Lattice Formulation of Two Dimensional Topological Field Theory

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K. Ohta, T.T Prog.Theor. Phys. 117 (2007) No2

[hep-lat /0611011] (and more)

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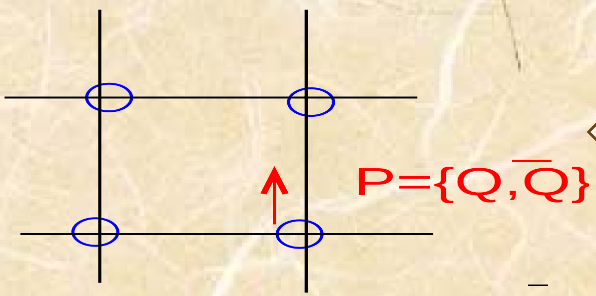
1. Introduction

Lattice construction of SUSY gauge theory is difficult.



$$\Longleftrightarrow \quad \{Q_\alpha, \bar{Q}_\beta\} = 2P_\mu(\gamma^\mu)_{\alpha\beta}$$

SUSY breaking ⟶ Fine-tuning problem

Difficult
- *taking continuum limit*
- *numerical study

Candidate to solve fine-tuning problem

A lattice model of Extended SUSY
preserving a partial SUSY Q

Q : does not include the translation
(BRST charge of TFT (topological field theory))

Candidate Models

CKKU models (Cohen-Kaplan-Katz-Unsal)

2-d N=(4,4),3-d N=4, 4-d N=4 etc. super Yang-Mills theories

(JHEP 08 (2003) 024, JHEP 12 (2003) 031, JHEP 09 (2005) 042)

Sugino models

(JHEP 01 (2004) 015, JHEP 03 (2004) 067, JHEP 01 (2005) 016 Phys.Lett. B635 (2006) 218-224)

Catterall models

(JHEP 11 (2004) 006, JHEP 06 (2005) 031)

(Relationship between them: T.T (JHEP 07 (2007) 010))

Do they really recover the target continuum theory ?

Perturbative studies

Lattice

continuum limit $a \to 0$

All right!

Target continuum theory

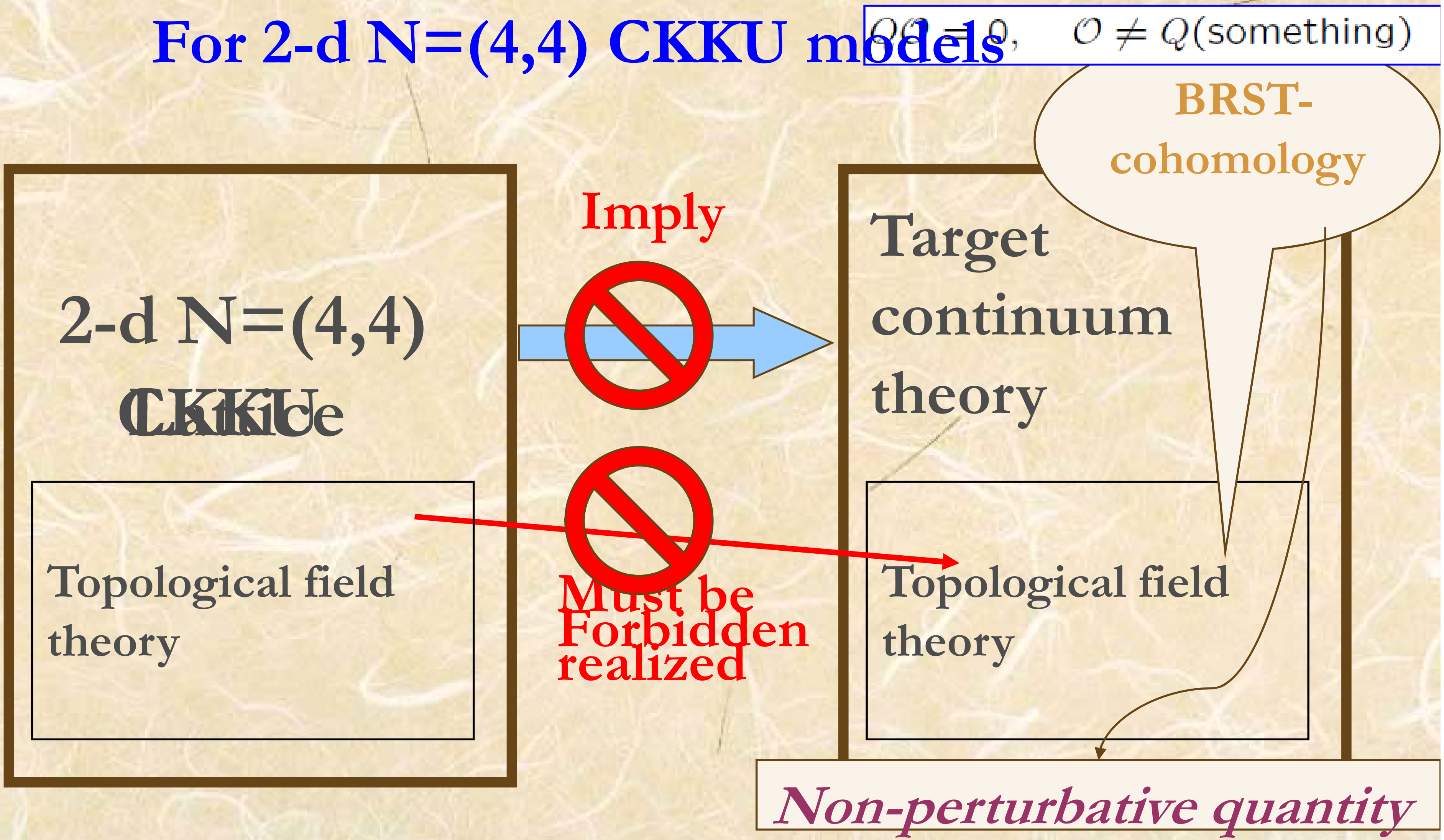
Non-perturbative studies

No sufficient study

I have done it
with the study of
Topological Field Theory

Non-perturbative study

For 2-d N=(4,4) CKKU models



2.1 The target continuum theory (2-d N=(4,4))

(Dijkgraaf and Moore, Commun. Math. Phys. 185 (1997) 411)

$$S = \frac{1}{g_2^2}\int d^2x Q\Xi(\vec{\mathcal{B}}, \vec{\mathcal{F}}, \Phi),$$

$$\Xi(\vec{\mathcal{B}}, \vec{\mathcal{F}}, \Phi) = \mathrm{Tr}\left[\frac{1}{4}\eta[\Phi, \bar{\Phi}] + \vec{\chi}\cdot(\vec{H} - i\vec{\mathcal{E}}) + \left\{-i\lambda_\mu D^\mu \bar{\Phi} + \xi_{s0}[s_0, \bar{\Phi}] + \xi_{s3}[s_3, \bar{\Phi}]\right\}\right],$$

A_μ : gauge field

(Set of Fields) $= \mathcal{A} \cup \{\Phi\}$

BRST transformation

BRST partner sets

$$\boxed{\mathcal{B}} \longleftrightarrow \boxed{\mathcal{F}}$$

$$
\begin{array}{lcl}
QA_\mu = \lambda_\mu, & \longleftrightarrow & Q\lambda_\mu = iD_\mu\Phi, \\
Qs_0 = \frac{1}{2}(\lambda + \lambda^\dagger) \equiv (\xi_{s0}), & \longleftrightarrow & Q\xi_{s0} = [\Phi, s_0], \\
Qs_3 = \xi_{s3}, & \longleftrightarrow & Q\xi_{s3} = [\Phi, s_3], \\
QH^{\mathbb{C}} = [\Phi, \chi^{\mathbb{C}}], & \longleftrightarrow & Q\chi^{\mathbb{C}} = H^{\mathbb{C}}, \\
QH^{\mathbb{C}\dagger} = [\Phi, \chi^{\mathbb{C}\dagger}], & \longleftrightarrow & Q\chi^{\mathbb{C}\dagger} = H^{\mathbb{C}\dagger}, \\
QH^{\mathbb{R}} - [\Phi, \chi^{\mathbb{R}}], & \longleftrightarrow & Q\chi^{\mathbb{R}} - H^{\mathbb{R}}, \\
Q\bar{\Phi} = \eta, & \longleftrightarrow & Q\eta = [\Phi, \bar{\Phi}], \\
Q\Phi = 0. & &
\end{array}
$$

$\ker Q = \{\Phi\}$ $\qquad$ $\boxed{Q\mathcal{A}}$

$\boxed{Q\mathcal{A}}$ is set of homogeneous linear function of $\mathcal{A}$

def

Q is homogeneous transformation of $\mathcal{A}$

(Φ : coefficient)

Questions

(I) Is BRST transformation homogeneous ?

(II) Does Q change the gauge transformation laws?

Answer for (I) and (II)

$$QA_\mu = \lambda_\mu, \quad \longleftrightarrow \quad Q\lambda_\mu = iD_\mu\Phi,$$
$$Qs_0 = \tfrac{1}{2}(\lambda + \lambda^\dagger) \equiv (\xi_{s0}), \longleftrightarrow Q\xi_{s0} = [\Phi, s_0],$$
$$Qs_3 = \xi_{s3}, \quad \longleftrightarrow \quad Q\xi_{s3} = [\Phi, s_3],$$
$$QH^{\mathbb{C}} = [\Phi, \chi^{\mathbb{C}}], \quad \longleftrightarrow \quad Q\chi^{\mathbb{C}} = H^{\mathbb{C}},$$
$$QH^{\mathbb{C}\dagger} = [\Phi, \chi^{\mathbb{C}\dagger}], \quad \longleftrightarrow \quad Q\chi^{\mathbb{C}\dagger} = H^{\mathbb{C}\dagger},$$
$$QH^{\mathbb{R}} = [\Phi, \chi^{\mathbb{R}}], \quad \longleftrightarrow \quad Q\chi^{\mathbb{R}} = H^{\mathbb{R}},$$
$$Q\bar{\Phi} = \eta, \quad \longleftrightarrow \quad Q\eta = [\Phi, \bar{\Phi}],$$
$$\boxed{Q\Phi = 0.}$$

(II)

Q change the gauge transformation law

$$A_\mu \to gA_\mu g^{-1} + g\partial_\mu g^{-1}$$

BRST Q

$$\lambda_\mu \to g\lambda_\mu g^{-1}$$

(I) Q is ***not*** homogeneous

$Q\lambda_\mu = i\partial_\mu\Phi - [\Phi, A_\mu]$: ***not*** homogeneous of $\mathcal{A}$

2.2 BRST cohomology in the continuum theory

(E.Witten, Commun. Math. Phys. 117 (1988) 353)

$$\begin{cases} \mathcal{W}_0 = \mathrm{Tr}\,\Phi^2 \\ \mathcal{W}_1 = \mathrm{Tr}\,\Phi\lambda \\ \mathcal{W}_2 = \mathrm{Tr}\,\Phi F + \lambda \wedge \lambda \end{cases} \Rightarrow$$

satisfying

descent relation

$$Q\mathcal{W}_k = d\mathcal{W}_{k-1}$$

Integration of $\mathcal{W}_k$ over k-homology cycle

$$\boxed{\mathcal{O}_k \equiv \int_{\gamma_k} \mathcal{W}_k} \Rightarrow$$

BRST-cohomology

$\mathcal{O}_k$ are **BRST cohomology** composed by $\lambda,\ A,\ \Phi$

Due to (II)

$\mathcal{O}_k$ ***can be BRST cohomology***

$\mathcal{O}_k \longrightarrow$ **formally BRST exact**

$$\int_{\gamma_1} \mathcal{W}_1 = \int_{\gamma_1} Q\boxed{\mathrm{Tr}\, A\Phi}$$
$$\int_{\gamma_2} \mathcal{W}_2 = \int_{\gamma_2} Q\boxed{\mathrm{Tr}\, \lambda \wedge A}$$

not BRST exact !

not gauge invariant

BRST exact $\longrightarrow$ Q**(gauge invariant quantity)**

Q **change the gauge transformation law(II)**

3.1 Two dimensional N=(4,4) CKKU action

(K.Ohta, T.T (2007))

BRST exact form

$$S=\frac{1}{g^2}\sum_{\mathbf{n}} Q\Xi(\vec{\mathcal{B}}_{\mathbf{n}},\vec{\mathcal{F}}_{\mathbf{n}},\Phi_{\mathbf{n}}),$$

$$\begin{aligned}\Xi(\vec{\mathcal{B}}_{\mathbf{n}},\vec{\mathcal{F}}_{\mathbf{n}},\Phi_{\mathbf{n}}) = \mathrm{Tr}\Bigg[&\frac{1}{4}\eta_{\mathbf{n}}[\Phi_{\mathbf{n}},\bar{\Phi}_{\mathbf{n}}]+\vec{\chi}_{\mathbf{n}}\cdot(\vec{H}_{\mathbf{n}}-i\vec{\mathcal{E}}_{\mathbf{n}})\\ &+\frac{1}{2}\Big\{\lambda_{\mathbf{n}}(X^{\dagger}_{\mathbf{n}}\bar{\Phi}_{\mathbf{n}}-\bar{\Phi}_{\mathbf{n}+\mathbf{i}}X^{\dagger}_{\mathbf{n}})+\lambda^{\dagger}_{\mathbf{n}-\mathbf{i}}(X_{\mathbf{n}-\mathbf{i}}\bar{\Phi}_{\mathbf{n}}-\bar{\Phi}_{\mathbf{n}-\mathbf{i}}X_{\mathbf{n}-\mathbf{i}})\\ &+\tilde{\lambda}_{\mathbf{n}}(Y^{\dagger}_{\mathbf{n}}\bar{\Phi}_{\mathbf{n}}-\bar{\Phi}_{\mathbf{n}+\mathbf{j}}Y^{\dagger}_{\mathbf{n}})+\tilde{\lambda}^{\dagger}_{\mathbf{n}-\mathbf{j}}(Y_{\mathbf{n}-\mathbf{j}}\bar{\Phi}_{\mathbf{n}}-\bar{\Phi}_{\mathbf{n}-\mathbf{j}}Y_{\mathbf{n}-\mathbf{j}})\Big\}\Bigg],\end{aligned}$$

Set of Fields $=\vec{\mathcal{A}}_{\mathbf{n}}\cup\Phi_{\mathbf{n}}$

BRST transformation on the lattice

$$Q\vec{\mathcal{A}}_{\mathbf{n}}$$

$$\begin{aligned}
&QX_{\mathbf{n}} = \lambda_{\mathbf{n}}, &&\longleftrightarrow && Q\lambda_{\mathbf{n}} = \Phi_{\mathbf{n}}X_{\mathbf{n}} - X_{\mathbf{n}}\Phi_{\mathbf{n}+\mathbf{i}},\\
&QY_{\mathbf{n}} = \tilde{\lambda}_{\mathbf{n}}, &&\longleftrightarrow && Q\tilde{\lambda}_{\mathbf{n}} = \Phi_{\mathbf{n}}Y_{\mathbf{n}} - Y_{\mathbf{n}}\Phi_{\mathbf{n}+\mathbf{j}},\\
&QH^{\mathbb{R}}_{\mathbf{n}} = [\Phi_{\mathbf{n}}, \chi^{\mathbb{R}}_{\mathbf{n}}], &&\longleftrightarrow && Q\chi^{\mathbb{R}}_{\mathbf{n}} = H^{\mathbb{R}}_{\mathbf{n}},\\
&QH^{\mathbb{C}}_{\mathbf{n}} = \Phi_{\mathbf{n}}\chi^{\mathbb{C}}_{\mathbf{n}} - \chi^{\mathbb{C}}_{\mathbf{n}}\Phi_{\mathbf{n}+\mathbf{i}+\mathbf{j}}, &&\longleftrightarrow && Q\chi^{\mathbb{C}}_{\mathbf{n}} = H^{\mathbb{C}}_{\mathbf{n}},\\
&Q\bar{\Phi}_{\mathbf{n}} = \eta_{\mathbf{n}}, &&\longleftrightarrow && Q\eta_{\mathbf{n}} = [\Phi_{\mathbf{n}}, \bar{\Phi}_{\mathbf{n}}],
\end{aligned}$$

$$\boxed{Q\Phi_{\mathbf{n}} = 0.}$$

BRST partner sets

(I)Homogeneous transformation of $\vec{\mathcal{A}}_{\mathbf{n}} = (\vec{\mathcal{B}}_{\mathbf{n}}, \vec{\mathcal{F}}_{\mathbf{n}})$

In continuum theory,

$$\begin{aligned}
&QA_\mu = \lambda_\mu, && Q\lambda_\mu = iD_\mu\Phi,\\
&Qs_0 = \tfrac{1}{2}(\lambda + \lambda^\dagger) \equiv (\xi_{s0}), && Q\xi_{s0} = [\Phi, s_0],\\
&Qs_3 = \xi_{s3}, && Q\xi_{s3} = [\Phi, s_3],\\
&QH^{\mathbb{C}} = [\Phi, \chi^{\mathbb{C}}], && Q\chi^{\mathbb{C}} = H^{\mathbb{C}},\\
&QH^{\mathbb{C}\dagger} = [\Phi, \chi^{\mathbb{C}\dagger}], && Q\chi^{\mathbb{C}\dagger} = H^{\mathbb{C}\dagger},\\
&QH^{\mathbb{R}} = [\Phi, \chi^{\mathbb{R}}], && Q\chi^{\mathbb{R}} = H^{\mathbb{R}},\\
&Q\bar{\Phi} = \eta, && Q\eta = [\Phi, \bar{\Phi}],\\
&Q\Phi = 0.
\end{aligned}$$

(I)Not Homogeneous transformation of $\mathcal{A}$

Homogeneous property

$Q \longrightarrow$ **tangent vector**

with $\tilde{Q} = \sum_{\mathrm{n}} X_{\mathrm{n}} \frac{\partial}{\partial \lambda_{\mathrm{n}}} + X_{\mathrm{n}}^{\dagger} \frac{\partial}{\partial \lambda_{\mathrm{n}}^{\dagger}} + Y_{\mathrm{n}} \frac{\partial}{\partial \tilde{\lambda}_{\mathrm{n}}} + Y_{\mathrm{n}}^{\dagger} \frac{\partial}{\partial \tilde{\lambda}_{\mathrm{n}}^{\dagger}} + \bar{\Phi}_{\mathrm{n}} \frac{\partial}{\partial \eta_{\mathrm{n}}} + \vec{\chi}_{\mathrm{n}} \cdot \frac{\partial}{\partial \vec{H}_{\mathrm{n}}}$

$Q = \sum_{\mathrm{n}} \Big[\lambda_{\mathrm{n}} \frac{\partial}{\partial \ldots} \ldots$

$+ (\Phi_{\mathrm{n}} Y_{\mathrm{n}} - Y_{\mathrm{n}} \Phi_{\mathrm{n+j}}) \frac{\partial}{\partial \tilde{\lambda}_{\mathrm{n}}} + (\Phi_{\mathrm{n}} Y_{\mathrm{n}}^{\dagger} - Y_{\mathrm{n}}^{\dagger} \Phi_{\mathrm{n-j}}) \frac{\partial}{\partial \tilde{\lambda}_{\mathrm{n}}^{\dagger}}$

$+ \vec{H}_{\mathrm{n}} \cdot \frac{\partial}{\partial \vec{\chi}_{\mathrm{n}}} + [\Phi_{\mathrm{n}}, \bar{\Phi}_{\mathrm{n}}] \frac{\partial}{\partial \eta_{\mathrm{n}}} \Big]$

Number operator $\boxed{\hat{N}_{\mathcal{A}}}$ as $\boxed{\{Q, \tilde{Q}\} = \hat{N}_{\mathcal{A}} \cdot}$

counts the number of fields in $\vec{\mathcal{A}}_{\mathrm{n}} = (\vec{\mathcal{B}}_{\mathrm{n}}, \vec{\mathcal{F}}_{\mathrm{n}})$

Q **closed term including the field of** $\vec{\mathcal{A}}_{\mathrm{n}} = (\vec{\mathcal{B}}_{\mathrm{n}}, \vec{\mathcal{F}}_{\mathrm{n}})$

$\longrightarrow Q$ ***exact form***

$$\boxed{h_{n_{\mathcal{A}}} =} n_{\mathcal{A}}^{-1} \hat{N}_{\mathcal{A}} \cdot h_{n_{\mathcal{A}}} = n_{\mathcal{A}}^{-1} \{Q, \tilde{Q}\} h_{n_{\mathcal{A}}} = \boxed{n_{\mathcal{A}}^{-1} Q \cdot (\tilde{Q} h_{n_{\mathcal{A}}})}$$

*(II)*Gauge symmetry under Q on the lattice

* ***(II) Gauge transformation laws do not change under BRST transformation***

$$A_\mu \to g A_\mu g^{-1} + g\partial_\mu g^{-1}$$

$\Downarrow$ Q BRST

$$\lambda_\mu \to g \lambda_\mu g^{-1}$$

in continuum theory

3.2 BRST cohomology on the lattice theory

(K.Ohta, T.T (2007))

BRST cohomology
cannot be realized!

Only the polynomial of Φ
can be BRST cohomology

Essence of the proof of the result

Q closed terms $h_{n_{\mathcal{A}}}$ **including the fields in** $\vec{\mathcal{A}}_{\mathrm{n}}$

$h_{n_{\mathcal{A}}}$ ***must be BRST exact*** .

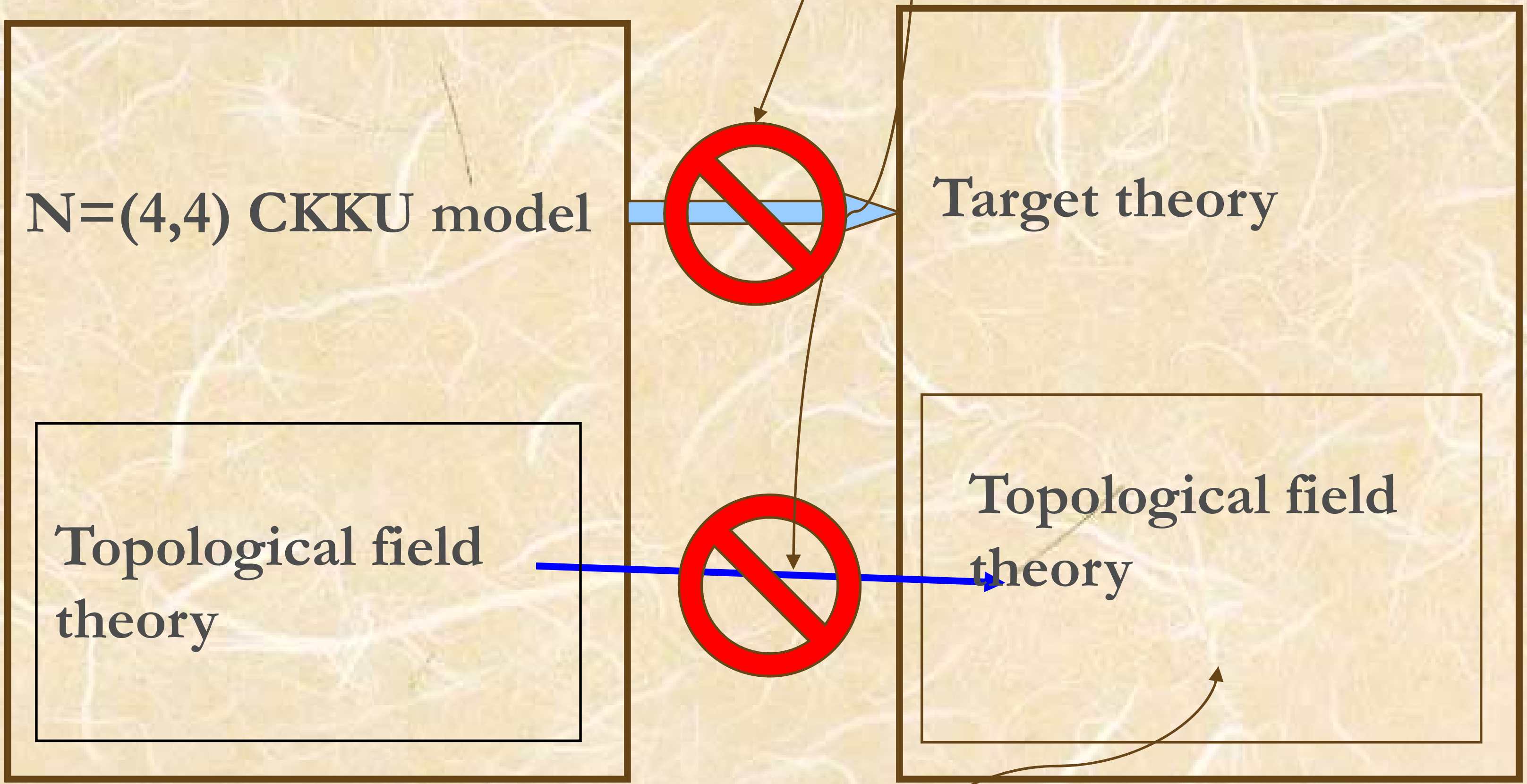
(I) Homogeneous property of Q

Only polynomial of Φ can be BRST cohomology

(II)

$h_{n_{\mathcal{A}}}$: gauge invariant $\Longleftrightarrow$ $\tilde{Q}\, h_{n_{\mathcal{A}}}$: gauge invariant

N=(4,4) CKKU model without mass term *would* not recover the target theory non-perturbatively



BRST cohomology are composed by λ, A, Φ

<u>5. Summary</u>

The topological property

(like as BRST cohomology)

could be used as

a non-perturbative criteria to judge

<u>whether supersymmetic lattice theories</u>

(which preserve BRST charge)

<u>have the desired continuum limit or not.</u>

- **We apply the criteria to N= (4,4) CKKU model without mass term**

The target continuum limit would not be realized

→ ***Implication by an explicit form.***

Perturbative study

→ did not show it

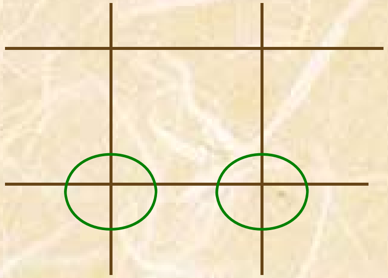
It can be a powerful criteria.

Comment on the No-go result

(I)and (II) plays the crucial role.

⇒ ***These relate with the gauge transformation law on the lattice.***

Gauge parameters are defined on each sites as the ***independent parameters***.



Vn **Vn+i**

$$A_\mu \to g A_\mu g^{-1} + \cancel{g \partial_\mu g^{-1}}$$

$$Q\lambda_\mu = \cancel{\partial_\mu \Phi} - [\Phi, A_\mu]$$

Contribute to the realization of BRST cohomology in the continuum theory

BRST cohomology $\mathcal{O}_k \equiv \int_{\gamma_k} \mathcal{W}_k$

→ **Topological quantity** defined by the inner product of homology and the cohomology

The realization is **difficult**
due to the independence of gauge parameters
(Singular gauge transformation**)**

→ **Admissibility condition etc. would be needed**