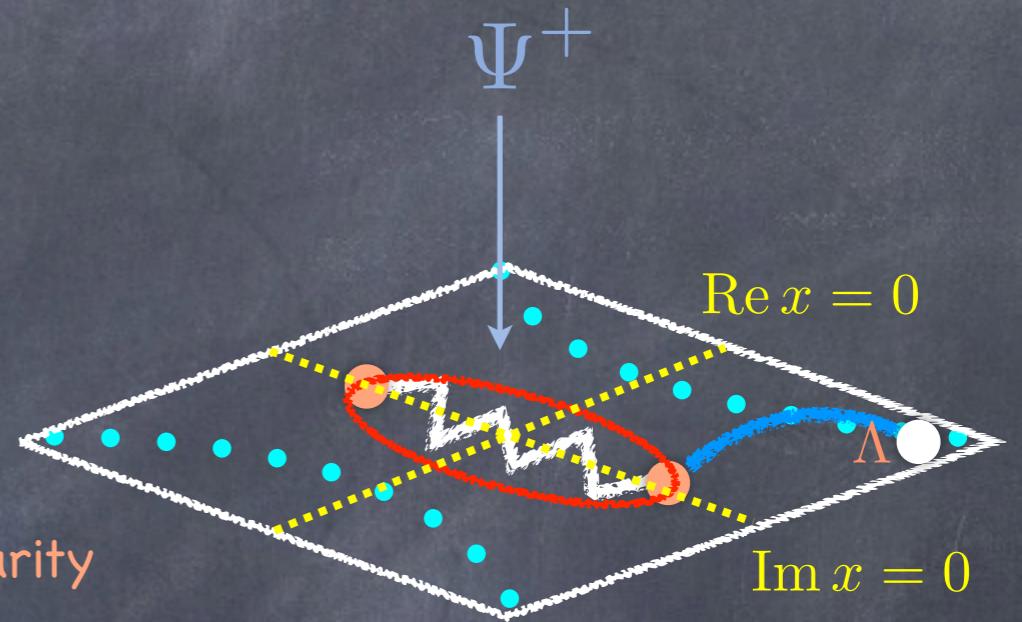
# Quantum Geometry

[Part III: D.K. '16]

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

$E(\Delta(N))$

Geometry develops a conifold singularity





# Quantum Geometry

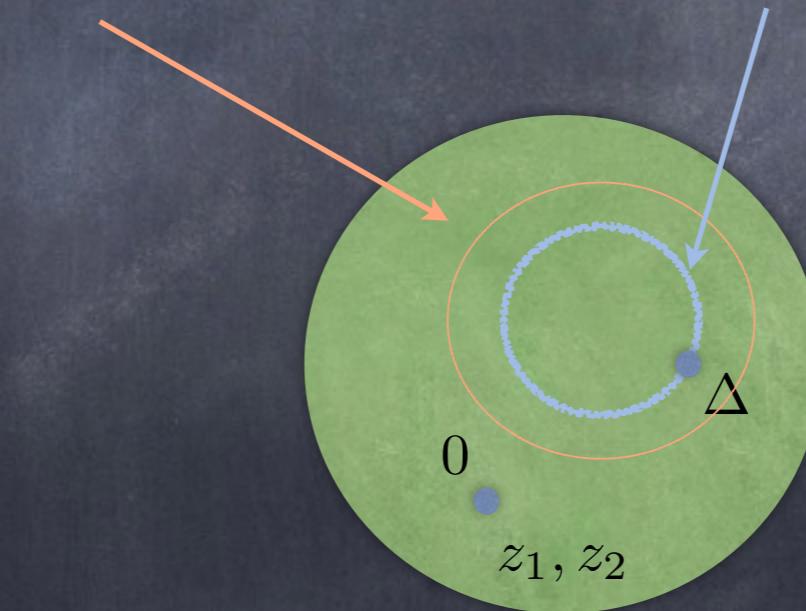
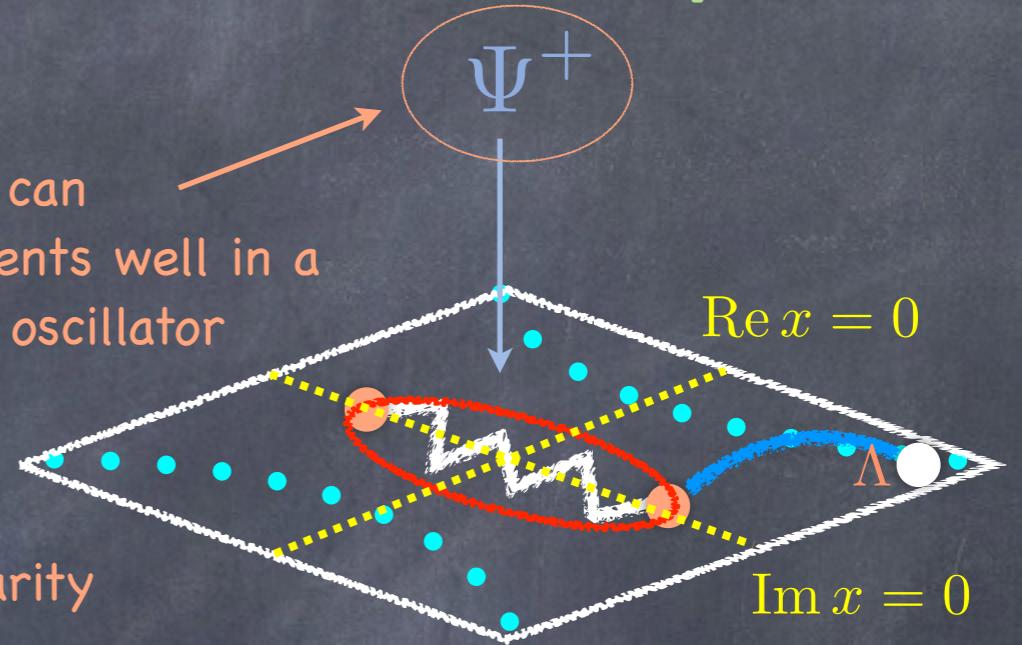
[Part III: D.K. '16]

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

$$E(\Delta(N))$$

At the conifold point we can approximate matrix elements well in a harmonic (or inharmonic) oscillator basis !

Geometry develops a conifold singularity





# Quantum Geometry

[Part III: D.K. '16]

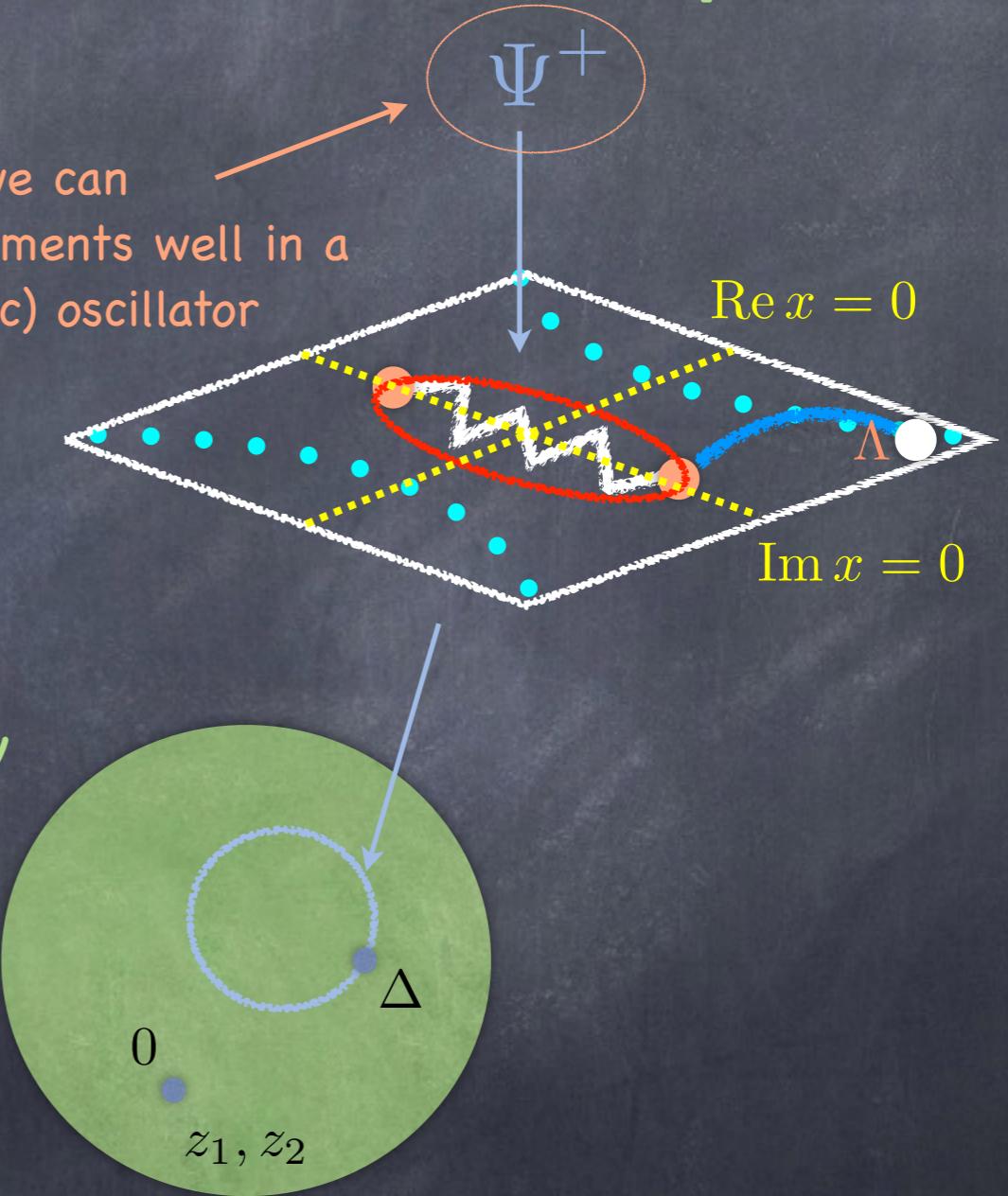
Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

$$E(\Delta(N))$$

At the conifold point we can approximate matrix elements well in a harmonic (or inharmonic) oscillator basis !



Groundstate  $E_0$  can be easily calculated numerically





# Quantum Geometry

[Part III: D.K. '16]

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

$$E(\Delta(N))$$

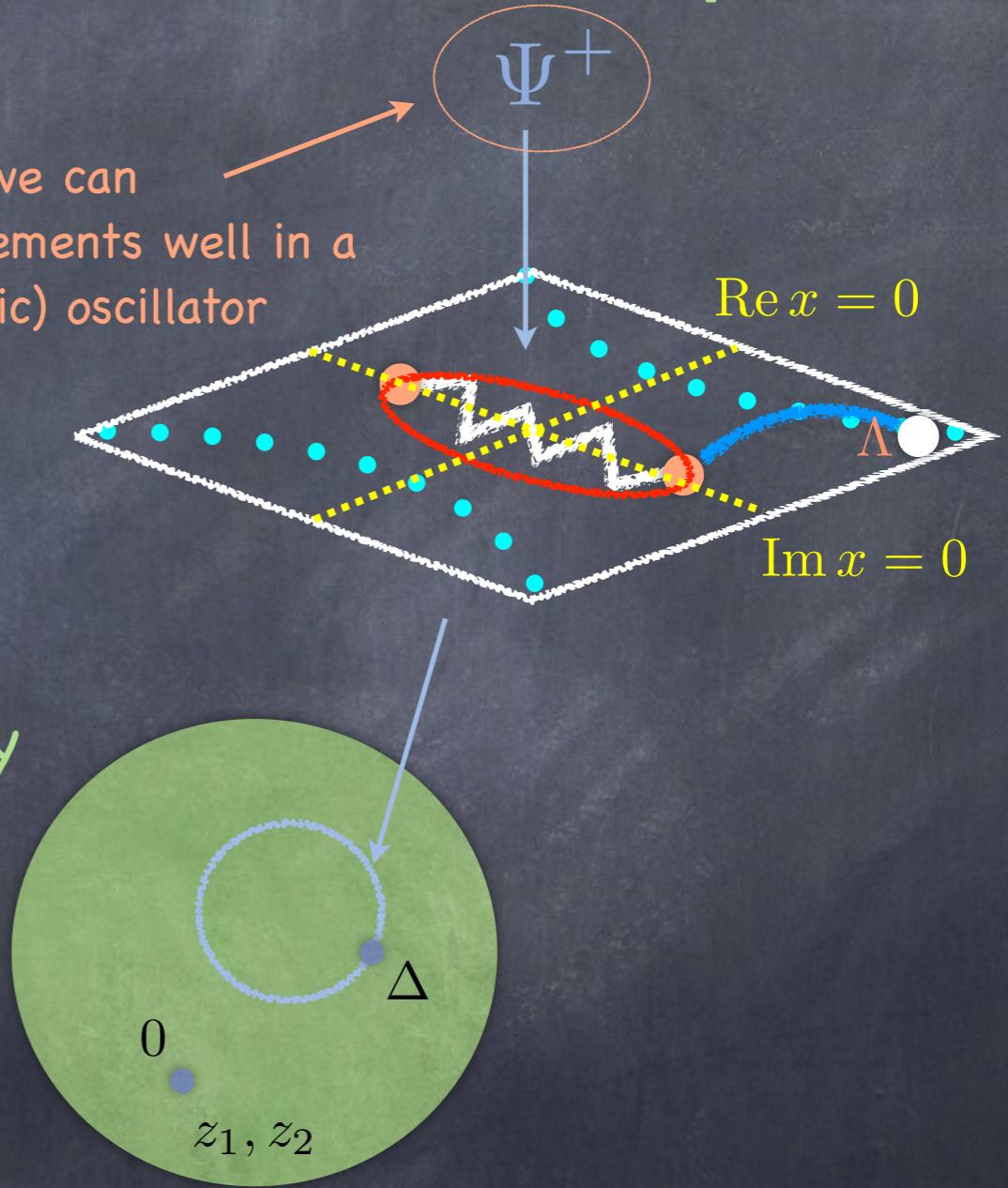
At the conifold point we can approximate matrix elements well in a harmonic (or inharmonic) oscillator basis !

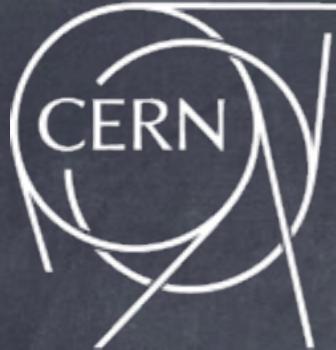


Groundstate  $E_0$  can be easily calculated numerically



The numerics is identical to [Huang+Wang '14] and other works in the literature. However, our insight leads to a fundamentally different story.





# Quantum Geometry

[Part III: D.K. '16]

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

$$(\mathcal{D} + 2\lambda \cos(x)) \Psi(x) = E\Psi(x)$$

$E(\Delta(N))$	$\hbar$	$E(\Delta_p(0) + \mathcal{O}(\hbar^4))$	$E(\Delta_p(0) + \mathcal{O}(\hbar^6))$	$E_0$
	0.10	<u>4.1012552083</u>	<u>4.1012553358</u>	4.1012553359
	0.25	<u>4.2578938802</u>	<u>4.2578987002</u>	4.2578987246
	0.50	<u>4.5319010416</u>	<u>4.5319739024</u>	4.5319753251
	0.75	<u>4.8225097656</u>	<u>4.8228570580</u>	4.8228719839
	1.00	<u>5.1302083333</u>	<u>5.1312377929</u>	5.1313156016
	1.25	<u>5.4554850260</u>	<u>5.4578319589</u>	5.4581090443
	1.50	<u>5.7988281250</u>	<u>5.8033496856</u>	5.8041260743

Numerical groundstate



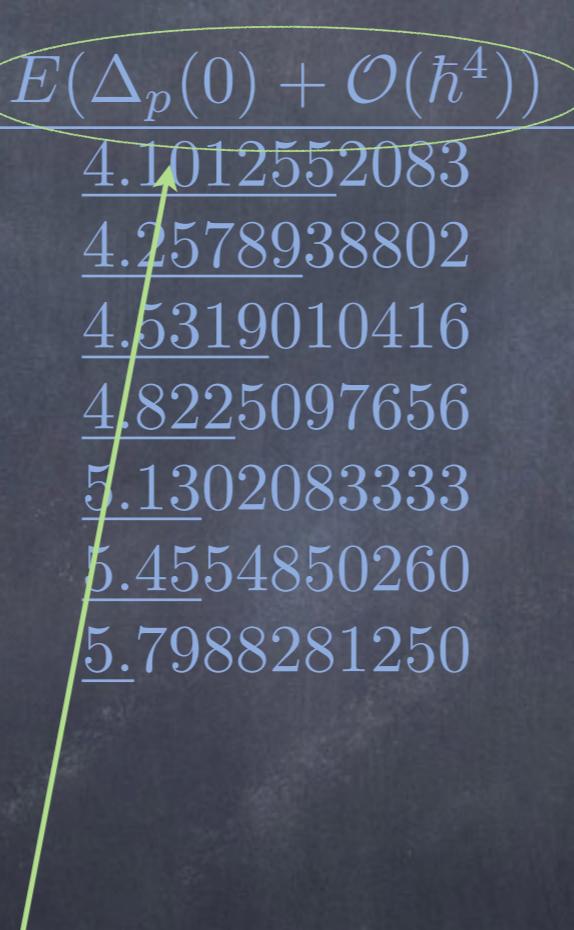
# Quantum Geometry

[Part III: D.K. '16]

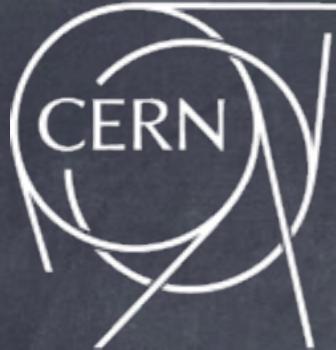
Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

$$(\mathcal{D} + 2\lambda \cos(x)) \Psi(x) = E\Psi(x)$$

$E(\Delta(N))$	$\hbar$	$E(\Delta_p(0) + \mathcal{O}(\hbar^4))$	$E(\Delta_p(0) + \mathcal{O}(\hbar^6))$	$E_0$
	0.10	4.1012552083	4.1012553358	4.1012553359
	0.25	4.2578938802	4.2578987002	4.2578987246
	0.50	4.5319010416	4.5319739024	4.5319753251
	0.75	4.8225097656	4.8228570580	4.8228719839
	1.00	5.1302083333	5.1312377929	5.1313156016
	1.25	5.4554850260	5.4578319589	5.4581090443
	1.50	5.7988281250	5.8033496856	5.8041260743



WKB ground state including  
up to order  $\hbar^4$  terms



# Quantum Geometry

[Part III: D.K. '16]

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

$$(\mathcal{D} + 2\lambda \cos(x)) \Psi(x) = E\Psi(x)$$

$E(\Delta(N))$	$\hbar$	$E(\Delta_p(0) + \mathcal{O}(\hbar^4))$	$E(\Delta_p(0) + \mathcal{O}(\hbar^6))$	$E_0$
	0.10	<u>4.1012552083</u>	<u>4.1012553358</u>	4.1012553359
	0.25	<u>4.2578938802</u>	<u>4.2578987002</u>	4.2578987246
	0.50	<u>4.5319010416</u>	<u>4.5319739024</u>	4.5319753251
	0.75	<u>4.8225097656</u>	<u>4.8228570580</u>	4.8228719839
	1.00	<u>5.1302083333</u>	<u>5.1312377929</u>	5.1313156016
	1.25	<u>5.4554850260</u>	<u>5.4578319589</u>	5.4581090443
	1.50	<u>5.7988281250</u>	<u>5.8033496856</u>	5.8041260743

WKB ground state including  
up to order  $\hbar^6$  terms



# Quantum Geometry

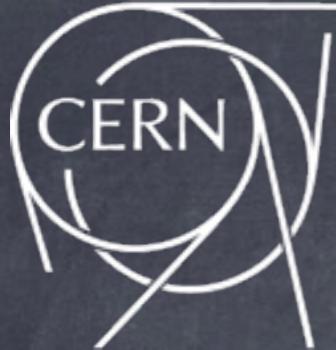
[Part III: D.K. '16]

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

$$(\mathcal{D} + 2\lambda \cos(x)) \Psi(x) = E\Psi(x)$$

$E(\Delta(N))$	$\hbar$	$E(\Delta_p(0) + \mathcal{O}(\hbar^4))$	$E(\Delta_p(0) + \mathcal{O}(\hbar^6))$	$E_0$
	0.10	<u>4.1012552083</u>	<u>4.1012553358</u>	4.1012553359
	0.25	<u>4.2578938802</u>	<u>4.2578987002</u>	4.2578987246
	0.50	<u>4.5319010416</u>	<u>4.5319739024</u>	4.5319753251
	0.75	<u>4.8225097656</u>	<u>4.8228570580</u>	4.8228719839
	1.00	<u>5.1302083333</u>	<u>5.1312377929</u>	5.1313156016
	1.25	<u>5.4554850260</u>	<u>5.4578319589</u>	5.4581090443
	1.50	<u>5.7988281250</u>	<u>5.8033496856</u>	5.8041260743

WKB converges with increasing order to the numerical results !



# Quantum Geometry

[Part III: D.K. '16]

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

$$(\mathcal{D} + 2\lambda \cos(x)) \Psi(x) = E\Psi(x)$$

$E(\Delta(N))$	$\hbar$	$E(\Delta_p(0) + \mathcal{O}(\hbar^4))$	$E(\Delta_p(0) + \mathcal{O}(\hbar^6))$	$E_0$
	0.10	4.1012552083	4.1012553358	4.1012553359
	0.25	4.2578938802	4.2578987002	4.2578987246
	0.50	4.5319010416	4.5319739024	4.5319753251
	0.75	4.8225097656	4.8228570580	4.8228719839
	1.00	5.1302083333	5.1312377929	5.1313156016
	1.25	5.4554850260	5.4578319589	5.4581090443
	1.50	5.7988281250	5.8033496856	5.8041260743

WKB converges with increasing order to the numerical results !



No NP corrections in this phase



# Quantum Geometry

[Part III: D.K. '16]

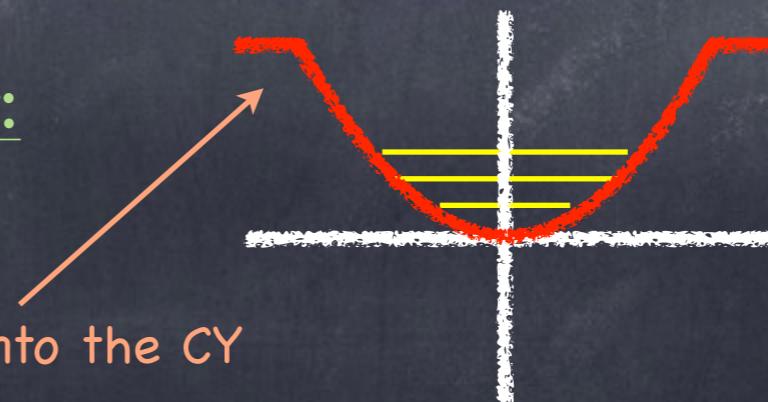
Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

$$(\mathcal{D} + 2\lambda \cos(x)) \Psi(x) = E\Psi(x)$$

$E(\Delta(N))$	$\hbar$	$E(\Delta_p(0) + \mathcal{O}(\hbar^4))$	$E(\Delta_p(0) + \mathcal{O}(\hbar^6))$	$E_0$
	0.10	4.1012552083	4.1012553358	4.1012553359
	0.25	4.2578938802	4.2578987002	4.2578987246
	0.50	4.5319010416	4.5319739024	4.5319753251
	0.75	4.8225097656	4.8228570580	4.8228719839
	1.00	5.1302083333	5.1312377929	5.1313156016
	1.25	5.4554850260	5.4578319589	5.4581090443
	1.50	5.7988281250	5.8033496856	5.8041260743

Intuitively clear:

Cutoff due to embedding into the CY



Bound state solutions

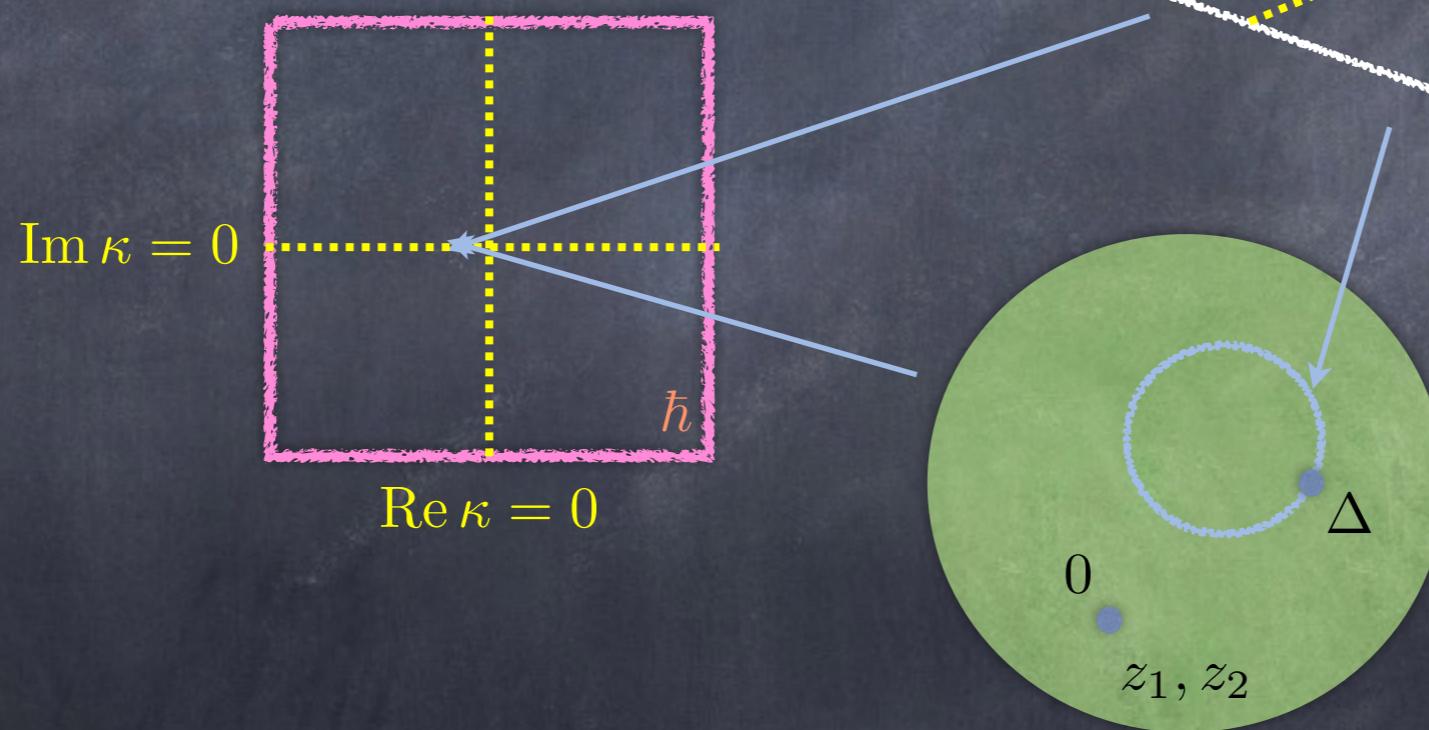


# Quantum Geometry

[Part III: D.K. '16]

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

Similar as for the pure deformed conifold, we have phase transitions over the extended moduli space



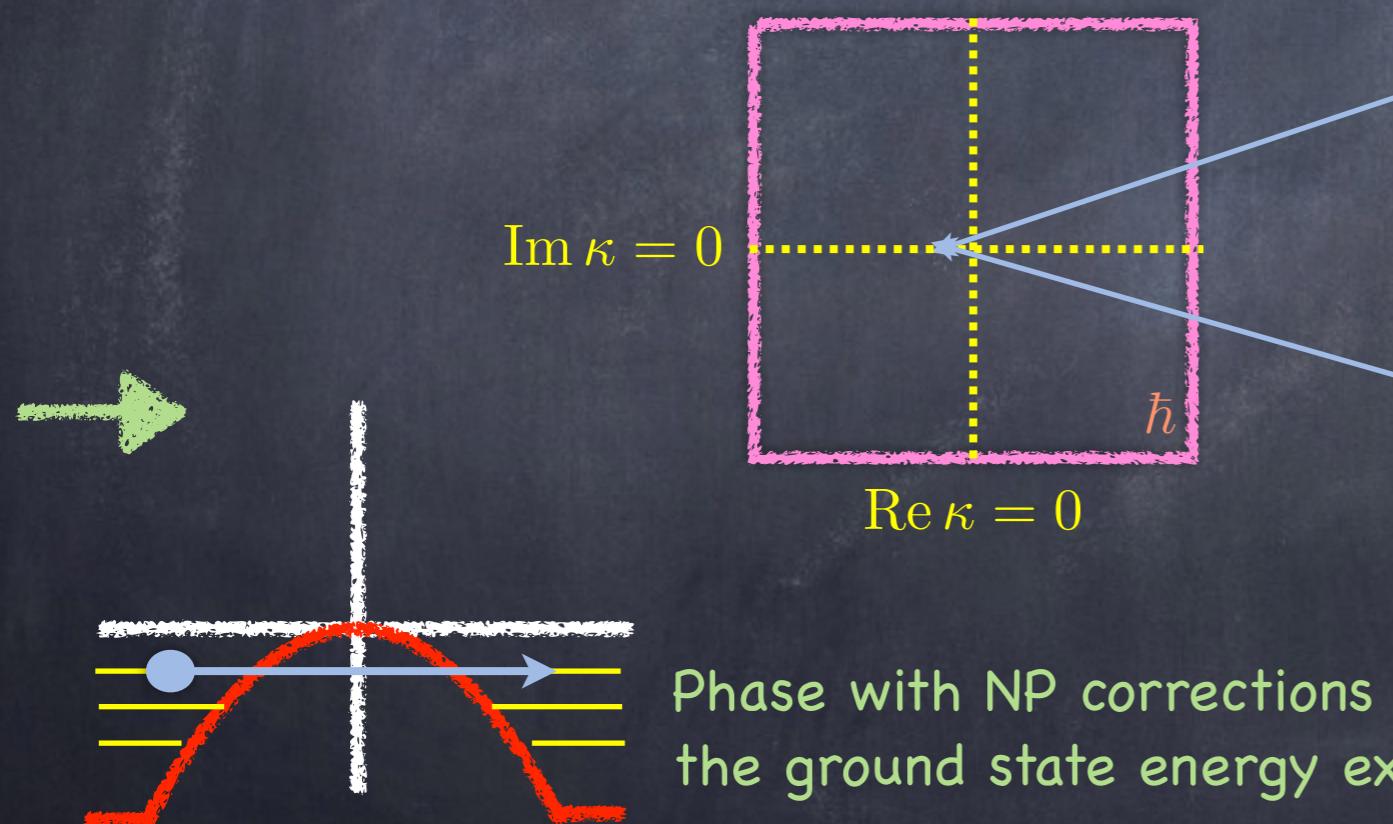


# Quantum Geometry

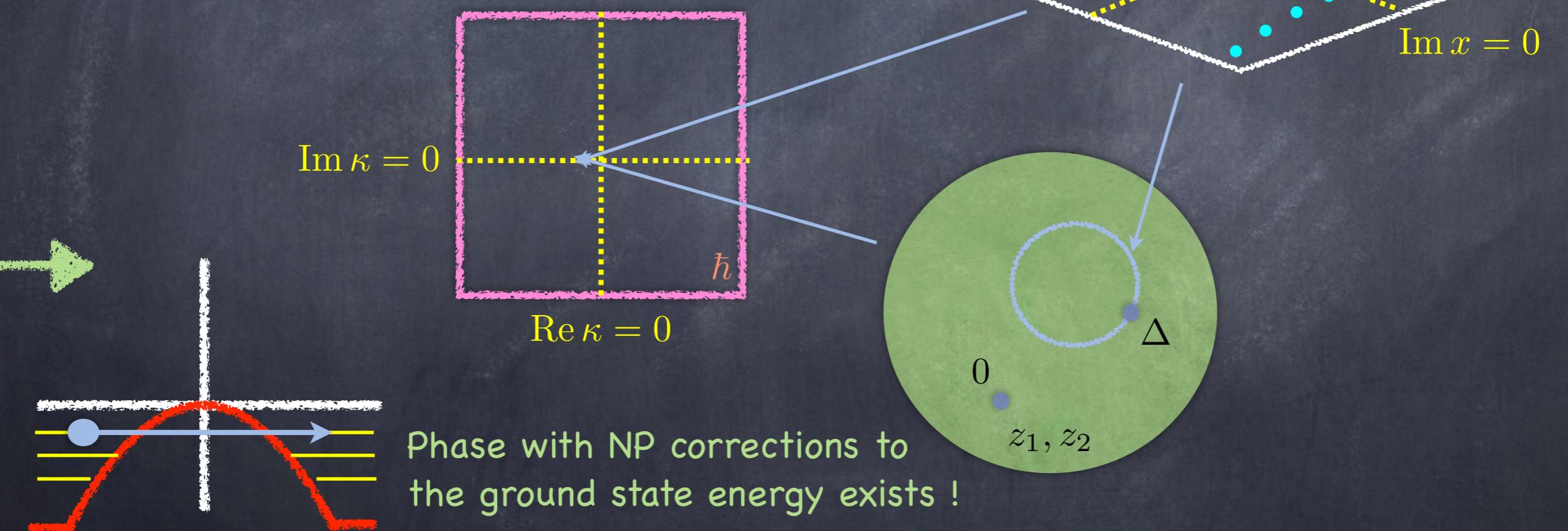
[Part III: D.K. '16]

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

Similar as for the pure deformed conifold, we have phase transitions over the extended moduli space



Phase with NP corrections to the ground state energy exists !



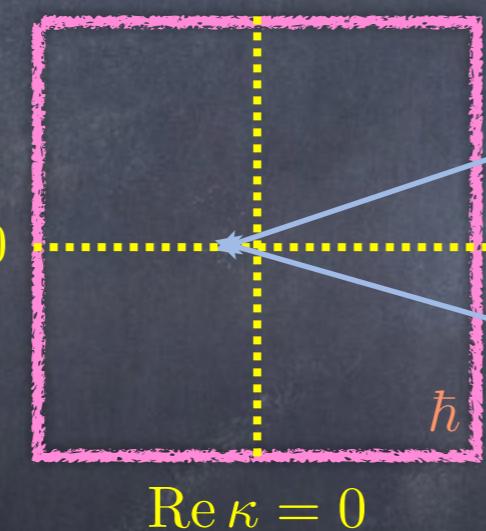
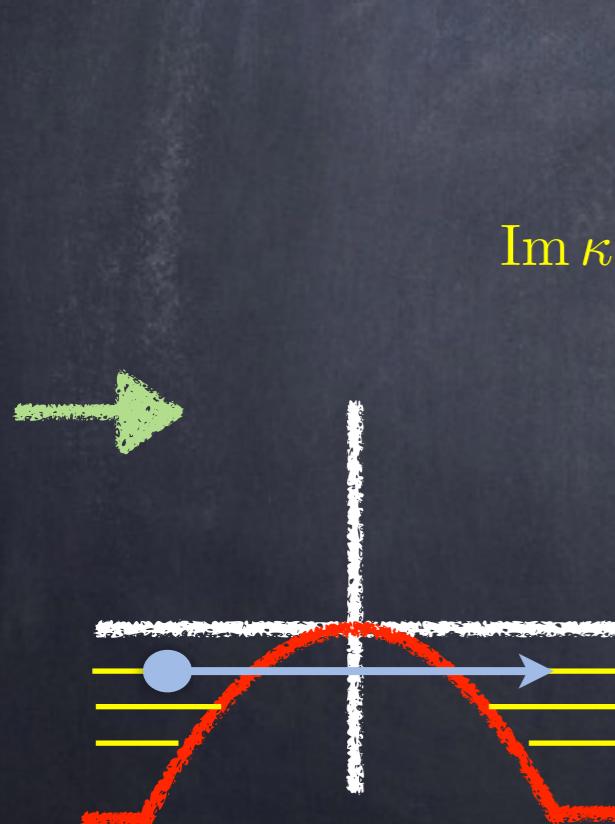


# Quantum Geometry

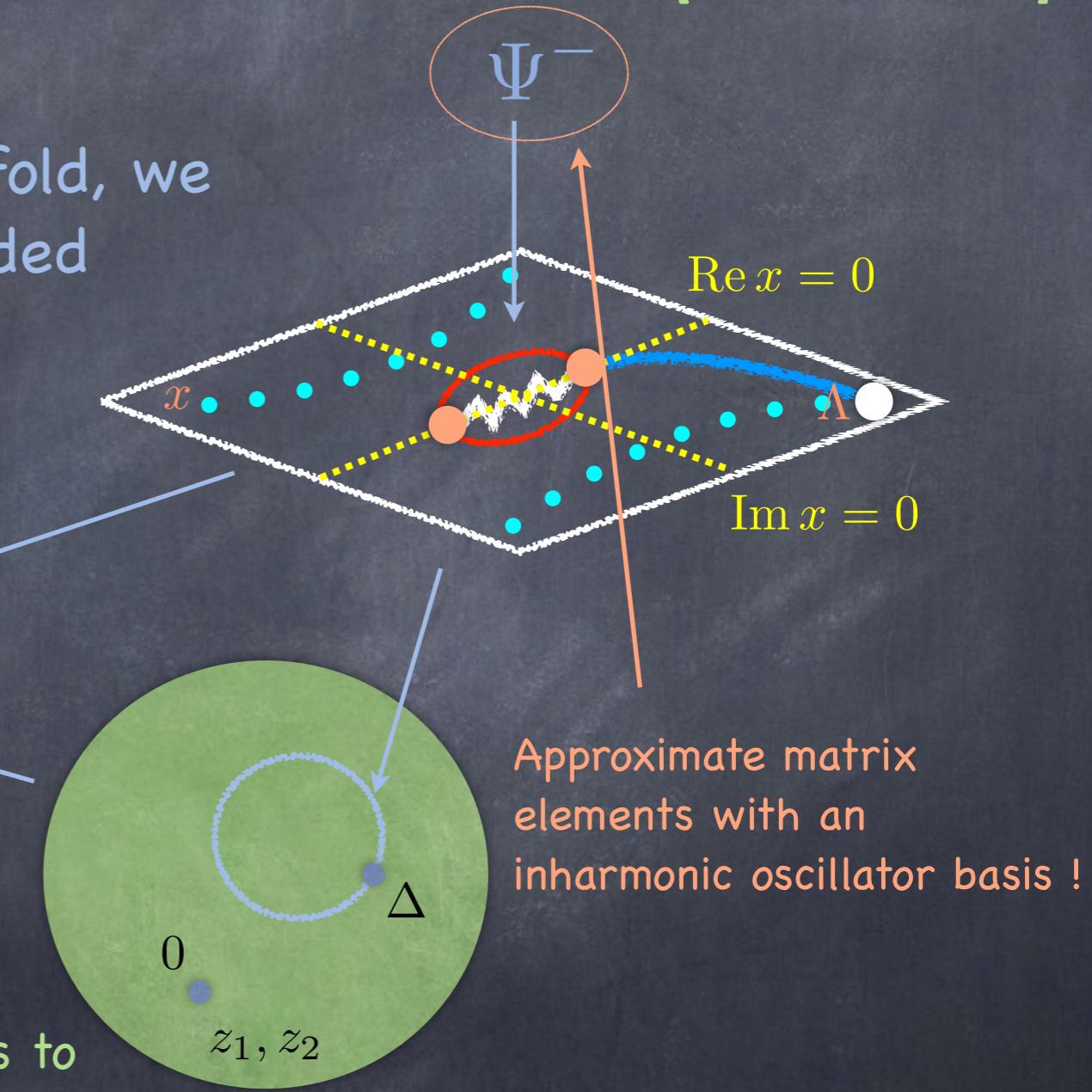
[Part III: D.K. '16]

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

Similar as for the pure deformed conifold, we have phase transitions over the extended moduli space



Phase with NP corrections to the ground state energy exists !



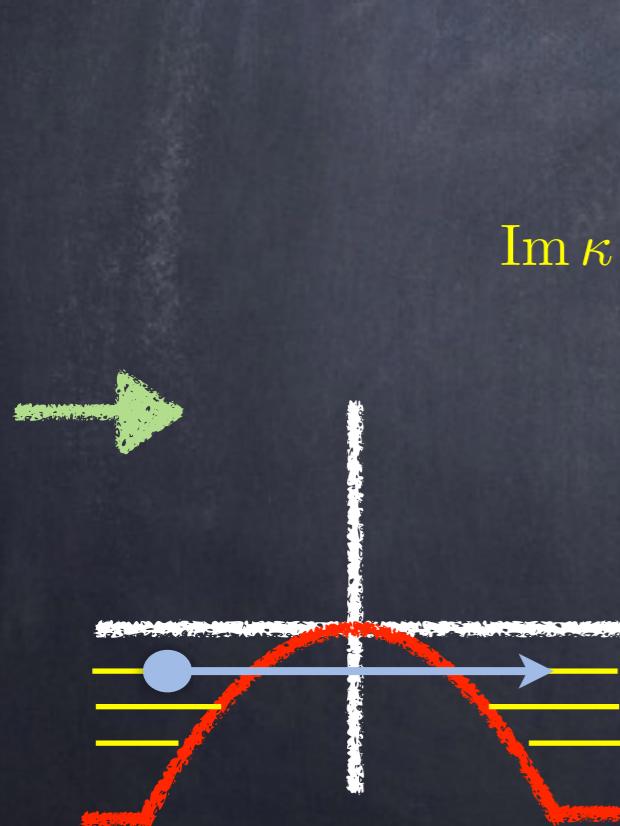


# Quantum Geometry

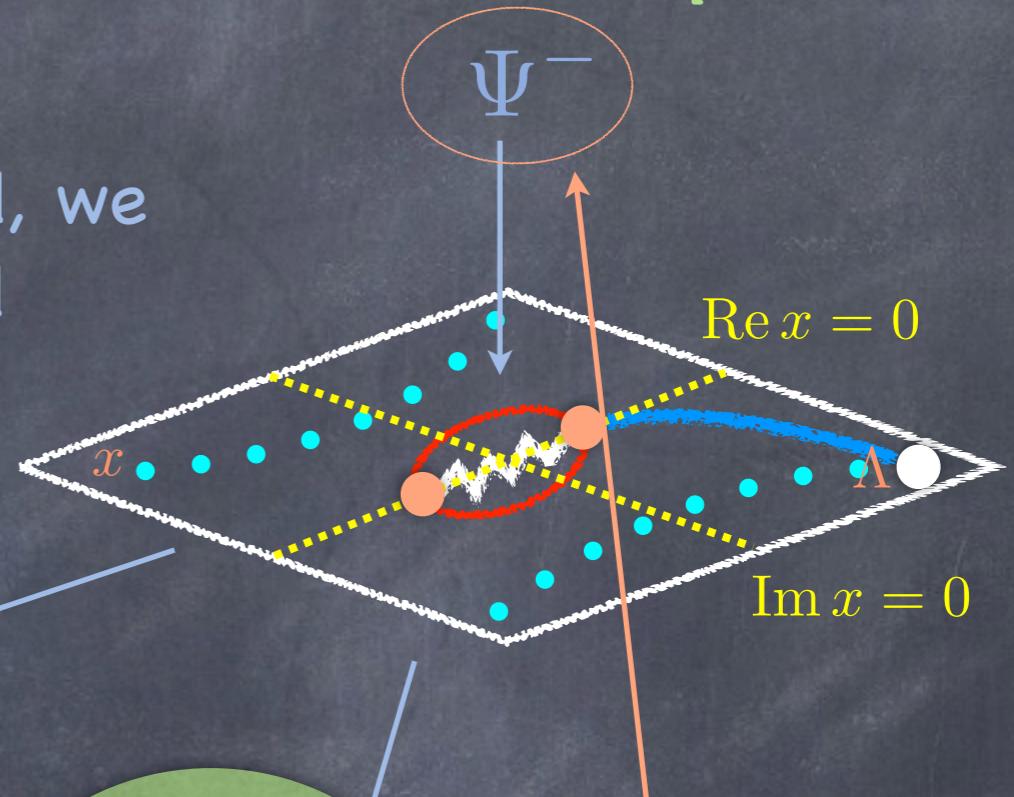
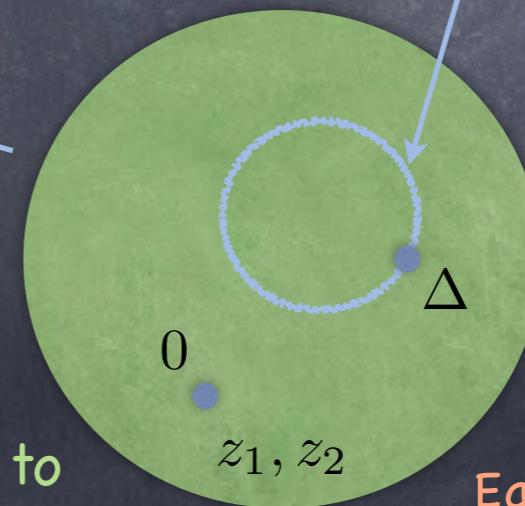
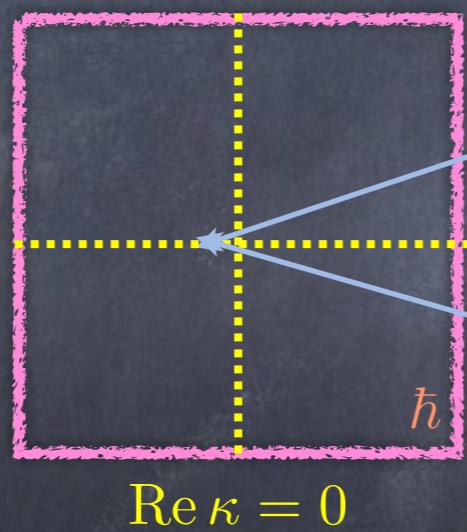
[Part III: D.K. '16]

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

Similar as for the pure deformed conifold, we have phase transitions over the extended moduli space

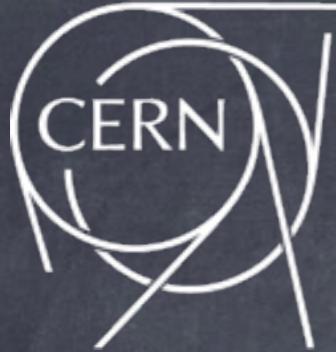


Phase with NP corrections to the ground state energy exists !



Approximate matrix elements with an inharmonic oscillator basis !

Easily achievable via suitable parameterization of  $\Sigma$



# Quantum Geometry

[Part III: D.K. '16]

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

$$(\mathcal{D} + 2\lambda \cos(x)) \Psi(x) = E\Psi(x)$$

$$E(\Delta(N))$$



# Quantum Geometry

[Part III: D.K. '16]

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

$$(\mathcal{D} + 2\lambda \cos(x)) \Psi(x) = E\Psi(x)$$

$$E(\Delta(N))$$



As we sit now in a different point in moduli space, different coordinate as before. But can be obtained in a similar fashion

$$E = \Delta$$

$$\Delta(N) = -(2N+1)\hbar + \frac{1 + 3N + 3N^2 + 2N^3}{48}\hbar^3 + \mathcal{O}(\hbar^5)$$



# Quantum Geometry

[Part III: D.K. '16]

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

$$(\mathcal{D} + 2\lambda \cos(x)) \Psi(x) = E\Psi(x)$$

$$E = \Delta$$

$\hbar$	$E(\Delta_p(0) + \mathcal{O}(\hbar^6))$	$E_0^\pm$	$\sim \Delta E_{10}$
0.25	<u>0.2496740341</u>	<u>0.24967403015559</u> <u>0.24967403015560</u>	$5 \times 10^{-15}$
0.50	<u>0.4973815917</u>	<u>0.49738099327675</u> <u>0.49738103560227</u>	$4 \times 10^{-8}$
0.75	<u>0.7411027908</u>	<u>0.74104438692828</u> <u>0.74107438072115</u>	$3 \times 10^{-5}$
1.00	<u>0.9787109375</u>	<u>0.97721595001824</u> <u>0.97766378718908</u>	$5 \times 10^{-4}$
1.25	<u>1.2079191207</u>	<u>1.19771240151976</u> <u>1.20066237802411</u>	$3 \times 10^{-3}$
1.50	<u>1.4262268066</u>	<u>1.39064094467481</u> <u>1.40031948070460</u>	$1 \times 10^{-2}$



# Quantum Geometry

[Part III: D.K. '16]

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

$$(\mathcal{D} + 2\lambda \cos(x)) \Psi(x) = E\Psi(x)$$

$$E = \Delta$$

$\hbar$	$E(\Delta_p(0) + \mathcal{O}(\hbar^6))$	$E_0^\pm$	$\sim \Delta E_{10}$
0.25	0.2496740341	0.24967403015559 0.24967403015560	$5 \times 10^{-15}$
0.50	0.4973815917	0.49738099327675 0.49738103560227	$4 \times 10^{-8}$
0.75	0.7411027908	0.74104438692828 0.74107438072115	$3 \times 10^{-5}$
1.00	0.9787109375	0.97721595001824 0.97766378718908	$5 \times 10^{-4}$
1.25	1.2079191207	1.19771240151976 1.20066237802411	$3 \times 10^{-3}$
1.50	1.4262268066	1.39064094467481 1.40031948070460	$1 \times 10^{-2}$

WKB ground state including  
up to order  $\hbar^6$  terms



# Quantum Geometry

[Part III: D.K. '16]

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

$$(\mathcal{D} + 2\lambda \cos(x)) \Psi(x) = E\Psi(x)$$

$$E = \Delta$$

$\hbar$	$E(\Delta_p(0) + \mathcal{O}(\hbar^6))$	$E_0^\pm$	$\sim \Delta E_{10}$
0.25	0.2496740341	0.24967403015559 0.24967403015560	$5 \times 10^{-15}$
0.50	0.4973815917	0.49738099327675 0.49738103560227	$4 \times 10^{-8}$
0.75	0.7411027908	0.74104438692828 0.74107438072115	$3 \times 10^{-5}$
1.00	0.9787109375	0.97721595001824 0.97766378718908	$5 \times 10^{-4}$
1.25	1.2079191207	1.19771240151976 1.20066237802411	$3 \times 10^{-3}$
1.50	1.4262268066	1.39064094467481 1.40031948070460	$1 \times 10^{-2}$

First two numerical energy levels



# Quantum Geometry

[Part III: D.K. '16]

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

$$(\mathcal{D} + 2\lambda \cos(x)) \Psi(x) = E\Psi(x)$$

$$E = \Delta$$

$\hbar$	$E(\Delta_p(0) + \mathcal{O}(\hbar^6))$	$E_0^\pm$	$\sim \Delta E_{10}$
0.25	0.2496740341	0.24967403015559 0.24967403015560	$5 \times 10^{-15}$
0.50	0.4973815917	0.49738099327675 0.49738103560227	$4 \times 10^{-8}$
0.75	0.7411027908	0.74104438692828 0.74107438072115	$3 \times 10^{-5}$
1.00	0.9787109375	0.97721595001824 0.97766378718908	$5 \times 10^{-4}$
1.25	1.2079191207	1.19771240151976 1.20066237802411	$3 \times 10^{-3}$
1.50	1.4262268066	1.39064094467481 1.40031948070460	$1 \times 10^{-2}$

Bandwidth



# Quantum Geometry

[Part III: D.K. '16]

Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

$$(\mathcal{D} + 2\lambda \cos(x)) \Psi(x) = E\Psi(x)$$

$$E = \Delta$$

$$\downarrow \\ \Delta^\pm$$

$\hbar$	$E(\Delta_p(0) + \mathcal{O}(\hbar^6))$	$E_0^\pm$	$\sim \Delta E_{10}$
0.25	<u>0.2496740341</u>	<u>0.24967403015559</u> <u>0.24967403015560</u>	$5 \times 10^{-15}$
0.50	<u>0.4973815917</u>	<u>0.49738099327675</u> <u>0.49738103560227</u>	$4 \times 10^{-8}$
0.75	<u>0.7411027908</u>	<u>0.74104438692828</u> <u>0.74107438072115</u>	$3 \times 10^{-5}$
1.00	<u>0.9787109375</u>	<u>0.97721595001824</u> <u>0.97766378718908</u>	$5 \times 10^{-4}$
1.25	<u>1.2079191207</u>	<u>1.19771240151976</u> <u>1.20066237802411</u>	$3 \times 10^{-3}$
1.50	<u>1.4262268066</u>	<u>1.39064094467481</u> <u>1.40031948070460</u>	$1 \times 10^{-2}$





# Quantum Geometry

[Part III: D.K. '16]

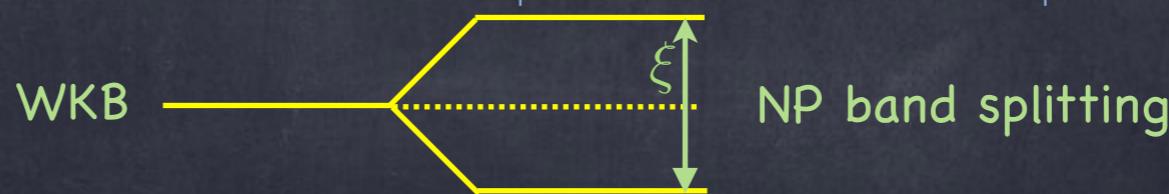
Example: local  $\mathbb{P}^1 \times \mathbb{P}^1$

$$(\mathcal{D} + 2\lambda \cos(x)) \Psi(x) = E\Psi(x)$$

$$E = \Delta$$

$\hbar$	$E(\Delta_p(0) + \mathcal{O}(\hbar^6))$	$E_0^\pm$	$\sim \Delta E_{10}$
0.25	<u>0.2496740341</u>	<u>0.24967403015559</u> <u>0.24967403015560</u>	$5 \times 10^{-15}$
0.50	<u>0.4973815917</u>	<u>0.49738099327675</u> <u>0.49738103560227</u>	$4 \times 10^{-8}$
0.75	<u>0.7411027908</u>	<u>0.74104438692828</u> <u>0.74107438072115</u>	$3 \times 10^{-5}$
1.00	<u>0.9787109375</u>	<u>0.97721595001824</u> <u>0.97766378718908</u>	$5 \times 10^{-4}$
1.25	<u>1.2079191207</u>	<u>1.19771240151976</u> <u>1.20066237802411</u>	$3 \times 10^{-3}$
1.50	<u>1.4262268066</u>	<u>1.39064094467481</u> <u>1.40031948070460</u>	$1 \times 10^{-2}$

Non-perturbatively  
corrected mirror map





# Quantum Geometry

[Part III: D.K. '16]

## Other examples:

Other toric Calabi-Yaus can be studied similarly, for instance local  $\mathbb{P}^2$



# Conclusion



# Quantum Geometry

[Part III: D.K. '16]

## Conclusion:

- ★ Presented simple and elegant framework based on quantum geometry to describe exactly the NS limit of physical theories, analytically



# Quantum Geometry

[Part III: D.K. '16]

## Conclusion:

- ★ Presented simple and elegant framework based on quantum geometry to describe exactly the NS limit of physical theories, analytically
- ★ Though we focused here on the conifold point, the results can be extended all over moduli space  
[preliminary results are given in Part II: D.K. '14 and in Basar+Dunne '15]



# Quantum Geometry

[Part III: D.K. '16]

## Conclusion:

- ★ Presented simple and elegant framework based on quantum geometry to describe exactly the NS limit of physical theories, analytically
- ★ Though we focused here on the conifold point, the results can be extended all over moduli space  
[preliminary results are given in Part II: D.K. '14 and in Basar+Dunne '15]
- ★ Stokes transitions over the extended moduli space are highly relevant



# Quantum Geometry

[Part III: D.K. '16]

## Conclusion:

- ★ Presented simple and elegant framework based on quantum geometry to describe exactly the NS limit of physical theories, analytically
- ★ Though we focused here on the conifold point, the results can be extended all over moduli space  
[preliminary results are given in Part II: D.K. '14 and in Basar+Dunne '15]
- ★ Stokes transitions over the extended moduli space are highly relevant
- ★ Between the lines I presented >3 possible non-trivial spin-off projects on a silver tablet



... Thank you ...