Methodical Accelerator Design
Highlights on Coupling Calculations
Part I – Initialization

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Objectives

- Establish equations given in the MAD8 Phy.G.
- Cross-check equations used by MAD-X code.
- Record assumptions and limitations.
- Discussion...

Sector map

The TRANSVERSE LINEAR COUPLING calculation for **stable motion** in MAD-X consists of finding *some* similarity R_M that transforms a **sector map** M into its normal form $M_\perp$ and apply *some* parametrization.

$$\vec{X}(s_2) = M\vec{X}(s_1),$$

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = R_M \begin{pmatrix} E & 0 \\ 0 & F \end{pmatrix} R_M^{-1} = R_M M_\perp R_M^{-1}.$$

Linear motion

A **linear motion** can be described by s-dependent quadratic Hamiltonian around some reference trajectory (bilinear matrix form):

$$\mathcal{H} = \frac{1}{2}\vec{X}^T H \vec{X},$$

where H is a real symmetrix matrix leading to the motion equation:

$$x_i' = \frac{\partial \mathcal{H}}{\partial p_i} \quad \text{and} \quad p_i' = -\frac{\partial \mathcal{H}}{\partial x_i} \quad \Rightarrow \quad \vec{X}' = SH\vec{X},$$

where S is the fundamental symplectic unit matrix:

$$S = \begin{pmatrix} S_2 & & 0 \\ & S_2 & \\ 0 & & \ddots \end{pmatrix}, \; S_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad S^{-1} = S^T = -S, \quad S^2 = -I.$$

➠ The sector map M is the solution of the motion equation.

Hamiltonian components in 4D

$$\mathcal{H}_{\text{drift}} = \frac{p_x^2 + p_y^2}{2(1+\delta_p)} \quad \text{and} \quad \delta_p^* = (1+\delta_p)^{-1} \;\Rightarrow\; H = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \delta_p^* & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_p^* \end{pmatrix},$$

$$\mathcal{H}_{\text{quad}} = \mathcal{H}_{\text{drift}} + \frac{k_1}{2}(x^2 - y^2) \;\Rightarrow\; H = \begin{pmatrix} k_1 & 0 & 0 & 0 \\ 0 & \delta_p^* & 0 & 0 \\ 0 & 0 & -k_1 & 0 \\ 0 & 0 & 0 & \delta_p^* \end{pmatrix},$$

$$\mathcal{H}_{\text{skew-quad}} = \mathcal{H}_{\text{drift}} + k_1^s xy \;\Rightarrow\; H = \begin{pmatrix} 0 & 0 & k_1^s & 0 \\ 0 & \delta_p^* & 0 & 0 \\ k_1^s & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_p^* \end{pmatrix},$$

$$\mathcal{H}_{\text{sbend}} = \mathcal{H}_{\text{drift}} + \frac{k_0}{2\rho}x^2 \quad \text{and} \quad h = \rho^{-1} \;\Rightarrow\; H = \begin{pmatrix} hk_0 & 0 & 0 & 0 \\ 0 & \delta_p^* & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_p^* \end{pmatrix},$$

$$\mathcal{H}_{\text{solenoid}} = \mathcal{H}_{\text{drift}} + k_s(yp_x - xp_y) + \frac{k_s^2}{2}(x^2 + y^2) \;\Rightarrow\; H = \begin{pmatrix} k_s^2 & 0 & 0 & -k_s \\ 0 & \delta_p^* & k_s & 0 \\ 0 & k_s & k_s^2 & 0 \\ -k_s & 0 & 0 & \delta_p^* \end{pmatrix}.$$

Hamiltonian of the system in 4D

The Hamiltonian of the system is (from previous examples):

$$\mathcal{H} = \frac{p_x^2 + p_y^2}{2(1+\delta_p)} + \frac{1}{2}(Fx^2 + Gy^2) + Kxy + L(yp_x - xp_y).$$

In matrix notation:

$$H = \begin{pmatrix} F & 0 & K & -L \\ 0 & \delta_p^* & L & 0 \\ K & L & G & 0 \\ -L & 0 & 0 & \delta_p^* \end{pmatrix},$$

where:

$$\begin{aligned} F &= hk_0 + k_1 + k_s^2, & G &= -k_1 + k_s^2, \\ K &= k_1^s, & L &= k_s. \end{aligned}$$

➠ Any bilinear Hamiltonian can be brought to this form by a suitable canonical transformation.

Symplectic condition

From two independent solutions $\vec{X}_1$ and $\vec{X}_2$, we can verify that the bilinear form $\vec{X}_2^T S \vec{X}_1$ is an invariant of the Hamiltonian flow:

$$\begin{aligned}
(\vec{X}_2^T S \vec{X}_1)' =& \vec{X}_2^{T\prime} S \vec{X}_1 + \vec{X}_2^T S \vec{X}_1' \\
=& (SH\vec{X}_2)^T S \vec{X}_1 + \vec{X}_2^T S (SH\vec{X}_1) \\
=& \vec{X}_2^T H^T S^T S \vec{X}_1 + \vec{X}_2^T SSH\vec{X}_1 \\
=& \vec{X}_2^T H^T \vec{X}_1 - \vec{X}_2^T H \vec{X}_1 = 0. \\
\vec{X}_2^T(s_2) S \vec{X}_1(s_2) =& \vec{X}_2^T(s_1) S \vec{X}_1(s_1). \qquad \text{(Liouville Theorem)}
\end{aligned}$$

Using the sector map:

$$\begin{aligned}
\vec{X}_1(s_2) =& M\vec{X}_1(s_1) \quad \text{and} \quad \vec{X}_2(s_2) = M\vec{X}_2(s_1), \\
\vec{X}_2^T(s_2) S \vec{X}_1(s_2) =& (M\vec{X}_2(s_1))^T S M \vec{X}_1(s_1) \\
=& \vec{X}_2^T(s_1) M^T S M \vec{X}_1(s_1). \qquad \text{(validity check)}
\end{aligned}$$

➠ Identification of terms leads to the **symplectic condition** $M^T S M = S$.

Symplectic structure

Assuming a **stable linearized motion** implies:

- $M \in \mathrm{Sp}(2n, \mathbb{R})$ (symplectic real matrix)
- $M^T S M = S$ (symplectic condition)
- $\det(M) = 1$ (symplectic matrix)
- $\{\,(\lambda_i, \lambda_i^{-1}), i = 1..n\,\}$ (symplectic matrix)
- $|\lambda| = 1$ (stable motion)
- Idem for M^{-1} (symplectic condition)
- Idem for $M_\perp$ (similarity property)

➠ Given that λ is an eigenvalue of $M \in \mathrm{Sp}(2n, \mathbb{R})$:

$$|M^T - \lambda I| = 0 \overset{\times SM}{\Longrightarrow} |S - \lambda SM| = 0 \overset{\lambda^{-1} S \times}{\Longrightarrow} |M - \frac{1}{\lambda} I| = 0.$$

Symplectic conjugate

From the symplectic condition:

$$M^T S M = S \quad \Rightarrow \quad M^{-1} = S^{-1} M^T S = -S M^T S = \bar{M}$$

The **symplectic conjugate** of a $2n \times 2n$ real matrix is defined by:

$$\bar{A} = -S A^T S \qquad\qquad (\ A \in \mathrm{Sp}(2n, \mathbb{R}) \ \Leftrightarrow \ \bar{A} = A^{-1}\ )$$

For a 2×2 real matrix,

$$\bar{A} = \overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{and} \quad \begin{array}{rcl} A\bar{A} = \bar{A}A & = & (ad - bc)I = \det(A)I, \\ A + \bar{A} & = & (a + d)I \ = \ \mathrm{tr}(A)I, \\ A + \bar{B} & = & \overline{\bar{A} + B}, \\ |A + \bar{B}| & = & |A| + |B| + \mathrm{tr}(AB). \end{array}$$

For $2n \times 2n$ real matrix,

$$\bar{A} = \overline{\left(A_{ij}\right)}_{1 \le i,j \le n} = \left(\bar{A}_{ji}\right)_{1 \le i,j \le n}$$

Symplectic relations

From the symplectic condition $M^T S M = M S M^T = S$:

$$A^T S A + C^T S C = S \overset{-S\times}{\Longrightarrow} \bar{A}A + \bar{C}C = I$$

$$A^T S B + C^T S D = S \overset{-S\times}{\Longrightarrow} \bar{A}B + \bar{C}D = 0$$

$$B^T S A + D^T S C = S \overset{-S\times}{\Longrightarrow} \bar{B}A + \bar{D}C = 0$$

$$B^T S B + D^T S D = S \overset{-S\times}{\Longrightarrow} \bar{B}B + \bar{D}D = I$$

$$A S A^T + B S B^T = S \overset{\times-S}{\Longrightarrow} A\bar{A} + B\bar{B} = I$$

$$A S C^T + B S D^T = S \overset{\times-S}{\Longrightarrow} A\bar{C} + B\bar{D} = 0$$

$$C S A^T + D S B^T = S \overset{\times-S}{\Longrightarrow} C\bar{A} + D\bar{B} = 0$$

$$C S C^T + D S D^T = S \overset{\times-S}{\Longrightarrow} C\bar{C} + D\bar{D} = I$$

$$M \in \mathrm{Sp}(4, \mathbb{R}) \;\Rightarrow\; |A| + |C| = |B| + |D| = |A| + |B| = |C| + |D| = 1,$$
$$\Rightarrow\; |A| = |D| \quad \text{and} \quad |B| = |C|.$$

Characteristic polynomial

From the symplectic condition: $M \in \mathrm{Sp}(2n, \mathbb{R})$

$$\begin{aligned} \mathrm{eig}(M) = \mathrm{eig}(M^{-1}) &= \{\, (\lambda_i, \lambda_i^{-1}), i = 1..n \,\}, \\ \mathrm{eig}(M + \bar{M}) &= \{\, \Lambda_i = \lambda_i + \lambda_i^{-1}, i = 1..n \,\}. \end{aligned}$$

Characteristic polynomial (coupled motion):

$$\begin{aligned} \det(M + \bar{M} - \Lambda I) &= \begin{vmatrix} A + \bar{A} - \Lambda I & B + \bar{C} \\ C + \bar{B} & D + \bar{D} - \Lambda I \end{vmatrix} \\ &= \begin{vmatrix} (\mathrm{tr}\, A - \Lambda) I & \overline{C + \bar{B}} \\ C + \bar{B} & (\mathrm{tr}\, D - \Lambda) I \end{vmatrix} \\ &= |(\mathrm{tr}\, A - \Lambda)(\mathrm{tr}\, D - \Lambda) I - (C + \bar{B})(\overline{C + \bar{B}})| \\ &= \left((\mathrm{tr}\, A - \Lambda)(\mathrm{tr}\, D - \Lambda) - |C + \bar{B}|\right)^2 |I| = 0. \\ &\Rightarrow (\mathrm{tr}\, A - \Lambda)(\mathrm{tr}\, D - \Lambda) - |C + \bar{B}| = 0. \end{aligned}$$

Determinant of block matrix:

$$\left| \begin{pmatrix} I & 0 \\ -CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right| = |AD - ACA^{-1}B| \overset{A \to \alpha I}{=} |AD - CB|.$$

Eigenmodes

Characteristic polynomial:

$$(\operatorname{tr} A - \Lambda)(\operatorname{tr} D - \Lambda) - |C + \bar{B}| = 0$$
$$\Lambda^2 - (\operatorname{tr} A + \operatorname{tr} D)\Lambda + \operatorname{tr} A \operatorname{tr} D - |C + \bar{B}| = 0.$$

Solving for Λ:

$$\Delta = (\operatorname{tr} A + \operatorname{tr} D)^2 - 4(\operatorname{tr} A \operatorname{tr} D - |C + \bar{B}|)$$
$$= (\operatorname{tr} A - \operatorname{tr} D)^2 + 4|C + \bar{B}|$$
$$\Lambda_A = \frac{1}{2}(\operatorname{tr} A + \operatorname{tr} D) + \frac{1}{2}\operatorname{sign}(\operatorname{tr} A - \operatorname{tr} D)\sqrt{\Delta}$$
$$\Lambda_D = \Lambda_A^{-1}\left(\operatorname{tr} A \operatorname{tr} D - |C + \bar{B}|\right) \qquad \text{(optional)}$$
$$\Lambda \in \mathbb{R} \;\Leftrightarrow\; |C + \bar{B}| \geq -\frac{1}{4}(\operatorname{tr} A - \operatorname{tr} D)^2 \qquad \text{(validity check)}$$

Solving for λ:

$$\Lambda = \lambda + 1/\lambda \;\Rightarrow\; \lambda^2 - \Lambda\lambda + 1 = 0 \;\Rightarrow\; \lambda = \frac{1}{2}\left(\Lambda \pm \sqrt{\Lambda^2 - 4}\right)$$
$$\lambda \in \mathbb{C}\backslash\mathbb{R} \;\Leftrightarrow\; \Lambda \in (-2, 2) \qquad \text{(validity check)}$$

Stable motion

From the characteristic polynomial:

$$\Lambda_{A,D} = \frac{1}{2}(\operatorname{tr} A + \operatorname{tr} D) \pm \frac{1}{2}\operatorname{sign}(\operatorname{tr} A - \operatorname{tr} D)\sqrt{\Delta}$$
$$|C + \bar{B}| = (\Lambda_{A,D} - \operatorname{tr} D)(\Lambda_{A,D} - \operatorname{tr} A) = \operatorname{tr} A \operatorname{tr} D - \Lambda_A \Lambda_D$$

Stable motion: $|\lambda| = 1 \;\Leftrightarrow\; \lambda^{-1} = \lambda^* \;\Leftrightarrow\; \lambda = e^{\pm i\mu}\;,\;\; +\mu \in [0, 2\pi)$

$$\Lambda_{A,D} = \lambda_{A,D} + 1/\lambda_{A,D} = 2\cos\mu_{A,D}$$
$$(\Lambda_A - \Lambda_D)^2 = 4(\cos\mu_A - \cos\mu_D)^2 = \Delta$$

Degenerated cases:

$\Delta < 0 \;\Rightarrow\; \Lambda \notin \mathbb{R}$	(unstability induced by coupling)
$\Lambda = \pm 2 \;\Rightarrow\; \lambda = \pm 1 \;\Rightarrow\; \mu = k\pi$	(undetermined solution)
$B = C = 0 \;\Rightarrow\; \Lambda_{A,D} = \operatorname{tr} A, D$	(uncoupled solution)
$\lvert C + \bar{B}\rvert = 0 \;\Leftrightarrow\; \Lambda_{A,D} = \operatorname{tr} A, D$	(uncoupled tunes)
$\operatorname{tr} A = \operatorname{tr} D \;\Rightarrow\; \lvert C + \bar{B}\rvert > 0$	(coupled tunes)
$\Lambda_A = \Lambda_D \;\Leftrightarrow\; \lvert C + \bar{B}\rvert = -\operatorname{tr}(A - D)^2/4$	(equal tunes)

Eigenvectors

For any non-zero eigenvectors $\vec{X}$ and $\vec{Y}$:

$$(M+\bar{M}-\Lambda_A I)\vec{X} = \begin{pmatrix}(\operatorname{tr}A-\Lambda_A)I & B+\bar{C}\\ C+\bar{B} & (\operatorname{tr}D-\Lambda_A)I\end{pmatrix}\begin{pmatrix}X\\ R_A X\end{pmatrix} = 0,$$

$$(M+\bar{M}-\Lambda_D I)\vec{Y} = \begin{pmatrix}(\operatorname{tr}A-\Lambda_D)I & B+\bar{C}\\ C+\bar{B} & (\operatorname{tr}D-\Lambda_D)I\end{pmatrix}\begin{pmatrix}R_D Y\\ Y\end{pmatrix} = 0$$

Leading to the solutions: (full coupling $R_{A,D} \approx \alpha I$)

$$R_A = \frac{C+\bar{B}}{\Lambda_A-\operatorname{tr}D}, \quad R_D = \frac{B+\bar{C}}{\Lambda_D-\operatorname{tr}A} = \frac{\overline{C+\bar{B}}}{-(\Lambda_A-\operatorname{tr}D)} = -\bar{R}_A$$

MAD8 Phy.G. eq. 7.6, $R=-R_A=\bar{R}_D$: (two typos)

$$R = -\left(\frac{1}{2}(\operatorname{tr}A-\operatorname{tr}D)+\frac{1}{2}\operatorname{sign}(\operatorname{tr}A-\operatorname{tr}D)\sqrt{\Delta}\right)^{-1}(C+\bar{B})$$

From R_A, R_D and $|B|=|C|$ we find that $BR=\bar{R}C$ and $RB=C\bar{R}$.

Normal form

The similarity R_M that block-diagonalizes M can be built from the eigenvectors found previously (MAD8 Phy.G. eq. 7.4, R $= R_M^{-1}$):

$$
\begin{aligned}
M_\perp = R_M^{-1} M R_M &= g^2 \bar{R}_M M R_M \\
&= g^2 \overline{\begin{pmatrix} I & R_D \\ R_A & I \end{pmatrix}} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I & R_D \\ R_A & I \end{pmatrix} \\
&= g^2 \begin{pmatrix} I & -R_D \\ -R_A & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I & R_D \\ R_A & I \end{pmatrix} \\
&= g^2 \begin{pmatrix} I & -\bar{R} \\ R & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I & \bar{R} \\ -R & I \end{pmatrix} = \begin{pmatrix} E & 0 \\ 0 & F \end{pmatrix},
\end{aligned}
$$

where g has to be determined such that $R_M^{-1} R_M = g^2 \bar{R}_M R_M = I$. Solving last equation for E and F gives:

$$
\begin{aligned}
E &= g^2 (A - BR - \bar{R}C + \bar{R}DR) \,, \quad 0 = B + A\bar{R} - \bar{R}D - \bar{R}C\bar{R} \\
F &= g^2 (D + RB + C\bar{R} + RA\bar{R}) \,, \quad 0 = C + RA - DR - RBR.
\end{aligned}
$$

Coupling strength and structure

In the previous normalization, the factor g^{-1} represents the strength of the coupling and gR its structure. To determine g, we solve:

$$g^2 \bar{R}_M R_M = g^2 \begin{pmatrix} I + \bar{R}R & 0 \\ 0 & I + R\bar{R} \end{pmatrix} = g^2 \left(1 + |R|\right) I = I$$

That is $g = (1 + |R|)^{-\frac{1}{2}} = |R_M|^{-\frac{1}{2}}$, gR_M is symplectic and so is $M_\perp$.

$$1 + R\bar{R} > 0 \;\Rightarrow\; |C + \bar{B}| > -(\Lambda_{A,D} - \operatorname{tr} D, A)^2 \quad \text{(validity check)}$$

From the characteristic polynomial using either $\Lambda_{A,D}$:

$$1 - R_A R_D = 1 - \frac{|C + \bar{B}|}{(\Lambda_A - \operatorname{tr} D)(\Lambda_D - \operatorname{tr} A)} = \frac{\Lambda_D - \Lambda_A}{\Lambda_D - \operatorname{tr} A} = \frac{\Lambda_A - \Lambda_D}{\Lambda_A - \operatorname{tr} D}$$

$$g = \left(\frac{\Lambda_D - \Lambda_A}{\Lambda_D - \operatorname{tr} A}\right)^{-\frac{1}{2}} = \left(\frac{\Lambda_A - \Lambda_D}{\Lambda_A - \operatorname{tr} D}\right)^{-\frac{1}{2}}$$

Edwards–Teng parametrization of gR_M is $g = \cos\theta$ and $D = -R\tan\theta$ from the 4D "symplectic" rotation of planes A and D by angle θ.

Normal form revisited

From previous block diagonal form:

$$
\begin{aligned}
0 &= (B + A\bar{R} - \bar{R}D - \bar{R}C\bar{R})R \\
&= (A - \bar{R}C)|R| + (BR - \bar{R}DR) \\
E &= g^2(A - \bar{R}C - (BR - \bar{R}DR)) \\
&= (1 + |R|)^{-1}(A - \bar{R}C)(1 + |R|) \\
&= A - \bar{R}C = A - BR
\end{aligned}
$$

$$
\begin{aligned}
0 &= (C + RA - DR - RBR)\bar{R} \\
&= (D + RB)|R| - (C\bar{R} + RA\bar{R}) \\
F &= g^2(D + RB + (C\bar{R} + RA\bar{R})) \\
&= (1 + |R|)^{-1}(D + RB)(1 + |R|) \\
&= D + RB = D + C\bar{R}
\end{aligned}
$$

MAD8 Phy.G. eq. 7.6: $E = A - BR$,
$F = D + \bar{R}C$. (typo)

MAD-X code: $E = A - BR$,
$F = D + RB$ or $F = D + C\bar{R}$. (correct)

Twiss parameters

Twiss parameters of E (and F) can be found by:

$$\det E = 1 \ , \quad |\operatorname{tr} E| \leq 2 \qquad \text{(validity check)}$$

$$E = \begin{pmatrix} E_{1,1} & E_{1,2} \\ E_{2,1} & E_{2,2} \end{pmatrix} = \begin{pmatrix} \cos\mu_A + \alpha_A \sin\mu_A & \beta_A \sin\mu_A \\ -\gamma_A \sin\mu_A & \cos\mu_A - \alpha_A \sin\mu_A \end{pmatrix}$$

$$\cos\mu_A = \frac{1}{2}\operatorname{tr} E, \quad \sin\mu_A = \operatorname{sign}(E_{1,2})\sqrt{-E_{1,2}E_{2,1} - \left(\frac{E_{1,1} - E_{2,2}}{2}\right)^2}$$

$$\beta_A = \frac{E_{1,2}}{\sin\mu_A}, \quad \gamma_A = -\frac{E_{2,1}}{\sin\mu_A}, \quad \alpha_A = \frac{E_{1,1} - E_{2,2}}{2\sin\mu_A}$$

$$M_\perp \in \mathrm{Sp}(4,\mathbb{R}) \ \Rightarrow \ \det E = 1 \ \Rightarrow \ \beta_A\gamma_A - \alpha_A^2 = 1 \qquad \text{(validity check)}$$

MAD-X code: $\gamma_A = \dfrac{1 + \alpha_A^2}{\beta_A}$

Ripken–Mais parameters

From the previous Twiss parameters, g and R, MAD-X computes:

$$\beta_{11} = g^2\beta_A, \quad \beta_{12} = \quad g^2(R_{2,2}^2\beta_D + 2R_{1,2}R_{2,2}\alpha_D + R_{1,2}^2\gamma_D)$$
$$\beta_{22} = g^2\beta_D, \quad \beta_{21} = \quad g^2(R_{1,1}^2\beta_A - 2R_{1,2}R_{1,1}\alpha_A + R_{1,2}^2\gamma_A)$$

$$\alpha_{11} = g^2\alpha_A, \quad \alpha_{12} = \quad g^2(R_{2,1}R_{2,2}\beta_D + (R_{1,2}R_{2,1} + R_{1,1}R_{2,2})\alpha_D + R_{1,2}R_{1,1}\gamma_D)$$
$$\alpha_{22} = g^2\alpha_D, \quad \alpha_{21} = -g^2(R_{2,1}R_{1,1}\beta_A - (R_{1,2}R_{2,1} + R_{1,1}R_{2,2})\alpha_A + R_{1,2}R_{2,2}\gamma_A)$$

$$\gamma_{11} = g^2\gamma_A, \quad \gamma_{12} = \quad \frac{(1-g^2)^2 + \alpha_{12}^2}{\beta_{12}} \;\text{ if } \beta_{12} \neq 0, \text{ otherwise } \gamma_{12} = 0$$
$$\gamma_{22} = g^2\gamma_D, \quad \gamma_{21} = \quad \frac{(1-g^2)^2 + \alpha_{21}^2}{\beta_{21}} \;\text{ if } \beta_{12} \neq 0, \text{ otherwise } \gamma_{21} = 0$$

Note that Ripken–Mais formalism uses eigenmodes (and eigenbasis):

$$\left\{ \frac{1}{2}\left(\lambda_{A,D} + \lambda_{A,D}^*\right) = \cos\mu_{A,D}, -\frac{1}{2i}\left(\lambda_{A,D} - \lambda_{A,D}^*\right) = -\sin\mu_{A,D} \right\}$$

Conclusions and comments

Requirements:

- ☞ Sector map M is symplectic (bug fixed in 2016).
- ☞ Stable linearized motion.

Things to look at:

- ☞ Why $|C + \bar{B}|$ should not be negative in MAD8 Phy.G.?
  i.e. Should be related to (unstable) sum of resonances...
- ☞ What about implementing the complete tracking of M,
  i.e. recompute $\Lambda_{A,D}$, R, E, and F from scratch.
- ☞ Add checks for symplectic conditions after each element.
- ☞ Handling of strong/full coupling, i.e. $R_A = \dfrac{C + \bar{B}}{\Lambda_A - \operatorname{tr} D} \approx \alpha I$.
- ☞ Check numerical instabilities,
  e.g. avoid catastrophic cancellation.