Counting BPS Solitons and Applications

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Introduction

BPS solitons in supersymmetric gauge theory play essential roles to understand the non-perturbative dynamics and properties.

Solitons	Codimensions
Instanton	4
Monopole	3
Voretex	2
Domain wall	1

It is very important to investigate the structure (topology, singularity, metric, etc.) of the moduli space of these solitons, but I would like to concentrate on calculus of the "volume" of the moduli space of solitons.

- Why is the "volume" important?

Application 1

Thermodynamical partition function for the BPS solitons

$$
\begin{aligned}
Z &= \frac{1}{(2\pi\hbar)^{2N}} \int_{T^*\mathcal{M}_N} e^{-H/T} \frac{\omega_{\mathrm{can}}^{2N}}{(2N)!} \\
&= \left(\frac{T}{2\pi\hbar^2}\right)^N \int_{\mathcal{M}_N} \frac{\omega^N}{N!} \\
&= \left(\frac{T}{2\pi\hbar^2}\right)^N \mathrm{Vol}(\mathcal{M}_N)
\end{aligned}
$$

Cotangent bundle over the moduli space of N solitons

Application 2

Non-perturbative effects of BPS instantons on the prepotential of the N=2 supersymmetric gauge theory.

$$
\begin{aligned}
Z_{\mathrm{inst}}(\vec{a};\Lambda;\epsilon) &= \sum_{k=0}^{\infty} \Lambda^{2rk} \int_{\mathcal{M}_{r,k}} \text{“1”} \\
&= \sum_{k=0}^{\infty} \Lambda^{2rk} \mathrm{Vol}(\mathcal{M}_{r,k})
\end{aligned}
$$

moduli space of $U(r)$ k-instantons

$$
\begin{aligned}
\Longrightarrow \mathcal{F}(\vec{a};\Lambda) &= \lim_{\epsilon \to 0} \epsilon^2 \log Z_{\mathrm{inst}}(\vec{a};\Lambda;\epsilon) \\
&= \sum_{k=0}^{\infty} \mathcal{F}_k(\vec{a}) \Lambda^{2rk}
\end{aligned}
$$

Prepotential

[Nekrasov (2002)]

In this talk, I explain how to calculate the volume of the moduli space of BPS solitons, in particular vortices in 2d, or the thermodynamical partition function.

I present a novel and simple method by using a statistical model of gas in 1d.

The Model

We consider the BPS vortices in the supersymmetric gauge theory with 8 supercharges. (G=$U(N_c)$ and N_f flavors)

$$
\begin{array}{ll}
A_\mu & \text{:gauge field} \\
\Sigma & \text{:adjoint (hermite) scalar} \\
H,\ \tilde{H} & \text{:hypermultiplets}
\end{array}
$$

Bosonic part of the Lagrangian

$$\mathcal{L} = \mathrm{Tr}\left[-\frac{1}{2g^2}F_{MN}F^{MN} + \mathcal{D}_M H \left(\mathcal{D}^M H\right)^\dagger - \frac{g^2}{2}\left(HH^\dagger - c\mathbf{1}_{N_\mathrm{c}}\right)^2\right]$$

BPS Vortices on T^2

From the static energy, we find the BPS equations for the vortices

$$\mathcal{D}_{\bar{z}} H = 0$$

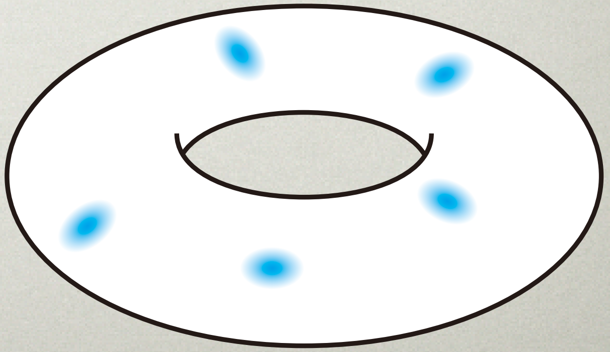
$$F_{12} + \frac{g^2}{2}\left(c\mathbf{1}_{N_{\rm c}} - HH^\dagger\right) = 0$$

Fayet-Iliopoulos (FI) parameter

If we consider the vortices on a torus, there exists a bound for the vortex number

$$k\boxed{\frac{1}{N_{\rm c}}\frac{4\pi}{g^2 c}} \le A$$

Number of vortices *Bradlow area* *Area of torus*



D-brane Realization of Vortices

BPS solitons can be realized by using D-brane bound states in superstring theory.

e.g. 3-dim model

$k\times$D0-branes + $N_c\times$D2-branes + $N_f\times$D6-branes in $\mathbf{R}^{1,2}\times\mathbf{C}^2/Z_2\times\mathbf{R}^3$

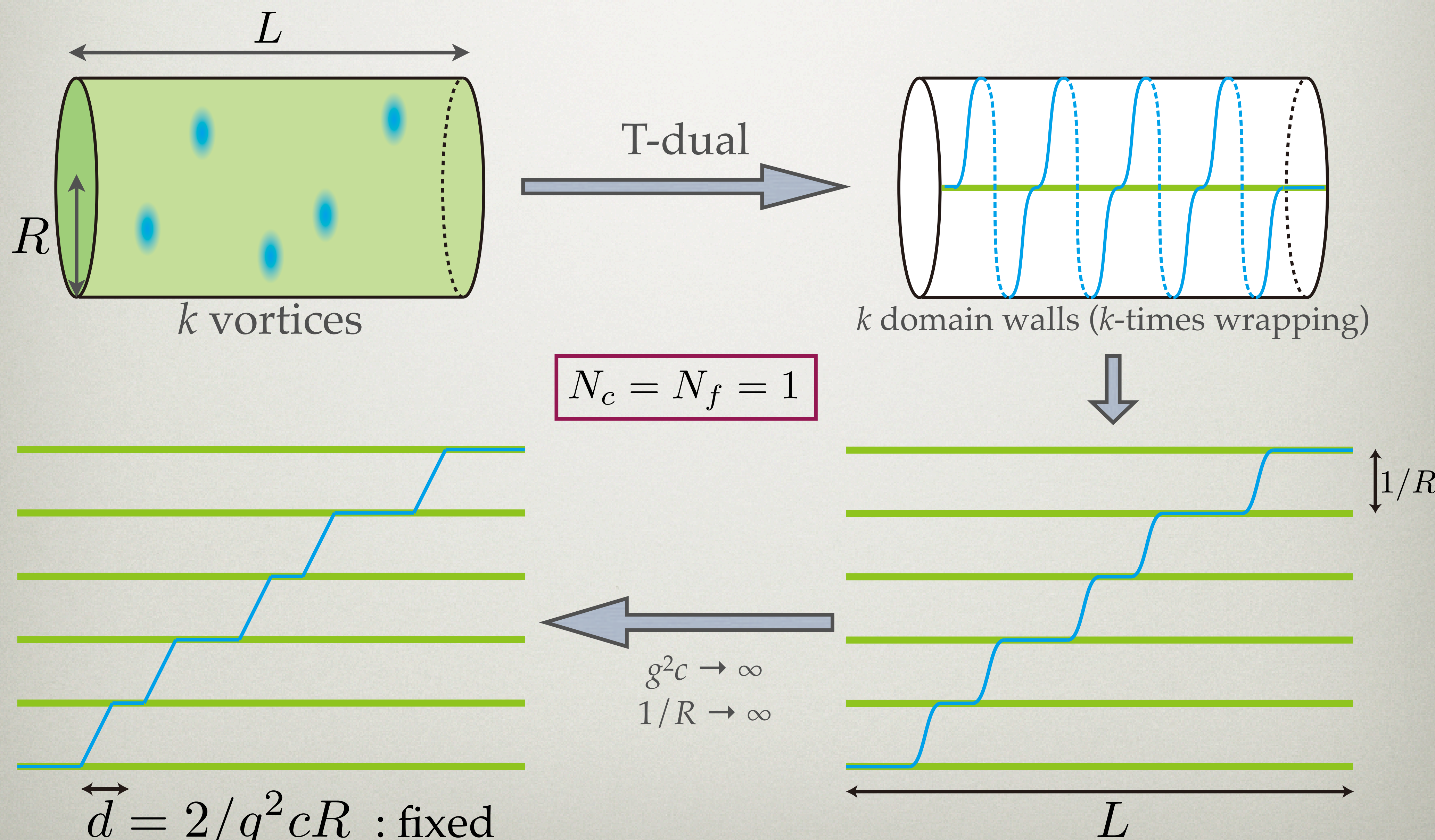
T-duality in superstring theory maps the vortex configuration to a domain-wall (kinky D-brane) configuration.

$$\hat{\Sigma}(x^1) = -\frac{1}{2\pi i R}\log\left[\mathbf{P}\exp\left(i\int_0^{2\pi R} dx^2\, A_2(x^1,x^2)\right)\right]$$

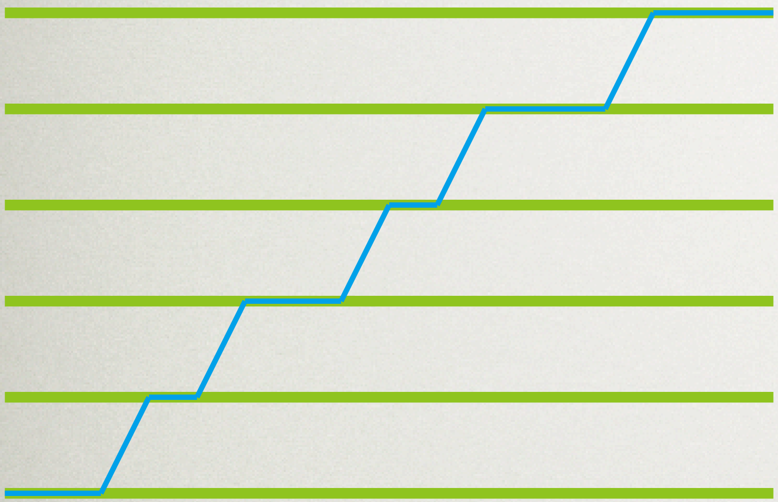
Vev of adjoint scalar

Wilson line of the gauge field

T-duality Mapping of Vortices

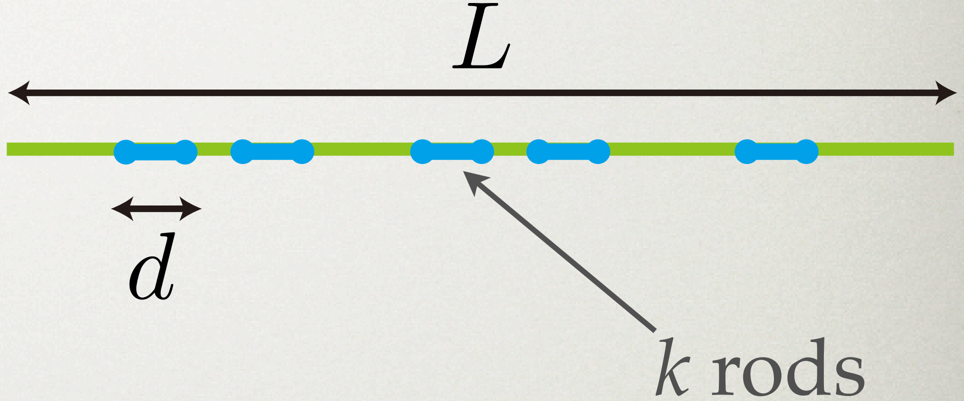


Gas of hard rods in 1D



Domain-wall kink configuration

$=$



k hard rods with length d on S^1

(Volume of the vortex moduli space)

= (Volume of dual domain-wall configuration space)

= (Volume of the configuration space of hard rods with length d on S^1)

$$= \frac{L(L-kd)^{k-1}}{k!}$$

Partition Function

The partition function for the gas of identical hard rods with mass m on S^1 with period L

$$Z_{\mathrm{rods}} = \left(\frac{mT}{2\pi}\right)^k \frac{L\,(L-kd)^{k-1}}{k!}$$

where $m=2\pi c$ and $d=2/g^2 cR$ in terms of the gauge theory vortices

T-duality ($c \to 2\pi Rc$)

The partition function of k-vortex system on a torus T^2 with area A

$$\begin{aligned} Z_{k,T^2}^{N_\mathrm{c}=N_\mathrm{f}=1} &= \frac{1}{k!}\,(cT)^k\,L\,(L-kd)^{k-1}\,(2\pi R)^k \\ &= \frac{1}{k!}\,(cT)^k\,A\left(A-k\frac{4\pi}{g^2 c}\right)^{k-1} \end{aligned}$$

where $A=2\pi RL$

Equation of State

Using the derived partition function, we obtain the van der Waals equation of state

$$P\left(A - k\boxed{\frac{4\pi}{g^2 c}}\right) = kT$$

Bradlow area

which agrees with [Manton-Nasir (1999), Manton-Sutcliffe (2004)] without any knowledge of the vortex moduli space metric!

$$k\frac{4\pi}{g^2 c} \leq A$$

Other Examples

Similarly, we can calculate

- k semi-local vortices with N_c=1 and general N_f

$$
\begin{aligned}
Z_{k,T^2}^{N_c=1,N_f} &= \left(\frac{T}{2\pi}\right)^{kN_f}(2\pi c)^{kN_f}(2\pi R)^{kN_f}\frac{1}{k}\frac{1}{(kN_f-1)!}L\,(L-dk)^{kN_f-1} \\
&= (cT)^{kN_f}\,\frac{1}{k}\frac{1}{(kN_f-1)!}A\left(A-\frac{4\pi k}{g^2c}\right)^{kN_f-1}
\end{aligned}
$$

- k local non-Abelian vortices with N_c=N_f=N

$$
Z_{k,T^2}^{N_c=N_f=N} = (cT)^{kN}\,\frac{1}{k!}\boxed{\left[\frac{A}{(N-1)!}\left(\frac{4\pi}{g^2c}\right)^{N-1}\right]^k}\left[1-D_N(k-1)\frac{k}{A}+\mathcal{O}\left(\left(\frac{4\pi}{g^2cA}\right)^2\right)\right]
$$

where $\dfrac{D_N}{4\pi/g^2c}=\dfrac{(2N-2)!!}{(2N-1)!!}=1,\ \dfrac{2}{3},\ \dfrac{8}{15},\ \dfrac{16}{35},\ \dfrac{128}{315},\cdots$

Large Area Limit

For N_c=N_f=1, in the limit of $A \to \infty$

$$\mathrm{Vol}(\mathcal{M}_k) = \frac{A(A-kd)^{k-1}}{k!} \simeq \frac{A^k}{k!}$$

This result agrees with Nekrasov-like localization method (Ω-background, Equivariant cohomology, etc.) as follows...

Reduced Matrix Model in Ω-background

Let us consider the following reduced matrix model partition function (from N=1 in 4d)

$$Z_k = \int [d\vec{\mathcal{B}}][d\vec{\mathcal{F}}][d\Phi] e^{-S(\vec{\mathcal{B}},\vec{\mathcal{F}},\Phi)}$$

where the action is BRST exact

$$S = \frac{1}{g^2} Q\Xi(\vec{\mathcal{B}},\vec{\mathcal{F}},\Phi)$$

and $\vec{\mathcal{B}} = (X, I, H)$, $\vec{\mathcal{F}} = (\lambda, \psi, \chi)$
which obey the following BRST transformations

$$\begin{aligned} &QX = \lambda, \quad Q\lambda = [\Phi, X] + \epsilon X \\ &QI = \psi, \quad Q\psi = \Phi I \\ &QH = [\Phi, \chi], \quad Q\chi = H \\ &Q\Phi = 0 \end{aligned}$$

Vortex Partition Function

The partition function reduces to the following residue integral with respect to eigenvalues of Φ.

$$Z_k = \oint \prod_{i=1}^{k} \frac{d\phi_i}{2\pi i \phi_i} \prod_{i<j} \frac{(\phi_i - \phi_j)^2}{(\phi_i - \phi_j)^2 - \epsilon^2}$$

The poles exist at $\phi_i = \epsilon(i-1)$

$$Z_k = \frac{1}{\epsilon^k k!}$$

In the limit of $\epsilon \to 0$, this gives the "volume" of the moduli space of *k* vortices in $\mathbb{C}$ with an identification of $A = 1/\epsilon$.

F-term Contribution

In general, we can obtain the non-perturbative contribution of *k*-vortices to the twisted superpotential of 2d *N*=(2,2) supersymmetric gauge theory from the partition function ("volume" of the moduli space of *k*-vortices) [Shadchin 2006]

$$Z_k(\vec{a},\epsilon) \quad = \quad \sum_{|\vec{k}|=k} \frac{\prod_{l=1}^{N_c}\prod_{f=1}^{N_f}\prod_{i_l=1}^{k_l}(a_l+m_f+\epsilon i_l)}{\prod_{l,n=1}^{N_c}\prod_{i_l=1}^{k_l}(a_l-a_n+\epsilon(k_l-k_m-i_l))}\Lambda^{2N_c k}$$

Conclusion

- We find a novel and simple method to compute the volume (partition function) of the BPS vortex moduli space.
    - It is equivalent to the configuration space of the hard rods system in 1-dimensional circle.
- This derivation does not need the detail structure of the moduli space like metric.
    - This is due to the “localization” property of the supersymmetric gauge theories.

Further Application

- Landscape (counting BPS vacua)
- Counting BPS states in SUGRA
- Kähler potential of Calabi-Yau manifold
- Themodynamics of vortices in early universe