Effective Action of Domain Wall Networks

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"Webs of Domain Walls in Supersymmetric Gauge Theories", Phys.Rev.**D72** (2005) 085004,

" Non-Abelian Webs of Walls", Phys.Lett.**B632** (2006) 384,

" Effective Action of Wall Loops ", Phys.Rev.**D75** (2007) 045010,

Contents

1 Introduction

D-branes: essential to study **non-perturbative** aspects of string theories

provide models for **brane-world scenario**

Domain walls $\sim$ D-branes : **1/2 BPS**

Junctions of walls $\sim$ **Junctions** of branes : **1/4 BPS**

Domain walls are expected to make a **network-like webs**

Our purpose: To construct **all solutions** of wall webs as 1/4 BPS states in $d = 4$, $\mathcal{N} = 2$ SUSY $U(N_{\rm C})$ gauge theories with $N_{\rm F}(> N_{\rm C})$ hypermultiplets in the fundamental representation with complex masses

Results:

1. **Webs of Domain Walls** are constructed as $\mathbf{1/4}$ BPS states
2. **Exact Solutions** of Webs of Walls are obtained for $g^2 \to \infty$.
3. Normalizable moduli of web of walls = **loops of walls**
4. **Metric** of a single triangle loop of walls is explicitly worked out and can be understood as **kinetic energy of walls and junctions**.

SUSY $U(N_{\rm C})$ Gauge Theory with $N_{\rm F}$ Flavors

$\mathcal{N}=2$ SUSY in 3+1 dim. $\mu,\nu=0,1,2,3,\ \alpha,\beta=1,2$

Vector multiplets : W_μ Gauge field, Σ_α 2 Real Scalars ($N_{\rm C}\times N_{\rm C}$ matrix)

Gauge coupling g for $U(N_{\rm C})$, Fayet-Iliopoulos (FI) parameter c

Hypermultiplets : $(H^i)^{rA}\equiv H^{irA}$ Complex Scalar ($N_{\rm C}\times N_{\rm F}$ matrix)

($i=1,2$; Color $r=1,\cdots,N_{\rm C}$; Flavor $A=1,\cdots,N_{\rm F}$)

Hypermultiplet Masses $(M_1)^A{}_B\equiv m_A\delta^A{}_B,(M_2)^A{}_B\equiv n_A\delta^A{}_B$

Non-degenerate masses: $m_A+in_A\neq m_B+in_B$

Minimal kinetic terms

$$\mathcal{L}=\mathrm{Tr}\left[-\frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu}+\frac{1}{g^2}\sum_{\alpha=1}^{2}\mathcal{D}_\mu\Sigma_\alpha\mathcal{D}^\mu\Sigma_\alpha+\mathcal{D}_\mu H^i\left(\mathcal{D}^\mu H^i\right)^\dagger\right]-V$$

$$V=\mathrm{Tr}\left[\sum_{\alpha=1}^{2}\left(H^iM_\alpha-\Sigma_\alpha H^i\right)\left(H^iM_\alpha-\Sigma_\alpha H^i\right)^\dagger-\frac{1}{g^2}\left[\Sigma_1,\Sigma_2\right]^2\right]$$

$$+\ \mathrm{Tr}\left[(Y^3)^2/g^2+g^2H^2H^{1\dagger}H^1H^{2\dagger}\right],$$

$$Y^3\equiv g^2\left(H^1H^{1\dagger}-H^2H^{2\dagger}-c\mathbf{1}_{N_{\rm C}}\right)/2$$

Color-flavor locking vacua $\langle A_1 A_2 \cdots A_{N_C} \rangle$

$$H^{1rA} = \sqrt{c}\, \delta^{A_r}{}_A, \quad H^{2rA} = 0$$

$$\Sigma \equiv \Sigma_1 + i\Sigma_2 = \mathrm{diag}\left(m_{A_1} + i n_{A_1}, m_{A_2} + i n_{A_2}, \cdots, m_{A_{N_C}} + i n_{A_{N_C}}\right)$$

${}_{N_F}C_{N_C} = N_F!/(N_C!(N_F - N_C)!)$ discrete SUSY vacua

Higgs Phase : Walls, Vortices are the only **elementary** solitons

Instantons, monopoles, junctions are **composite** solitons

2 1/4 BPS Equations

Dependence on x^1, x^2, $\quad$ 2 D Poincaré invariance $\longrightarrow W_{0,3} = 0$

Bogomol'nyi completion of Energy density (assuming $H^2 = 0, H^1 \equiv H$)

$$\begin{aligned}
\mathcal{E} \;=\; & \mathrm{Tr}\left[\frac{1}{g^2}\left(F_{12} - i\left[\Sigma_1, \Sigma_2\right]\right)^2 + \frac{1}{g^2}\left(\mathcal{D}_1\Sigma_2 - \mathcal{D}_2\Sigma_1\right)^2\right. \\
& + \sum_{\alpha=1,2}\left(\mathcal{D}_\alpha H - H M_\alpha + \Sigma_\alpha H\right)\left(\mathcal{D}_\alpha H - H M_\alpha + \Sigma_\alpha H\right)^\dagger \\
& \left. + \frac{1}{g^2}\left(\mathcal{D}_1\Sigma_1 + \mathcal{D}_2\Sigma_2 - Y^3\right)^2\right] + \mathcal{Z}_1 + \mathcal{Z}_2 + \mathcal{Y} + \sum_{\alpha=1,2}\partial_\alpha J_\alpha
\end{aligned}$$

$$\mathcal{Z}_1 \equiv c\partial_1 \mathrm{Tr}\Sigma_1, \quad \mathcal{Z}_2 \equiv c\partial_2 \mathrm{Tr}\Sigma_2, \quad \mathcal{Y} \equiv \frac{2}{g^2}\partial_\alpha \mathrm{Tr}\left(\epsilon^{\alpha\beta}\Sigma_2 \mathcal{D}_\beta \Sigma_1\right)$$

1/4 BPS equations for domain wall webs

$$F_{12} = i\left[\Sigma_1, \Sigma_2\right], \quad \mathcal{D}_1\Sigma_2 = \mathcal{D}_2\Sigma_1$$
$$(\mathcal{D}_1 + \Sigma_1)H = HM_1, \quad (\mathcal{D}_2 + \Sigma_2)H = HM_2$$
$$\mathcal{D}_1\Sigma_1 + \mathcal{D}_2\Sigma_2 = Y^3$$

Integrability condition

$$F_{12} = i\left[\Sigma_1, \Sigma_2\right], \mathcal{D}_1\Sigma_2 = \mathcal{D}_2\Sigma_1 \longrightarrow \left[\mathcal{D}_1 + \Sigma_1, \mathcal{D}_2 + \Sigma_2\right] = 0$$

$\longrightarrow N_{\mathrm{C}} \times N_{\mathrm{C}}$ non-singular matrix $S(x^\alpha)$ as simultaneous solution

$$W_1 - i\Sigma_1 = -iS^{-1}\partial_1 S, \; W_2 - i\Sigma_2 = -iS^{-1}\partial_2 S$$

Hypermultiplet BPS equations are solved by $S(x^\alpha)$ as

$$H = S^{-1}H_0 e^{M_1 x^1 + M_2 x^2}$$

Moduli matrix H_0: $N_{\mathrm{C}} \times N_{\mathrm{F}}$ constant complex matrix of rank N_{C}

Master equation in terms of a **gauge invariant** matrix $\Omega \equiv SS^\dagger$

$$\frac{1}{cg^2}\left[\partial_1\left(\partial_1\Omega\Omega^{-1}\right) + \partial_2\left(\partial_2\Omega\Omega^{-1}\right)\right] = 1_{N_{\mathrm{C}}} - \Omega_0\Omega^{-1}$$

$$\Omega_0 \equiv c^{-1} H_0 e^{2(M_1 x^1 + M_2 x^2)} H_0^\dagger$$

Moduli matrix H_0 contains all moduli parameters

(H_0, S) and (H_0', S') give the same configurations, if related by

$$H_0 \to H_0' = V H_0, \quad S \to S' = V S, \quad V \in GL(N_{\rm C}, {\rm C})$$

Independent moduli are equivalence class defined by $(H_0, S) \sim (H_0', S')$

The **total moduli space**: the **complex Grassmann manifold**

$$\mathcal{M}_{\rm tot}^{\rm webs} \simeq G_{N_{\rm F}, N_{\rm C}} = \{H_0 \mid H_0 \sim V H_0,\ V \in GL(N_{\rm C}, {\rm C})\}$$

Existence and uniqueness of solutions of the master equation to be proved

Exact Solution at $g \to \infty$: NLSM

Strong coupling limit $g^2 c/\Delta m \gg 1$: BPS Eq. for $\Omega \longrightarrow$ Algebraic equation

$$\Omega^{g\to\infty} = \Omega_0 = c^{-1} H_0 e^{2(M_1 x^1 + M_2 x^2)} H_0^\dagger$$

Abelian gauge theory ($N_{\rm C} = 1$): configurations of scalar fields are

$$H^A = \sqrt{c} \frac{H_0^A e^{m_A x^1 + n_A x^2}}{\sqrt{\sum_{B=1}^{N_{\rm F}} |H_0^B|^2 e^{2(m_B x^1 + n_B x^2)}}}$$

3 Webs of Walls

Take $U(1)$ gauge theory as a simple example

Moduli matrix: $H_0 = \sqrt{c}(e^{a_1+ib_1}, \cdots, e^{a_{N_F}+ib_{N_F}})$

$\log\Omega \sim \log\Omega_0$ outside the core of the wall

Position of the domain wall: equal weights of the vacua i, j

$$(m_i - m_j)x^1 + (n_i - n_j)x^2 + a_i - a_j = 0$$

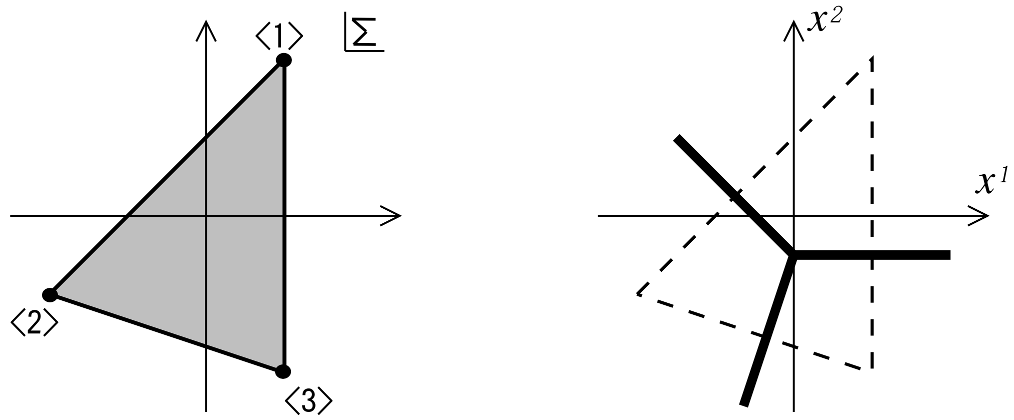


Figure 1: The minimal model for 3-pronged wall junction. The left one is the **grid diagram** in the complex Σ plane and the right one is the **web diagram** in the configuration space.

Wall Junction

$N_{\rm F} = 3$ with 3 discrete vacua labeled by $\langle A \rangle$ $(A = 1, 2, 3)$

$1/4$ BPS wall junction: a triangle with 3 vertices at $m_A + i n_A$ in Σ

Polygons in the Σ plane = *grid diagrams* ((p, q) string/5-brane webs)

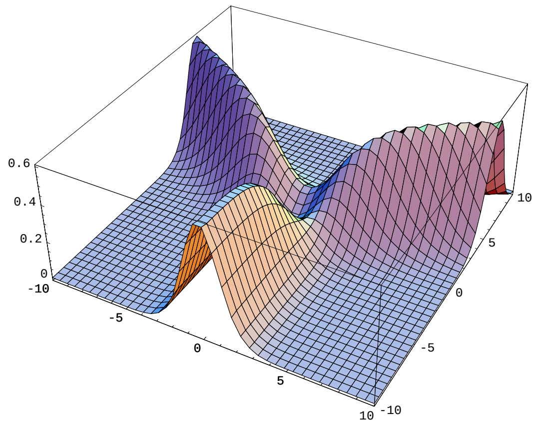


Figure 2: Binding energy at the junction point: The energy density is numerically evaluated for the moduli matrix $H_0 e^{m \cdot x} = \left(e^{x^2},\ e^{\sqrt{3}x^1/2 - x^2/2},\ e^{-\sqrt{3}x^1/2 - x^2/2}\right)$, gauge coupling $g = 1$ and FI parameter $c = 1$.

Boundary conditions for 1/4 BPS webs: $\longrightarrow$ walls at $x^1, x^2 \to \infty$

Walls with tension $T^{AB} = (Z_2^{AB}, -Z_1^{AB})$ pull junction along the wall

Central charge: $(Z_1^{AB}, Z_2^{AB}) \equiv c(m_B - m_A, n_B - n_A)$

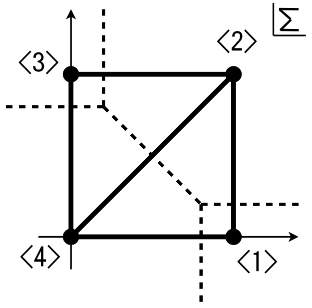
Junction configuration: **web diagram** dual to the **grid diagram**

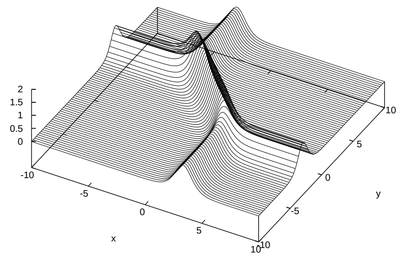
Edges of grid diagram $\leftrightarrow \boldsymbol{x}^1, \boldsymbol{x}^2 \longrightarrow \infty$ (Walls)

Area of triangle $\leftrightarrow$ junction charge $\boldsymbol{Y}$: contributes to the energy

negatively ($\boldsymbol{Y} < \mathbf{0}$) in $\boldsymbol{U}(\mathbf{1})$ gauge theories (**binding energy**)

positively ($\boldsymbol{Y} > \mathbf{0}$) in **nonAbelian junctions**





(a) grid diagram (b) energy density ($\boldsymbol{g}^2 \longrightarrow \infty$)

Figure 3: Wall web with 4 external legs of walls. Grid diagram:(a), and energy density:(b). ($[\boldsymbol{m}_A, \boldsymbol{n}_A] = \{[\mathbf{1}, \mathbf{0}], [\mathbf{1}, \mathbf{1}], [\mathbf{0}, \mathbf{1}], [\mathbf{0}, \mathbf{0}]\}$)

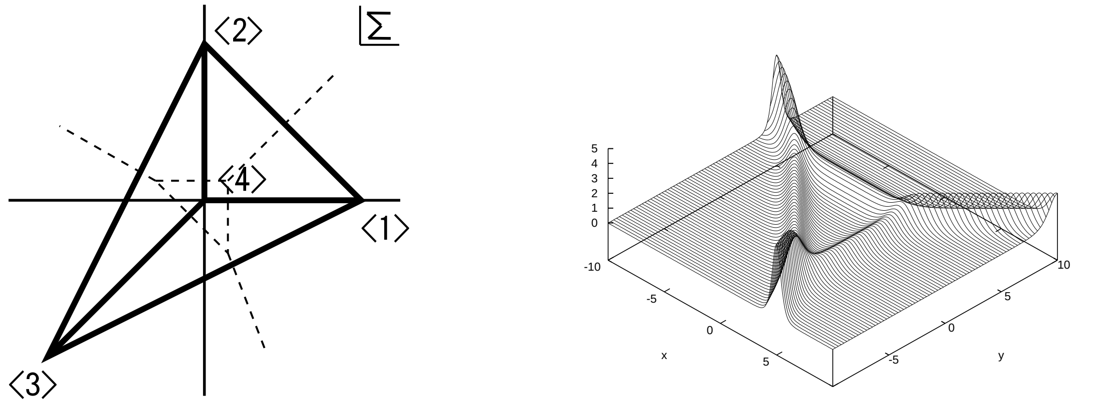
Web of Walls

$\boldsymbol{U}(\mathbf{1})$ gauge theories with $\boldsymbol{N}_{\mathbf{F}}$ flavors (Web diagrams with $\boldsymbol{N}_{\mathbf{F}}$ faces)

Two kinds of web diagrams: **tree** diagram, and diagram with **loops**

$\boldsymbol{N}_{\mathbf{F}} = \mathbf{4}$ model: Moduli matrix: $\boldsymbol{H}_0 = \sqrt{\boldsymbol{c}}\left(\boldsymbol{e}^{a_1+ib_1}, \boldsymbol{e}^{a_2+ib_2}, \boldsymbol{e}^{a_3+ib_3}, \boldsymbol{e}^{a_4+ib_4}\right)$

Homogeneous coordinates of the total moduli space $\mathbf{C}\boldsymbol{P}^3$



(a) grid diagram (b) energy density ($g^2 \to \infty$)

Figure 4: Web with 1 loop in the $N_{\mathrm{F}} = 4$ model. Grid diagram:(a), and energy density:(b). ($[m_A, n_A] = [1, 0],\ [0, 1],\ [-1, -1],\ [0, 0]$)

Wall Loops

Different choices of mass in the $N_{\mathrm{F}} = 4$ model $\longrightarrow$ wall web with a **loop**

4 Effective Action of Wall Loops

Size of the Loop (and associated phase) is the **normalizable moduli**

$$H_0 = \sqrt{c}(1, 1, 1, \phi) \quad \text{with} \quad \phi = e^{r+i\theta}$$

$$\mathcal{L}^{eff} = K_{ij^*}(\phi, \phi^*)\partial^\mu \phi^i \partial_\mu \phi^{j*}, \quad K(\phi, \phi^*) = K_w(\phi, \phi^*) + K_g(\phi, \phi^*)$$

$$K_w(\phi, \phi^*) \equiv \int d^2x\, c\, \mathrm{logdet}\Omega, \quad K_g(\phi, \phi^*) \equiv \int d^2x\, \frac{1}{2g^2}\mathrm{Tr}(\Omega^{-1}\partial_\alpha \Omega)^2$$

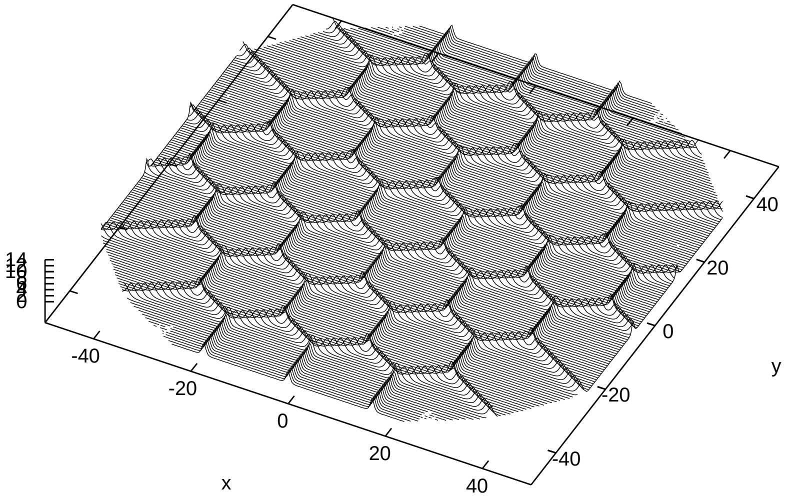


Figure 5: Honeycomb web of domain walls which divides 37 vacua ($\mathbf{C}\boldsymbol{P}^{36}$) with 18 external walls and 19 internal loops of walls.

Take $\boldsymbol{U(1)}$ gauge theory with $\boldsymbol{N_{N_\mathrm{F}} = 4}$, $\boldsymbol{[m_4, n_4] = [0, 0]}$

Metric at Small Loops in strong coupling limit $\boldsymbol{g^2 \to \infty}$

$$\boldsymbol{K_w \equiv c \int d^2x \left[\log \Omega_0 - \log \tilde{\Omega}_0\right] = c \int d^2x \, \log\left(1 + \frac{|\phi|^2}{\tilde{\Omega}_0}\right)}$$

$$\boldsymbol{\Omega_0 = e^{2m_1\cdot x} + e^{2m_2\cdot x} + e^{2m_3\cdot x} + |\phi|^2}$$

$$\boldsymbol{\tilde{\Omega}_0 \equiv e^{2m_1\cdot x} + e^{2m_2\cdot x} + e^{2m_3\cdot x}, \quad m_A \cdot x \equiv m_A x^1 + n_A x^2}$$

Areas of triangles in field space $\boldsymbol{\Delta_{[123]}}$

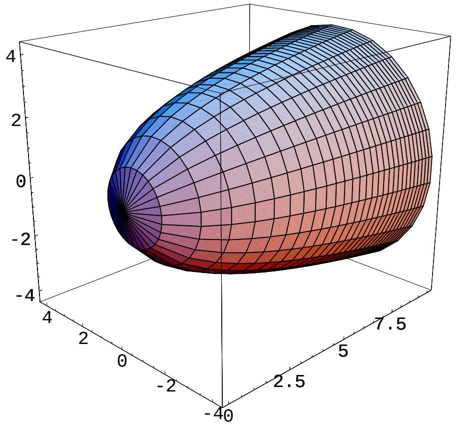


Figure 6: The moduli space of single triangle loop around $\phi = 0$ where the loop shrinks. $U(1)$ isometry is the phase modulus. The other direction is the size modulus of the loop.

For $|\phi|^2 \leq \exp\left(-\sum \alpha_i \log \alpha_i\right)$, $\quad (\alpha_i \equiv (m_j \times m_k)/\Delta_{[123]})$

$$K_w = \frac{c}{4\Delta_{[123]}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{\Gamma(\alpha_1 n)\Gamma(\alpha_2 n)\Gamma(\alpha_3 n)}{\Gamma(n)} |\phi|^{2n}$$

Scalar curvature is finite (**nonsingular**) even at $\phi = 0$ (vanishing loop)

$$R = \frac{16\Delta_{[123]}}{c} \frac{\Gamma(2\alpha_1)\Gamma(2\alpha_2)\Gamma(2\alpha_3)}{\left(\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\right)^2} + \mathcal{O}(|\phi|^2)$$

Metric at Large Loops (arbitrary gauge coupling g)

Tropical limit: Retaining the largest term in each region

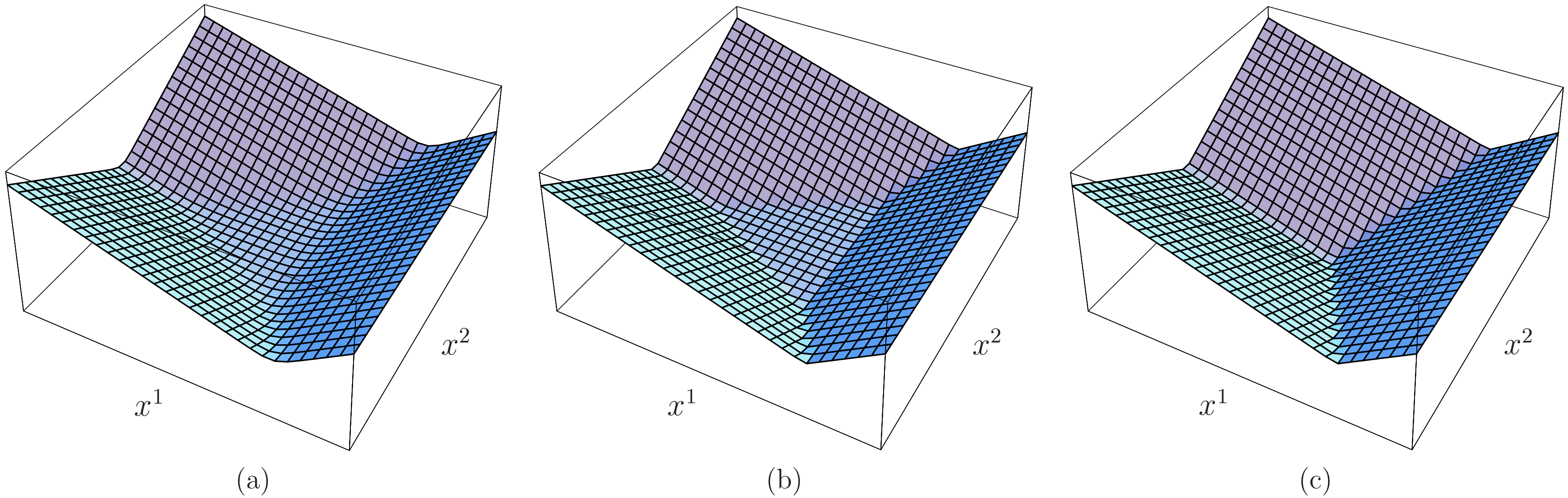


Figure 7: (a) plot of $\log \Omega_0$, (b) tropical limit of $\log \Omega_0$, (c) tropical limit of $\log \tilde{\Omega}_0$

K_w is given by c times volume of the tetrahedron

$$K_w^{trop} = \frac{c}{24\Delta_{[123]}} \frac{1}{\alpha_1\alpha_2\alpha_3} (\log|\phi|^2)^3$$

Total Kähler metric

$$ds^2 = \frac{c}{\Delta_{[123]}} \left[\frac{r}{\alpha_1\alpha_2\alpha_3} - \frac{1}{g^2 c} \left(\frac{|m_{12}|^2}{\alpha_3} + \frac{|m_{23}|^2}{\alpha_1} + \frac{|m_{31}|^2}{\alpha_2} \right) \right] (m^2 dr^2 + d\theta^2)$$

Kinetic energies of walls and junctions due to the moduli motion

5 Conclusion

1. **Webs of Domain Walls** are constructed as $\mathbf{1/4}$ BPS states in $\mathcal{N} = \mathbf{2}$ SUSY $U(N_{\rm C})$ **Non-Abelian Gauge Theories** in $\mathbf{4}$ dimensions with $N_{\rm F}$ hypermultiplets in the fundamental representation.

2. **Total moduli space** of the webs of walls is given by a **complex Grassmann manifold** described by the **moduli matrix** H_0

$$G_{N_{\rm F},N_{\rm C}} \simeq SU(N_{\rm F})/[SU(N_{\rm F}-N_{\rm C}) \times SU(N_{\rm C}) \times U(1)]$$

3. **Exact Solutions** of Webs of Walls are obtained for $g^2 \to \infty$.

4. A General Formula for the **Effective Lagrangian** is obtained.

5. **Abelian junction** has negative junction charge (binding energy). **Non-Abelian Junction** has positive junction charge (Hitchin system).

6. Normalizable moduli of web of walls = **loops of walls**

7. **Metric** of a single triangle loop of walls is explicitly worked out and can be understood as **kinetic energy of walls and junctions**.