Fundamental experimental objects

$\Gamma(a_1 \rightarrow b_1 b_2 \ldots b_n)$ **Decay width = 1/lifetime**

$\sigma(a_1 a_2 \rightarrow b_1 b_2 \ldots b_n)$ **Cross section**

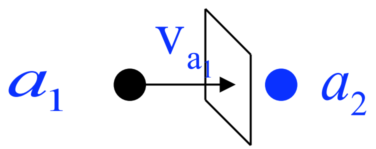
$$\Pi_{i=1}^{n} \frac{V d^3 p}{(2\pi)^3}$$

$$\text{Cross section} = \frac{\text{Transition rate x Number of final states}}{\text{Initial flux}}$$

$$\frac{|v_{a_1}|}{V} \times \frac{1}{V}$$

particles passing through unit area in unit time

target particles per unit volume



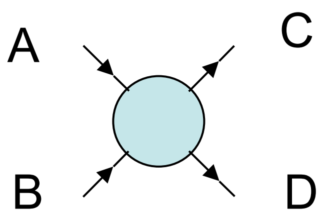
(Lab frame)

The transition rate

$$T_{fi} = -\int d^4x\, \phi_f^*(x) V(x) \phi_i(x) + \ldots$$

$$\phi_{i,f} \rightarrow f_p^{\pm} = e^{(-,+)ip.x} \frac{1}{\sqrt{2p^0 V}} \equiv \frac{N}{\sqrt{V}} e^{(-,+)ip.x}$$

e.g.



Transition rate per unit volume

$$W_{fi} = \frac{|T_{fi}|^2}{TV}$$

$$\phi_{f,i} = e^{(-,+)ip.x}$$

$$T_{fi} = -\frac{N_A N_B N_C N_D}{V^2} (2\pi)^4 \delta^4(p_C + p_D - p_A - p_B) \mathfrak{M}_{fi}$$

$$W_{fi} = (2\pi)^4 \frac{\delta^4(p_C + p_D - p_A - p_B) |\mathfrak{M}|^2}{V^4} \left(\frac{1}{2E_A}\right)\left(\frac{1}{2E_B}\right)\left(\frac{1}{2E_C}\right)\left(\frac{1}{2E_D}\right)$$

The cross section

Cross section = $\dfrac{\text{Transition rate} \times \text{Number of final states}}{\text{Initial flux}}$

$$d\sigma = \frac{V^2}{|\mathbf{v_A}|2E_A 2E_B}\frac{1}{V^4}|\mathfrak{M}|^2\frac{(2\pi)^4}{(2\pi)^6}\delta^4(p_C+p_D-p_A-p_B)\frac{d^3p_C}{2E_C}\frac{d^3p_D}{2E_D}V^2$$

$$d\sigma = \frac{|\mathfrak{M}|^2}{F}dQ$$

$$dQ = (2\pi)^4\delta^4(p_C+p_D-p_A-p_B)\frac{d^3p_C}{(2\pi)^3 2E_C}\frac{d^3p_D}{(2\pi)^3 2E_D}$$

Lorentz Invariant Phase space

$$F = |\mathbf{v_A}|2E_A 2E_B$$
$$= 4((p_A.p_B)^2 - m_A^2 m_B^2)^{1/2}$$

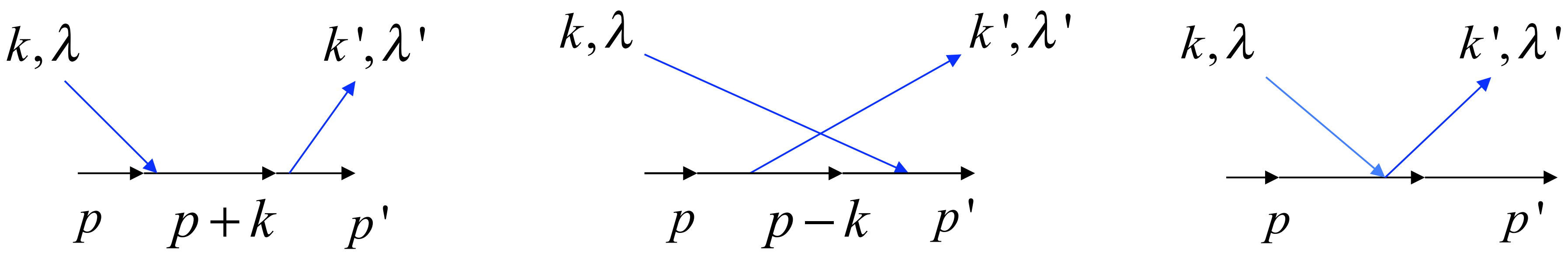
The decay rate

$$d\Gamma = \frac{1}{2E_A} |\mathfrak{M}|^2 \, dQ$$

$$dQ = (2\pi)^4 \delta^4 (p_A - p_{B_1} \ldots - p_{B_n}) \frac{d^3 p_{B_1}}{(2\pi)^3 2E_{B_1}} \cdots \frac{d^3 p_{B_n}}{(2\pi)^3 2E_{B_n}}$$

Compton scattering of a π meson

$$\gamma\pi \to \gamma\pi$$

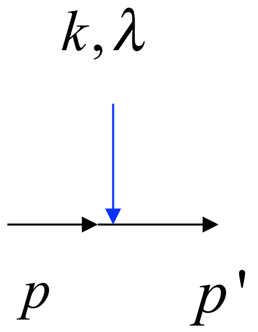


Feynman rules

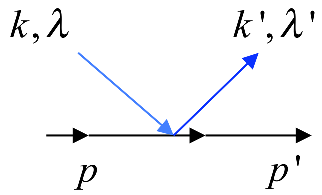
Klein Gordon

$$(\partial_\mu \partial^\mu + m^2)\psi = -V\psi$$

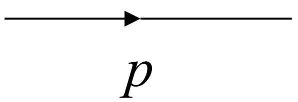
$$V = -ie(\partial_\mu A^\mu + A^\mu \partial_\mu) - e^2 A^2$$



$$-ie(p_\lambda + p'_\lambda)$$



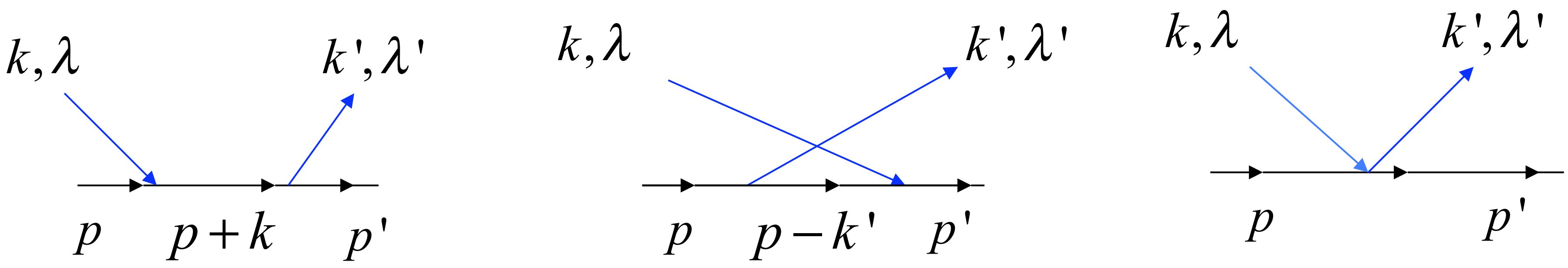
$$ie^2$$



$$\frac{i}{p^2 - m^2}$$

External photon ε^λ

Compton scattering of a π meson

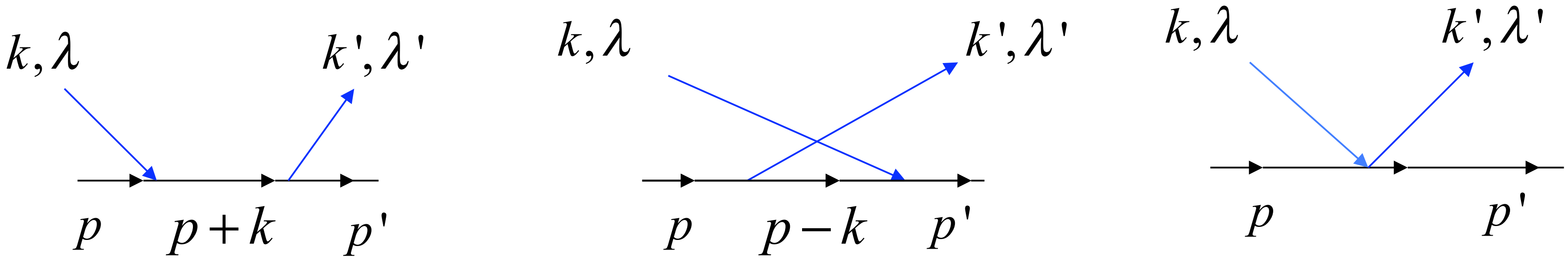


$$i\mathfrak{M}_{fi} = (-ie)^2[\varepsilon.(2p+k)\frac{i}{(p+k)^2-m^2}\varepsilon'.(2p'+k')$$

$$+\varepsilon.(2p'-k)\frac{i}{(p-k')^2-m^2}\varepsilon'.(2p-k')-2i\varepsilon.\varepsilon']$$

$$d\sigma = \frac{V^2}{|\mathbf{v_A}|2E_A 2E_B}\frac{1}{V^4}|\mathfrak{M}|^2\frac{(2\pi)^4}{(2\pi)^6}\delta^4(p_C+p_D-p_A-p_B)\frac{d^3p_C}{2E_C}\frac{d^3p_D}{2E_D}V^2$$

Compton scattering of a π meson



$$\frac{1}{\mathfrak{e}^2}\mathfrak{M}_{fi} = \varepsilon.(2p+k)\frac{i}{(p+k)^2-m^2}\varepsilon'.(2p'+k') + \varepsilon.(2p'-k')\frac{i}{(p'-k')^2-m^2}\varepsilon'.(2p'-k') - 2i\varepsilon.\varepsilon'$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \frac{\alpha^2}{m^2}\frac{(\varepsilon.\varepsilon')^2}{\left[1+\frac{k}{m}(1-\cos\theta)\right]^2}$$

Transverse polarisation $\varepsilon.p = \varepsilon'.p = 0$

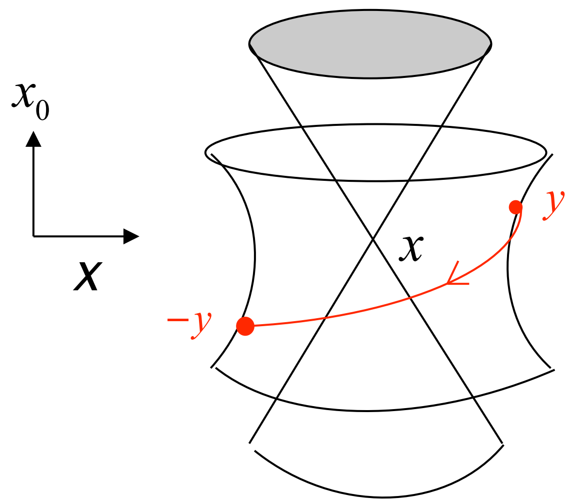
$$\sigma_{total}\big|_{k=0} = \int\frac{d\sigma}{d\Omega}.d\Omega = \frac{8\pi\alpha^2}{3m_\pi^2} \simeq 8.10^{-2}\,GeV^{-2} = 3.10^{-2}\,mb$$

$$\sigma_{total}\big|_{k/m>>1} \simeq \frac{2\pi\alpha^2}{mk}$$

Causality?

QM : $U(x'-x) \propto e^{-m\sqrt{(\mathbf{x}'-\mathbf{x})^2-(t'-t)^2}}$ ✗

See Peskin & Schroeder
"Quantum Field Theory" p28



Field theory :

$$\Delta_F(x'-x) = -i\int \frac{d^3p}{(2\pi)^3 2\omega_p} e^{-i\omega_p|t'-t|-i\mathbf{p}.(\mathbf{x}'-\mathbf{x})}$$

$$= D(x-y) - D(y-x)$$

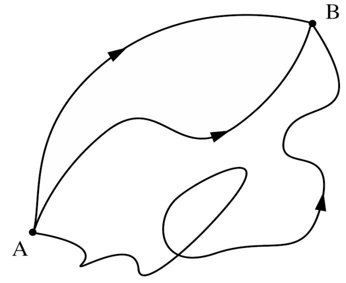
When $(x-y)^2 < 0$, we can perform a Lorentz transformation taking $(x-y) \rightarrow -(x-y)$
...causality preserved $(e^{-m|r|} - e^{-m|r|})$

No (continuous) transformation possible for $(x\text{-}y)^2 > 0$
...and amplitude nonvanishing $(e^{-imt} - e^{imt})$

Construction of a relativistic field theory

Lagrangian $$L = T - V$$ (Nonrelativistic mechanics)

Action $$S = \int_{t_1}^{t_2} L \, dt$$



- Classical path … minimises action

- Quantum mechanics … sum over all paths with amplitude $\propto e^{iS/\hbar}$

Lagrangian invariant under all the symmetries of nature

-makes it easy to construct viable theories

Lagrangian formulation of the Klein Gordon equation

$$L = \int \mathfrak{L}\, d^3x, \quad \mathfrak{L} \quad \text{lagrangian density}$$

Klein Gordon field $\phi(x)$

$$\mathfrak{L} = \underbrace{\left(\partial_\mu \phi(x)\right)^\dagger \partial^\mu \phi(x)}_{T} - \underbrace{m^2 \phi(x)^\dagger \phi(x)}_{V}$$

Manifestly Lorentz invariant

Lagrangian formulation of the Klein Gordon equation

$$L = \int \mathfrak{L}\, d^3x, \quad \mathfrak{L}$$ lagrangian density

Klein Gordon field $\phi(x)$

$$\mathfrak{L} = \left(\partial_\mu \phi(x)\right)^\dagger \partial^\mu \phi(x) - m^2 \phi(x)^\dagger \phi(x)$$

Manifestly Lorentz invariant

Classical path : $\dfrac{\delta S}{\delta \phi} = 0$

$$\delta\mathfrak{L} = \frac{\partial \mathfrak{L}}{\partial \phi}\delta\phi + \frac{\partial \mathfrak{L}}{\partial(\partial^\mu \phi)}\delta(\delta\partial^\mu \phi)$$

$$= \left[\frac{\partial \mathfrak{L}}{\partial \phi} - \partial^\mu\left(\frac{\partial \mathfrak{L}}{\partial(\partial^\mu \phi)}\right)\right]\delta\phi + \partial^\mu\left(\frac{\partial \mathfrak{L}}{\partial(\partial^\mu \phi)}\delta\phi\right)$$

surface integral in S..vanishes

$$\delta S = 0 \quad \Rightarrow \quad \frac{\partial \mathfrak{L}}{\partial \phi} - \partial^\mu \frac{\partial \mathfrak{L}}{\partial(\partial^\mu \phi)} = 0$$

Euler Lagrange equation

$$\Rightarrow \quad (\partial_\mu \partial^\mu + m^2)\phi = 0$$

Klein Gordon equation

New symmetries

$$\mathfrak{L} = \left(\partial_\mu \phi(x)\right)^\dagger \partial^\mu \phi(x) - m^2 \phi(x)^\dagger \phi(x)$$

Is invariant under $\phi(x) \rightarrow e^{i\alpha}\phi(x)$ …an Abelian (U(1)) gauge symmetry

A symmetry implies a conserved current and charge.

e.g. Translation ⇒ Momentum conservation

Rotation ⇒ Angular momentum conservation

What conservation law does the U(1) invariance imply?

Noether current

$$\mathfrak{L} = \left(\partial_\mu \phi(x)\right)^\dagger \partial^\mu \phi(x) - m^2 \phi(x)^\dagger \phi(x)$$

Is invariant under $\phi(x) \rightarrow e^{i\alpha}\phi(x)$ …an Abelian (U(1)) gauge symmetry

$i\alpha\phi$ $\quad$ $i\alpha\partial_\mu\phi$

$$0 = \delta\mathfrak{L} = \frac{\partial\mathfrak{L}}{\partial\phi}\delta\phi + \frac{\partial\mathfrak{L}}{\partial(\partial^\mu\phi)}\delta(\partial^\mu\phi) + (\phi \leftrightarrow \phi^\dagger)$$

0 (Euler lagrange eqs.)

$$= i\alpha\left[\frac{\partial\mathfrak{L}}{\partial\phi} - \partial^\mu\left(\frac{\partial\mathfrak{L}}{\partial(\partial^\mu\phi)}\right)\right]\phi + i\alpha\partial^\mu\left(\frac{\partial\mathfrak{L}}{\partial(\partial^\mu\phi)}\phi\right) - (\phi \leftrightarrow \phi^\dagger)$$

$$\Rightarrow \quad \partial^\mu j_\mu = 0, \quad j_\mu = \frac{ie}{2}\left(\frac{\partial\mathfrak{L}}{\partial(\partial^\mu\phi)}\phi - \frac{\partial\mathfrak{L}}{\partial(\partial^\mu\phi^\dagger)}\phi^\dagger\right)$$

Noether current

The Klein Gordon current

$$\mathfrak{L} = \left(\partial_\mu \phi(x)\right)^\dagger \partial^\mu \phi(x) - m^2 \phi(x)^\dagger \phi(x)$$

Is invariant under $\psi(x) \to e^{i\alpha}\psi(x)$ …an Abelian (U(1)) gauge symmetry

$$\partial^\mu j_\mu = 0, \quad j_\mu = \frac{ie}{2}\left(\frac{\partial \mathfrak{L}}{\partial(\partial^\mu \phi)}\phi - \frac{\partial \mathfrak{L}}{\partial(\partial^\mu \phi^\dagger)}\phi^\dagger\right)$$

$$j_\mu^{KG} = -ie\left(\phi^* \partial_\mu \phi - \phi\, \partial_\mu \phi^*\right)$$

This is of the form of the electromagnetic current we used for the KG field

The Klein Gordon current

$$\mathfrak{L} = \left(\partial_\mu \phi(x)\right)^\dagger \partial^\mu \phi(x) - m^2 \phi(x)^\dagger \phi(x)$$

Is invariant under $\phi(x) \rightarrow e^{i\alpha}\phi(x)$ …an Abelian (U(1)) gauge symmetry

$$\partial^\mu j_\mu = 0, \quad j_\mu = \frac{ie}{2}\left(\frac{\partial \mathfrak{L}}{\partial(\partial^\mu \phi)}\phi - \frac{\partial \mathfrak{L}}{\partial(\partial^\mu \phi^\dagger)}\phi^\dagger \right)$$

$$j_\mu^{KG} = -ie\left(\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*\right)$$

This is of the form of the electromagnetic current we used for the KG field

$Q = \int d^3x \; j^0$ is the associated conserved charge

Suppose we have two fields with different U(1) charges :

$$\phi_{1,2}(x) \rightarrow e^{i\alpha Q_{1,2}} \phi_{1,2}(x)$$

$$\mathfrak{L} = \left(\partial_\mu \phi_1(x)\right)^\dagger \partial^\mu \phi_1(x) - m^2 \phi_1(x)^\dagger \phi_1(x)$$
$$+ \left(\partial_\mu \phi_2(x)\right)^\dagger \partial^\mu \phi_2(x) - m^2 \phi_2(x)^\dagger \phi_2(x)$$

..no cross terms possible (corresponding to charge conservation)

Additional terms

Terms allowed by U(1) symmetry

$$\mathfrak{L} = \left(\partial_\mu \phi(x)\right)^\dagger \partial^\mu \phi(x) - m^2 \phi(x)^\dagger \phi(x) + \lambda \left|\phi\right|^4 + \frac{\lambda'}{M^2}\left|\phi\right|^6 + \dots$$

Renormalisable $D \leq 4$

If $M \gg 10^3 GeV$, "Effective" Field theory approximately renormalisable

U(1) local gauge invariance and QED

$$\phi(x) \to e^{i\alpha(x)Q}\phi(x)$$

$$\mathfrak{L} = \left(\partial_\mu \phi(x)\right)^\dagger \partial^\mu \phi(x) - m^2 \phi(x)^\dagger \phi(x)$$ not invariant due to derivatives

$$\partial_\mu \phi \to \partial_\mu e^{i\alpha(x)Q}\phi = e^{i\alpha(x)Q}\partial_\mu \phi + iQe^{i\alpha(x)Q}\phi\, \partial_\mu \alpha(x)$$

To obtain invariant Lagrangian look for a modified derivative transforming covariantly

$$D_\mu \phi \to e^{i\alpha(x)Q} D_\mu \phi$$

U(1) local gauge invariance and QED

$$\phi(x) \to e^{i\alpha(x)Q}\phi(x)$$

$$\mathfrak{L} = \left(\partial_\mu \phi(x)\right)^\dagger \partial^\mu \phi(x) - m^2 \phi(x)^\dagger \phi(x)$$ not invariant due to derivatives

$$\partial_\mu \phi - iQA_\mu \phi \to \partial_\mu e^{i\alpha(x)Q}\phi = e^{i\alpha(x)Q}(\partial_\mu \phi - iQA_\mu \phi) + iQe^{i\alpha(x)Q}\phi\,\partial_\mu \alpha(x) - iQe^{i\alpha(x)Q}\phi\,\partial_\mu \alpha(x)$$

To obtain invariant Lagrangian look for a modified derivative transforming covariantly

$$D_\mu \phi \to e^{i\alpha(x)Q} D_\mu \phi$$

Need to introduce a new *vector* field $A_\mu \to A_\mu + \partial_\mu \alpha$

$$D_\mu = \partial_\mu - iQA_\mu$$

$$\phi(x) \to e^{iQ\alpha(x)}\phi(x)$$

$$D_\mu\phi \to e^{i\alpha(x)Q} D_\mu\phi$$

$$A_\mu \to A_\mu + \partial_\mu\alpha$$

$$\mathfrak{L} = \left(D_\mu\phi(x)\right)^\dagger D^\mu\phi(x) - m^2\phi(x)^\dagger\phi(x)$$ is invariant under local U(1)

Note : $\partial_\mu \to D_\mu = \partial_\mu - iQA_\mu$ is equivalent to $p^\mu \to p^\mu + eA^\mu$

universal coupling of electromagnetism *follows* from local gauge invariance

The Euler lagrange equation give the KG equation:

$$(\partial_\mu\partial^\mu + m^2)\psi = -V\psi \quad where \quad V = -ie(\partial_\mu A^\mu + A^\mu\partial_\mu) - e^2A^2$$

$$\phi(x) \to e^{iQ\alpha(x)}\phi(x)$$

$$D_\mu\phi \to e^{i\alpha(x)Q} D_\mu\phi$$

$$A_\mu \to A_\mu + \partial_\mu\alpha$$

$$\mathfrak{L} = \left(D_\mu\phi(x)\right)^\dagger D^\mu\phi(x) - m^2\phi(x)^\dagger\phi(x)$$ is invariant under local U(1)

Note : $\partial_\mu \to D_\mu = \partial_\mu - iQA_\mu$ is equivalent to $p^\mu \to p^\mu + eA^\mu$

universal coupling of electromagnetism *follows* from local gauge invariance

i.e. $\mathfrak{L} = \mathfrak{L}^{\mathrm{KG}} = \left(\partial_\mu\phi(x)\right)^\dagger \partial^\mu\phi(x) - m^2\phi(x)^\dagger\phi(x) - j_\mu^{KG} A^\mu + O(e^2)$

The electromagnetic Lagrangian

$$\begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$F_{\mu\nu} \to F_{\mu\nu}, \qquad A_\mu \to A_\mu + \partial_\mu \alpha$$

$$\mathfrak{L}^{EM} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu$$

$M^2 A^\mu A_\mu$ Forbidden by gauge invariance

The Euler-Lagrange equations give Maxwell equations !

$$\frac{\partial \mathfrak{L}}{\partial A^\nu} - \partial^\mu \frac{\partial \mathfrak{L}}{\partial(\partial^\mu A^\nu)} = 0$$

$$\partial_\mu F^{\mu\nu} = j^\nu$$

$\left(\text{N.B. } \varepsilon_{\mu\nu\rho\sigma} \partial^\mu F^{\rho\sigma} = 0\right)$

$$\equiv$$

$$\nabla.\mathbf{E} = \rho, \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla.\mathbf{B} = 0, \qquad \nabla \times \mathbf{B} - \frac{\partial E}{\partial t} = \mathbf{j}$$

EM dynamics follows from a local gauge symmetry!!

The photon propagator

- The propagators determined by terms quadratic in the fields, using the Euler Lagrange equations.

$$\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \partial^\nu (\partial^\mu A_\mu) = j^\nu$$

The Klein Gordon propagator (reminder)

$$(\partial_\mu \partial^\mu + m^2)\psi = -V\psi \quad where \quad V = -ie(\partial_\mu A^\mu + A^\mu \partial_\mu) - e^2 A^2$$

$$(\partial_\mu \partial^\mu + m^2)\Delta_F(x' - x) = \delta^4(x' - x)$$

In momentum space:

$$\Delta'_F(p) = \frac{i}{-p^2 + m^2 \pm i\varepsilon}$$

With normalisation convention used in Feynman rules = inverse of momentum space operator multiplied by -i

The photon propagator

- The propagators determined by terms quadratic in the fields, using the Euler Lagrange equations.

$$\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \partial^\nu(\partial^\mu A_\mu) = j^\nu$$

Choose as

$$-\frac{1}{\xi}\partial^\mu A_\mu$$

(gauge fixing)

Gauge ambiguity

$$A_\mu \to A_\mu + \partial_\mu \alpha \qquad \partial^\mu A_\mu \to \partial^\mu A_\mu + \partial^2 \alpha$$

i.e. with suitable "gauge" choice of α ("ξ" gauge) want to solve

$$\partial_\mu \partial^\mu A^\nu - (1-\frac{1}{\xi})\partial^\nu(\partial_\mu A^\mu) \equiv (g^{\nu\mu}\partial^2 - (1-\frac{1}{\xi})\partial^\nu \partial_\mu)A^\mu = j^\nu$$

In momentum space the photon propagator is

$$-i\left(g^{\mu\nu}p^2 - (1-\frac{1}{\xi})p^\mu p^\nu\right)^{-1} = \frac{i}{p^2}\left(-g_{\mu\nu} + (1-\xi)\frac{p_\mu p_\nu}{p^2}\right)$$

('t Hooft Feynman gauge ξ=1)