Supersymmetric partition functions and the 3d A-twist

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CERN
January 17, 2017

Based on 1701.03171
with H. Kim and B. Willett

Supersymmetry across dimensions

Supersymmetric QFT presents us with unique opportunities to probe the strong-coupling regime.

In particular, supersymmetry implies some wonderful simplification of certain path integrals. This is known as supersymmetric localization.

I would like to discuss supersymmetric QFT with four supercharges:

$$4\text{d}\ \mathcal{N}=1 \quad \to \quad \boxed{3\text{d}\ \mathcal{N}=2} \quad \to \quad 2\text{d}\ \mathcal{N}=(2,2)$$

We will consider gauge theories with a $U(1)_R$ symmetry.
They often flow to interesting *superconformal* theories in the IR.

Supersymmetry in three dimensions

In this talk, we will focus on three-dimensional theories with $\mathcal{N} = 2$ supersymmetry (four supercharges).

Due to the lack of local anomalies in 3d, we have fewer tools to characterize theories non-perturbatively. An important object, in any 3d CFT, is the quantity:

$$F_{S^3} = -\log |Z_{S^3}|$$

It plays the role of the central charge c in 2d (or a in 4d).

It can be computed exactly in any 3d $\mathcal{N} = 2$ SCFT that can be obtained in the IR of an $\mathcal{N} = 2$ gauge theory.

Supersymmetric partition functions

This Z_{S^3} is an example of a supersymmetric partition function.

It is independent on the Weyl factor of the metric, and therefore RG invariant. Thus we can compute it in the UV, and obtain F_{S^3} of the IR theory.

More generally, we would like to consider partition functions on a large class of three-manifolds:

$$\mathcal{M}_3 \mapsto Z_{\mathcal{M}_3}$$

We choose to preserve two supercharges of opposite R-charge on $\mathcal{M}_3$, such that:

$$\{\mathcal{Q}, \tilde{\mathcal{Q}}\} = \mathcal{L}_K \ ,$$

with K a real Killing vector. Then $\mathcal{M}_3$ is a Seifert manifold.

[C.C., Dumitrescu, Festuccia, Komargodski, 2012]

Supersymmetric partition functions

Note that:

- The partition function are supersymmetric—they are defined as the path integral of the theory on a susy-preserving geometric background. [Festuccia, Seiberg, 2011]
- In particular, fermions are periodic along any one-cycle. That is, supersymmetry dictates a choice of spin structure on $\mathcal{M}_3$.
- We also consider insertion of certain half-BPS Wilson loops:

$$\langle W_1 W_2 \cdots \rangle_{\mathcal{M}_3}$$

as well as other loop operators. They must be parallel to K, to preserve supersymmetry. Note that:

$$Z_{\mathcal{M}_3} = \langle 1 \rangle_{\mathcal{M}_3}$$

Supersymmetric partition functions

Examples:

- The S^3 partition function. Schematically:
  [Kapustin, Willett, Yaakov, 2010; Jafferis, 2010; Hama, Hosomichi, Lee, 2010]

$$Z_{S^3} = \int d\sigma \, e^{\pi i k \sigma^2} \, \mathcal{Z}^{\text{1-loop}}_{S^3}(\sigma)$$

  Here k is a CS level. The integral is over the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g} = \mathrm{Lie}(G)$.

- The twisted $S^2 \times S^1$ partition function of [Benini, Zaffaroni, 2015]:

$$Z_{S^2\times S^1} = \sum_{\mathfrak{m}\in\Gamma_{\mathbf{G}^\vee}} \oint_{\mathrm{JK}} \frac{du}{2\pi i} \, e^{2\pi i k u} \, \mathcal{Z}^{\text{1-loop}}_{S^2\times S^1}(u)$$

  with $u = i\sigma - a_0$. Sum over gauge fluxes.
  Contour integral (JK residue).

Supersymmetric partition functions

More generally, we study 3d $\mathcal{N} = 2$ theories on $\mathcal{M}_{g,p}$ a $U(1)$ principal bundle over a Riemann surface:

$$S^1 \longrightarrow \mathcal{M}_{g,p} \xrightarrow{\pi} \Sigma_g$$

Here p is the first Chern class of the bundle.

Note that:

$$\mathcal{M}_{0,0} \cong S^2 \times S^1 \ , \qquad \mathcal{M}_{0,1} \cong S^3 \ , \qquad \mathcal{M}_{1,0} \cong T^3$$

Assumption: The K of the SUSY algebra generates the S^1 fiber.
$\Rightarrow$ We don't allow "squashing" of $\mathcal{M}_{0,p} \cong S^3/\mathbb{Z}_p$, where K would have a component along the S^2 base.

Note that [Ohta, Yoshida, 2012] considered the same setup, however our final results differ.

Outline

3d $\mathcal{N} = 2$ on a circle

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3d $\mathcal{N}=2$ on a circle

Fibering operator

Application: Dualities

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Application: Dualities

Localization

3d $\mathcal{N}=2$ gauge theories

Consider a 3d $\mathcal{N}=2$ gauge theory, which consists of:

- Vector multiplet $\mathcal{V}$ for a gauge group $\mathbf{G}$, with $\mathfrak{g}=\mathrm{Lie}(\mathbf{G})$.
- Chiral multiplets Φ_i in representations $\mathfrak{R}_i$ of $\mathfrak{g}$.

We may also have interactions dictated by a superpotential $W(\Phi)$ that preserves the R-symmetry $U(1)_R$.

We also have supersymmetric CS terms:

$$S_{\mathrm{CS}}=\frac{k}{4\pi}\int_{\mathcal{M}_3} d^3x\sqrt{g}(i\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho-2\sigma D+\tilde{\lambda}\lambda)$$

The level k is integer-quantized, $k\in\mathbb{Z}$. That includes FI terms.

Circle reduction

Consider the theory on $\mathbb{R}^2 \times S^1$, with S^1 a circle of radius β. We can expand all fields into KK modes:

$$\phi = \sum_{n \in \mathbb{Z}} \phi_n(z, \bar{z}) e^{in\psi}$$

and consider the 3d theory as a 2d theory with an infinite number of fields, in 2d $\mathcal{N} = (2,2)$ susy multiplets.

In particular, we have a 2d vector multiplet from the lowest mode of the 3d vector multiplet: $\mathcal{V}_{\rm 2d} = (a_\mu, u, \tilde{u}, \lambda, \tilde{\lambda}, D)$

Here we defined the dimensionless field:

$$u = i\beta(\sigma + i a_0)$$

It is the lowest component of a twisted chiral multiplet:

$$[Q_-, U] = [\bar{Q}_+, U] = 0$$

A-twist on $\Sigma_g \times S^1$

It is natural to compactify $\mathbb{R}^2$. We can define the theory on any (closed, oriented) Riemann surface Σ_g by the so-called topological A-twist, which precisely preserves Q_- and $\bar{Q}_+$. [Witten, 1988]

Note that we have the identification:

$$u \sim u + 1$$

due to large gauge transformations. Thus a more natural variable is:

$$x = e^{2\pi i u}$$

The observables of this 3d A-model are the topological correlators:

$$\langle W_1(x) W_2(x) \cdots \rangle_{\Sigma_g \times S^1}$$

$W(x)$ are loop operators along S^1, local operators on Σ_g.

The Coulomb branch

The 3d theory has a classical Coulomb branch corresponding to VEVs of σ and of the dual photon. The natural Coulomb branch coordinates are the monopole operators:

$$T_a^{\pm} \sim e^{\pm\phi} \;, \qquad \phi_a = -\frac{2\pi}{e^2}\sigma_a + i\varphi_a$$

which lie in 3d chiral multiplets.

On $\mathbb{R}^2 \times S^1$, we consider instead the "2d Coulomb branch" spanned by:

$$u_a \;, \qquad a = 1, \cdots, \mathrm{rk}(\mathbf{G})$$

The variables ϕ_a and u_a are T-dual. [Aganagic, Hori, Karch, Tong, 2001]

In either variables, the classical Coulomb branch takes the form:

$$\mathfrak{M} \cong \tilde{\mathfrak{M}}/W_{\mathbf{G}} \;, \qquad \tilde{\mathfrak{M}} \cong (\mathbb{C}^*)^{\mathrm{rk}(\mathbf{G})}$$

The Coulomb branch

At a generic point of $\tilde{\mathfrak{M}}$, the theory is abelian:

$$\mathbf{G} \to \mathbf{H} \cong \prod_{a=1}^{\mathrm{rk}(\mathbf{G})} U(1)_a$$

Integrating out all massive modes, one obtains the two-dimensional effective twisted superpotential:

$$\mathcal{W}(u) = \mathcal{W}_{CS}(u) + \mathcal{W}_{\text{1-loop}}(u)$$

which governs the dynamics of the low-energy modes u_a.
The 'vacuum equations' are called the Bethe equations:

$$\exp\left(2\pi i \frac{\partial \mathcal{W}}{\partial u_a}\right) = 1 \;, \qquad\qquad w \cdot \hat{u} \neq \hat{u}, \;\; \forall w \in W_G$$

The solutions are the "Bethe vacua". [Nekrasov, Shatashvili, 2009]

The twisted superpotential: CS terms

The classical piece in the twisted superpotential comes from the 3d CS interactions. Schematically:

$$\mathcal{W}_{CS}(u) = \sum_a \frac{k_{aa}}{2} u_a(u_a+1) + \sum_{a>b} k_{ab} u_a u_b + \frac{1}{24} k_g$$

It corresponds to:

$$\begin{aligned} S_{CS} &= \frac{k_{aa}}{4\pi}\int (i a_a \wedge d a_a + \cdots) + \frac{k_{ab}}{2\pi}\int (i a_a \wedge d a_b + \cdots) \\ &\quad + \frac{k_g}{192\pi}\int (i\omega \wedge d\omega + \cdots) \end{aligned}$$

Note the constant term in $\mathcal{W}_{\rm CS}$, which is identified with the gravitational CS level in 3d.

The twisted superpotential: CS terms

The 3d twisted superpotential is defined modulo:

$$\mathcal{W} \sim \mathcal{W} + n^a u_a + n^0 \ , \qquad n^a, n^0 \in \mathbb{Z}$$

In particular, for a $U(1)_k$ CS term, we have:

$$\mathcal{W}_{\rm CS} = \frac{k}{2}(u^2 + u)$$

The linear term is physical for k an odd integer. This terms arises because of the non-trivial spin structure imposed by supersymmetry on $\Sigma_g \times S^1$. In the presence of a flux $\mathfrak{m} \in \mathbb{Z}$ on Σ_g, we have:

$$e^{-S_{CS}} = (-x)^{\mathfrak{m}k}$$

This corrects previous sign mistakes in the literature.
Related comments appeared in [Seiberg, Senthilb, Wang, Witten, 2016].

The twisted superpotential: Chiral multiplets

Consider next the integrating out of the chiral multiplets. For a chiral multiplet Φ of gauge charge Q under $\mathbf{G}=U(1)$, we have:

$$\mathcal{W}_\Phi(u) = -\frac{1}{2\pi i}\sum_{n\in\mathbb{Z}}(Qu+n)\left(\log\left(Qu+n\right)-1\right)$$

Regularizing the sum over the KK modes, we find:

$$\mathcal{W}_\Phi(u) = \frac{1}{(2\pi i)^2}\mathrm{Li}_2(e^{2\pi i Qu})$$

The only scheme dependence of this result is through the addition of CS terms.

Aside: The parity anomaly

In 3d, there is a "parity anomaly" in quantizing a Dirac fermion coupled to a (background) gauge field a_μ: we cannot preserve both gauge invariance and 3d "parity".

Of course, we choose to preserve gauge invariance.

[Alvarez-Gaumé, Della Pietra, Moore, 1985]

Consider Φ with $Q=1$. We claim that:

$$\mathcal{W}_\Phi = \frac{1}{(2\pi i)^2}\mathrm{Li}_2(x)$$

is a "$U(1)_{-\frac{1}{2}}$ quantization." We have the contact terms:

$$\kappa = -\frac{1}{2}\,, \qquad \kappa_g = -1$$

in two-point functions of conserved currents.

[C.C., Dumitrescu, Festuccia, Komargodski, Seiberg, 2012]

Aside: The parity anomaly

CS terms shift these contact terms by integers:

$$\kappa = -\frac{1}{2} + k \ , \qquad \kappa_g = -1 + k_g$$

The superpotential $\mathcal{W}_\Phi$ has the limits:

$$\lim_{\sigma \to \infty} \mathcal{W}_\Phi = 0 \ , \qquad \lim_{\sigma \to -\infty} \mathcal{W}_\Phi = -\frac{1}{2} u(u+1) - \frac{1}{12} \ ,$$

that correspond to

$$\kappa = \kappa_g = 0 \qquad \text{or} \qquad \kappa = -1 \ , \quad \kappa_g = -2$$

The correct treatment of parity anomalies clears some confusions in the localization literature.

The twisted superpotential

The full effective twisted superpotential is:

$$\begin{aligned}\mathcal{W} &= \sum_a \frac{k_{aa}}{2} u_a(u_a+1) + \sum_{a>b} k_{ab} u_a u_b + \sum_\alpha \frac{k_{\alpha\alpha}}{2}\nu_\alpha(\nu_\alpha+1) \\ &+ \sum_{\alpha>\beta} k_{\alpha\beta}\nu_\alpha\nu_\beta + \sum_{a,\alpha} k_{a\alpha} u_a \nu_\beta + \frac{k_g}{24} \\ &+ \frac{1}{(2\pi i)^2}\sum_i \sum_{\rho_i\in\mathfrak{R}_i} \mathrm{Li}_2(x^{\rho_i} y_i)\end{aligned}$$

with:

- The gauge parameters u_a and $x_a = e^{2\pi i u}$ as above.
- ν_α and $y_\alpha = e^{2\pi i \nu_\alpha}$ for the flavor symmetry.
- All possible gauge, flavor, and flavor-gauge CS levels.
- ρ_i are the weights of $\mathfrak{R}_i$ for the chiral Φ_i.

$\Sigma_g \times S^1$ compactification

Going to the cohomology of Q_-, $\bar{Q}_+$, we have a two-dimensional topological quantum field theory (TQFT). This is defined on any Σ_g via the A-twist.

Up to Q-exact terms, the TQFT effective action reads:

$$S_{\text{TQFT}} = \int_{\Sigma_g} \left(-2 f_a \frac{\partial \mathcal{W}(u)}{\partial u_a} + \tilde{\Lambda}^a \Lambda^b \frac{\partial^2 \mathcal{W}(u)}{\partial u_a \partial u_b} \right) + \frac{i}{2} \int_{\Sigma_g} d^2x \sqrt{g}\, \Omega(u) R$$

with f_a the abelian field strength of a_μ and R the Ricci scalar.
see *e.g.* [Witten, 1993; Nekrasov, Shatashvili, 2014]

The effective dilaton

The effective dilaton $\Omega(u)$ captures the coupling of the theory to curved space. We have a classical contribution:

$$\Omega_{\rm CS}(u) = \sum_a k_{aR} u_a + \sum_\alpha k_{\alpha R} \nu_\alpha + \frac{1}{2} k_{RR}$$

which come from supersymmetric $U(1)_R$ CS terms:

$$S_{\rm CS} = \frac{k_{aR}}{2\pi} \int (i a_a \wedge dA^{(R)} + \cdots) + \frac{k_{\alpha R}}{2\pi} \int (i a_\alpha \wedge dA^{(R)} + \cdots) + \frac{k_{RR}}{4\pi} \int (i A^{(R)} \wedge dA^{(R)} + \cdots)$$

[C.C., Dumitrescu, Festuccia, Komargodski, Seiberg, 2012]

Note that Ω is defined modulo integers, $\Omega \sim \Omega + n$.

The effective dilaton

We also have one-loop contribution from the matter fields, and from the W-bosons:

$$\begin{aligned}\Omega(u) &= \sum_a k_{aR} u_a + \sum_F k_{FR}\nu_F + \frac{1}{2}k_{RR} \\ &\quad -\frac{1}{2\pi i}\sum_i (r_i - 1)\sum_{\rho_i\in\mathfrak{R}_i}\log(1-x^{\rho_i}y_i) \\ &\quad -\frac{1}{2\pi i}\sum_{\alpha\in\mathfrak{g}}\log(1-x^{\alpha})\end{aligned}$$

The two locally holomorphic functions $\mathcal{W}(u)$, $\Omega(u)$ fully determine the A-model associated to any given 3d gauge theory.

Charge quantization and gauge invariance

For the A-twist on Σ_g, we have:

$$-\frac{1}{8\pi}\int_{\Sigma_g} d^2x\sqrt{g}R = \frac{1}{2\pi}\int dA^{(R)} = g-1$$

Dirac quantization: We take all R-charges to be integers. All other gauge and flavor charges are similarly quantized. It follows that all CS levels are integers.

This ensures that $\mathcal{W}$, Ω are invariant under large gauge transformations, $u_a \sim u_a + 1$ or $\nu_\alpha \sim \nu_\alpha + 1$, modulo the ambiguities:

$$\mathcal{W} \sim \mathcal{W} + n^a u_a + n^\alpha \nu_\alpha + n_0 \ , \qquad \Omega \sim \Omega + n \ ,$$

with $n^a, n^\alpha, n^0, n \in \mathbb{Z}$. *Note:* the linear term in $\mathcal{W}_{CS}$ is crucial.

Flux operator

In the presence of flavor symmetries, it is natural to consider non-trivial background vector multiplets $\mathcal{V}_F$. We may turn on:

$$u_F = \nu_F \ , \qquad \qquad \frac{1}{2\pi}\int_{\Sigma_g} da_F = \mathfrak{n}_F$$

while preserving supersymmetry. This adds a piece to the effective action:

$$S_{\text{flux}} = \int_{\Sigma_g} \left(-2 f_F \frac{\partial \mathcal{W}(u,\nu)}{\partial \nu_F} \right)$$

with $f_F = da_F$.

Flux operator

In particular, if we take

$$f_F = 2\pi\, \mathfrak{n}_F\, \delta^2(x - x_0)\ ,$$

turning on background flux is equivalent to the insertion of a local operator:

$$\Pi_F(u,\nu)^{\mathfrak{n}_F}$$

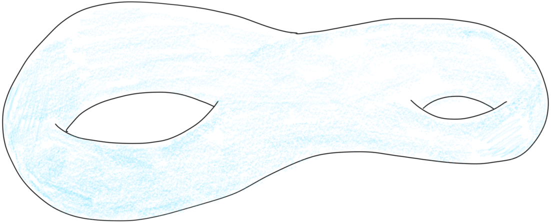
in the path integral, with

$$\boxed{\Pi_F(u,\nu) = \exp\left(2\pi i \frac{\partial \mathcal{W}}{\partial \nu_F}\right)}$$

We call Π_F the flux operator for the flavor symmetry $U(1)_F$.

Handle-gluing operator

Any 2d TQFT has a "handle-gluing operator" $\mathcal{H}$:



The explicit form of $\mathcal{H}$ was known for the simplest LG models from [Vafa, 1990], but the generalization to 2d gauge theories was only investigated recently [Melnikov, Plesser, 2005; Nekrasov, Shatashvili, 2014].

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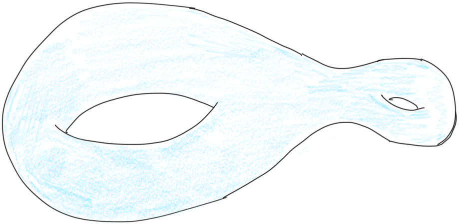

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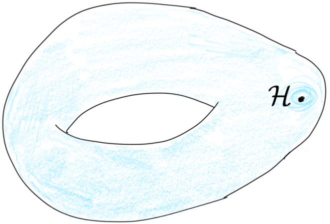

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Handle-gluing operator

In any 2d TQFT, we have:

$$\langle \mathcal{O} \rangle_{\Sigma_g} = \langle \mathcal{O}\mathcal{H}^g \rangle_{\mathbb{C}P^1} = \mathrm{Tr}_V \left(\mathcal{H}^{g-1} \mathcal{O} \right)$$

where V is the TQFT Hilbert space.

In the A-twisted theory, the handle-gluing operator can be seen as a "flux operator for $U(1)_R$". It is given by: [Nekrasov, Shatashvili, 2014]

$$\boxed{\mathcal{H}(u, \nu) = e^{2\pi i \Omega(\sigma)} \det_{ab} \left(\frac{\partial^2 \mathcal{W}}{\partial u_a \partial u_b} \right)}$$

Note that Π_F and $\mathcal{H}$ are rational functions of the fugacities x, y.

The $\Sigma_g \times S^1$ index

Let us define the set of Bethe vacua:

$$\mathcal{S}_{BE} = \left\{ \hat{u}_a \,\middle|\, \Pi_a(\hat{u},\nu) = 1 \ , \ \ \forall a \ , \quad w \cdot \hat{u} \neq \hat{u}, \ \ \forall w \in W_G \right\} / W_{\mathbf{G}}$$

and $\hat{x} = e^{2\pi i \hat{u}}$. Note that Π_a are the gauge flux operators.

We then find the $\Sigma_g \times S^1$ twisted index:

$$Z_{\Sigma_g \times S^1}(y;\mathfrak{n}) = \sum_{\hat{x} \in \mathcal{S}_{\rm BE}} \mathcal{H}(\hat{x},y)^{g-1} \prod_\alpha \Pi_\alpha(\hat{x},y)^{\mathfrak{n}_\alpha}$$

This is the supersymmetric partition function on $\Sigma_g \times S^1$ in the presence of the flavor background fluxes $\mathfrak{n}_\alpha$. It can also be derived using supersymmetric localization.

[C.C., Kim, 2016; Benini, Zaffaroni, 2015, 2016]

Wilson loops on $\Sigma_g \times S^1$

We can similarly insert Wilson loops along S^1:

$$W_{\mathfrak{R}} = \mathrm{Tr}_{\mathfrak{R}}\, \mathrm{Pexp}\left(-i\int_{S^1} dx^\mu \left(a_\mu - i\eta_\mu \sigma\right)\right) = \mathrm{Tr}_{\mathfrak{R}}\,(x)$$

They reduce to twisted chiral operators on Σ_g (thus, independent of the position on Σ_g).

We then have:

$$\Big\langle W \Big\rangle_{\Sigma_g \times S^1} = \sum_{\hat{x}\in \mathcal{S}_{\mathrm{BE}}} W(\hat{x})\, \mathcal{H}(\hat{x}, y)^{g-1} \prod_\alpha \Pi_\alpha(\hat{x}, y)^{\mathfrak{n}_\alpha}$$

This allows us to understand the fusion algebra of half-BPS Wilson loops systematically. [CC, Kim, 2016]

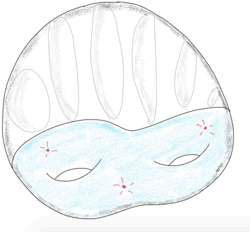
Seifert manifold compactification

This TQFT logic can be generalized. Let us consider a $U(1)$ principal bundle:

$$S^1 \longrightarrow \mathcal{M}_{g,p} \xrightarrow{\pi} \Sigma_g \ .$$

See also [Ohta, Yoshida, 2012].

This is the simplest example of a Seifert manifold.



Supersymmetry is preserved by a pull-back of the A-twist on Σ.

The fibering operator

The introduction of a non-trivial fibration over Σ_g corresponds to a local operator on Σ_g (or to a loop operator on $\Sigma_g \times S^1$).

The 2d $\mathcal{N} = (2, 2)$ theory obtained from 3d has a flavor symmetry:

$$\mathbf{G}_F \times U(1)_{KK}$$

The $U(1)_{KK}$ charge is the S^1 momentum. The "graviphoton" $\mathcal{C}_\mu^{\mathrm{KK}}$ for $U(1)_{KK}$ sits in a two-dimensional vector multiplet. In particular, there is a twisted mass:

$$m_{KK} = \beta^{-1}$$

A non-trivial fibration introduces a non-zero flux on the base:

$$\frac{1}{2\pi} \int_{\Sigma_g} d\mathcal{C}^{\mathrm{KK}} = p \in \mathbb{Z}$$

The fibering operator

There exists a flux operator for $U(1)_{KK}$, which we call the fibering operator.

Reinstating dimensions, we find:

$$\mathcal{F}(u,\nu) \equiv \exp\left(2\pi i \frac{\partial}{\partial m_{KK}}\Big(m_{KK}\mathcal{W}(u,\nu)\Big)\right)$$

This leads to the explicit expression:

$$\boxed{\mathcal{F}(u,\nu) = \exp\Big(2\pi i(\mathcal{W} - u_a\partial_{u_a}\mathcal{W} - \nu_\alpha\partial_{\nu_\alpha}\mathcal{W})\Big)}$$

Note that the ambiguities of $\mathcal{W}$ cancel in $\mathcal{F}$.

The fibering operator

Importantly, the fibering operator is not gauge invariant. Instead, we have the difference equations:

$$\mathcal{F}(u_a - \mathfrak{m}_a, \nu_\alpha - \mathfrak{n}_\alpha) = \mathcal{F}(u,\nu) \prod_a \Pi_a(u,\nu)^{\mathfrak{m}_a} \prod_\alpha \Pi_\alpha(u,\nu)^{\mathfrak{n}_\alpha} ,$$

$\forall \mathfrak{m}_a, \mathfrak{n}_\alpha \in \mathbb{Z}$. It is, however, gauge invariant on the Bethe vacua, where $\Pi_a(\hat{u}) = 1$.

For instance, consider a $U(1)_k$ CS theory:

$$\Pi_{\text{CS}} = q(-x)^k , \qquad \mathcal{F}_{\text{CS}} = e^{-\pi i k u^2 - 2\pi i \tau u}$$

for the gauge flux and fibering operators, with $q = e^{2\pi i \tau}$ and τ the FI term. We see that they satisfy the difference equation:

$$\mathcal{F}_{\text{CS}}(u-1,\tau) = \Pi_{\text{CS}}(u,\tau)\mathcal{F}_{\text{CS}}(u,\tau)$$

The fibering operator for a chiral multiplet

The contribution of a chiral multiplet to the fibering operator is:

$$\mathcal{F}_\Phi(u) = \exp\left(\frac{1}{2\pi i}\mathrm{Li}_2\left(e^{2\pi i u}\right) + u\log\left(1-e^{2\pi i u}\right)\right)$$

This is a meromorphic function of u with poles at $u=-1,-2,\cdots$ and zeros at $z=1,2,\cdots$.

We also have:

$$\Pi_\Phi(u) = \frac{1}{1-x}\;, \qquad \mathcal{F}_\Phi(u-1) = \Pi_\Phi(u)\mathcal{F}_\Phi(u)$$

$\mathcal{F}_\Phi(u)$ is closely related to the chiral multiplet S^3 partition function of [Jafferis, 2010; Hama, Hosomichi, Lee, 2010].

The $Z_{\mathcal{M}_{g,p}}$ partition function

We can then write the $Z_{\mathcal{M}_{g,p}}$ partition function as a sum over Bethe vacua:

$$Z_{\mathcal{M}_{g,p}}(\nu;\mathfrak{n}) = \sum_{\hat{u}\in\mathcal{S}_{\rm BE}} \mathcal{F}(\hat{u},\nu)^p\,\mathcal{H}(\hat{u},\nu)^{g-1}\prod_\alpha \Pi_\alpha(\hat{u},\nu)^{\mathfrak{n}_\alpha}$$

We should note that:

- We have $\mathfrak{n}_\alpha \in \mathbb{Z}_p$ (*torsion* fluxes on $\mathcal{M}_{g,p}$).
- Under large gauge transformations for any $U(1)_F$,

$$(\nu,\mathfrak{n}) \sim (u+1,\mathfrak{n}+p)$$

and $Z_{\mathcal{M}_{g,p}}$ is invariant due to the difference equations.

A 3d quasi-TQFT

More generally, we find:

$$\langle W(x)\rangle_{\mathcal{M}_{g,p}} = \sum_{\hat{u}\in\mathcal{S}_{\rm BE}} W(\hat{x})\,\mathcal{F}(\hat{u})^p\,\mathcal{H}(\hat{u})^{g-1}\prod_\alpha \Pi(\hat{u})^{\mathfrak{n}_\alpha}\;.$$

It implies:

$$\Big\langle W\Big\rangle_{\mathcal{M}_{g,p}} = \Big\langle \mathcal{F}(x)^p\, W(x)\Big\rangle_{\Sigma_g\times S^1}$$

This generalizes results about pure CS theory [Blau, Thompson, 2006] to any 3d $\mathcal{N}=2$ gauge theories with a $U(1)_R$ symmetry.

These $\mathcal{N}=2$ theories are "quasi-topological"—they only depend on the Seifert structure, not on the metric.

S^3 vs. $S^2 \times S^1$

In particular, the S^3 partition function and the $S^2 \times S^1$ twisted index are very closely related:

$$Z_{S^3} = \left\langle \mathcal{F} \right\rangle_{S^2 \times S^1}$$

We also find the general equality:

$$\begin{aligned} \langle W \rangle_{S^3} &= \int d\sigma\, W(x) e^{\pi i k \sigma^2}\, \mathcal{Z}^{\text{1-loop}}_{S^3}(\sigma) \\ &= \sum_{\hat{u} \in \mathcal{S}_{\rm BE}} W(\hat{x})\, \mathcal{F}(\hat{u})^p\, \mathcal{H}(\hat{u})^{-1} \end{aligned}$$

where $x = e^{-2\pi\sigma}$ in the first line. This can be proven relatively straightforwardly.

R-charge dependence on $\mathcal{M}_{g,p}$

To preserve supersymmetry on $\mathcal{M}_{g,p}$, we need a non-trivial R-symmetry line bundle with:

$$c_1(L^{(R)}) = g-1 \in \mathbb{Z}_p \subset H^2(\mathcal{M}_{g,p}, \mathbb{Z})$$

We then have a Dirac quantization on the R-charge, like in 2d. We need $r \in \mathbb{Z}$ for the quasi-topological story to apply.

The $U(1)_R$ line bundle is trivial if $g-1=0 \mod p$. We can then vary the R-charges *continuously*. This is important to study $\mathcal{N}=2$ SCFTs on S^3.

Our results naturally reproduce the expected R-charge dependence.

[C.C., Dumitrescu, Festuccia, Komargodski, 2014]

R-charge dependence on $\mathcal{M}_{g,p}$

Example: On the round S^3, we have:

$$Z_\Phi(\sigma, r) = \mathcal{F}_\Phi(i\sigma + r - 1)$$

for a free chiral.

Comparing to the result of [Jafferis, 2010; Hama, Hosomichi, Lee, 2010]:

$$\tilde{Z}^{\Phi}_{S^3}(\sigma) = e^{-\frac{\pi i}{2}(i\sigma+r-1)^2+\frac{\pi i}{12}}\,\mathcal{F}_\Phi(i\sigma + r - 1)$$

for any $r \in \mathbb{R}$. Disagreement is just in the choice of quantization scheme. (Previous results not gauge invariant.)

Field theory dualities

The Bethe-equation formula leads to a simple way to match supersymmetric partition functions across field theory dualities (Seiberg duality, 3d mirror symmetry,...).

Given a duality between theories $\mathcal{T}$ and $\mathcal{T}_D$, two operators are dual:

$$\mathcal{O} \in \mathcal{T} \qquad \leftrightarrow \qquad \mathcal{O}_D \in \mathcal{T}_D$$

if and only if

$$\mathcal{O}(\hat{u}) = \mathcal{O}_D(\hat{u}^D)$$

for any pairs $\hat{u}$ and $\hat{u}^D$ of dual vacua (dual solutions to the Bethe equations).

Field theory dualities

Given the mapping of Bethe equations (which can be analysed in flat space), the duality statement in curved space reads:

$$\mathcal{F}(\hat{u}) = \mathcal{F}_D(\hat{u}^D)$$

$$\mathcal{H}(\hat{u}) = \mathcal{H}_D(\hat{u}^D)$$

Flux operators are similarly matched:

$$\Pi_F(\hat{u}) = \Pi_{F,D}(\hat{u}^D)$$

but that follows from the equality for fibering operators.

Example: $U(1)_{\frac{1}{2}}$ with Φ dual to free chiral T

Consider a chiral Φ of charge 1 under $U(1)_{\frac{1}{2}}$. We have:

$$\mathcal{W}(u,\tau) = \frac{1}{(2\pi i)^2}\mathrm{Li}_2(x) + \tau u + \frac{1}{2}u(u+1)$$

The single Bethe equation gives:

$$\hat{x} = (1-q)^{-1} \ , \qquad\qquad q \equiv e^{2\pi i \tau}$$

We then have the on-shell superpotential (with $\hat{\mathcal{W}} = (2\pi i)^2 \mathcal{W}$):

$$\hat{\mathcal{W}}(\hat{x}(q), q) = \mathrm{Li}_2\left(\frac{1}{1-q}\right) - \log(q)\log(1-q) + \frac{1}{2}\left(\log^2(q-1) + \pi^2\right)$$

By a well-known dilog identity, this is equal to:

$$\hat{\mathcal{W}}_D(q) = \mathrm{Li}_2(q) + \frac{\pi^2}{6}$$

Example: $U(1)_{\frac{1}{2}}$ with Φ dual to free chiral T

This implies:

$$\mathcal{F}(\hat{x},q)=\mathcal{F}_D(q)\ , \qquad \Pi_T(\hat{x},q)=\Pi_{T,D}(q)$$

Similarly, we have:

$$\Omega(u,\tau)=-\frac{1}{2\pi i}(r-1)\log(1-x)$$

and

$$\Omega_D(\tau)=-\frac{1}{2\pi i}(-r)\log(1-q)+k^D_{TR}\tau+\frac{1}{2}k^D_{RR}$$

so $k^D_{TR}=k^D_{RR}=-r$. This implies

$$\mathcal{H}(\hat{x},q)=\mathcal{H}_D(q)$$

The integral formula

By a localization argument in the UV, one can obtain (for $p \neq 0$):

$$Z_{\mathcal{M}_{g,p}} = \frac{(-1)^{\mathbf{r}}}{|W_{\mathbf{G}}|} \sum_{\mathfrak{m} \in \mathbb{Z}_p^{\mathbf{r}}} \int_{\mathcal{C}^\eta} d^{\mathbf{r}} u \, \mathcal{F}(u)^p \, \Pi_a(u)^{\mathfrak{m}_a} \, \Pi_\alpha(u)^{\mathfrak{n}_\alpha} \, e^{2\pi i (g-1)\Omega(u)} \, H(u)^g$$

- Sum over torsion fluxes $\mathfrak{m}$. ($\mathcal{M}_{g,p}$ topological sectors.)
- $\mathcal{F}$ the fibering operator.
- Π_a, Π_α gauge and flavor flux.
- $\mathcal{C}^\eta$, with $\eta \in i\mathfrak{h}^*$ is a "JK contour". The contour is a conjecture for higher-rank, well-understood for $\mathbf{r}=1$.

Argument similar to [Benini, Eager, Hori, Tachikawa, 2013; Hori, Kim, Yi, 2014]

Conclusions

- For theories with 4 supercharges, "half-BPS" curved-space supersymmetry is intimately related to the two-dimensional A-twist. [Dumitrescu, Festuccia, Seiberg, 2012, CC, Dumitrescu, Festuccia, Komargodski, 2012]
- We used this fact to compute supersymmetric partition functions of 3d $\mathcal{N} = 2$ gauge theories. We obtained new localization result on the family $\mathcal{M}_{g,p}$. This unifies some previous results.
- $Z_{\mathcal{M}_{g,p}}$ is fully determined by $\mathcal{W}, \Omega$.
- Considering a larger family of backgrounds lead us to clarify a number of subtle points about localization results, especially some subtle phases related to CS contact terms.

Outlook

- There is an analog story for 4d $\mathcal{N}=1$ theories on T^2! Fibering operator given in terms of elliptic gamma functions.
- Bethe formula similar to surgery prescription in pure CS theory. Can we push this point of view? This might lead to results for any Seifert manifold.
- At genus $g=0$, it would be interesting to generalize our 2d approach to the "squashing" of $\mathcal{M}_{0,p} \cong S^3/\mathbb{Z}_p$.
- Relation to holomorphic blocks? [Beem, Dimofte, Pasquetti, 2012]
- Our results lead to interesting challenges for the 3d/3d correspondence. [Dimofte, Gaiotto, Gukov, 2011]
  What is the TQFT dual of $Z_{\mathcal{M}_{g,p}}$?