Methodical Accelerator Design
Highlights on Coupling Calculations
Part II – Propagation

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Objectives

- ☞ Establish equations given in the MAD8 Phy.G.
- ☞ Cross-check equations used by MAD-X code.
- ☞ Record assumptions and limitations.
- ☞ Discussion...

Propagation of the map

Given the known decoupled sector map $M_{1\perp}$ at point 1 (from initialization) and the transport map M_{12} from points 1 to 2, we look for the unknown decoupled sector map $M_{2\perp}$ at point 2:

$$\begin{aligned} M_2 &= M_{12} M_1 M_{12}^{-1} \\ &= M_{12} \left(R_{M_1} M_{1\perp} R_{M_1}^{-1} \right) M_{12}^{-1} \end{aligned}$$

$$\begin{aligned} M_{2\perp} &= R_{M_2}^{-1} M_2 R_{M_2} \\ &= \left(R_{M_2}^{-1} M_{12} R_{M_1} \right) M_{1\perp} \left(R_{M_1}^{-1} M_{12}^{-1} R_{M_2} \right) \\ &= W_{12} M_{1\perp} W_{12}^{-1} \end{aligned}$$

where we have $W_{12} W_{12}^{-1} = I$.

Block diagonal propagation

Since the eigen modes 1 & 2 in $M_{1\perp}$ and $M_{2\perp}$ are independent from each other, W_{12} is either **block** or anti-block diagonal:

$$R_{M_2} W_{12} = M_{12} R_{M_1}$$

$$g_2 \begin{pmatrix} I & \bar{R}_2 \\ -R_2 & I \end{pmatrix} \begin{pmatrix} E_{12} & 0 \\ 0 & F_{12} \end{pmatrix} = g_1 \begin{pmatrix} A_{12} & B_{12} \\ C_{12} & D_{12} \end{pmatrix} \begin{pmatrix} I & \bar{R}_1 \\ -R_1 & I \end{pmatrix}$$

Solving for E_{12}, F_{12} and R_2 gives (MAD8 Phys.G. eq. 7.10 and 7.11):

$$\begin{aligned} E_{12} &= \quad g_{12}(A_{12} - B_{12}R_1) \\ F_{12} &= \quad g_{12}(D_{12} + C_{12}\bar{R}_1) && \text{(typo)} \\ R_2 &= -g_{12}(C_{12} - D_{12}R_1)E_{12}^{-1} \\ &= -g_{12}(C_{12} - D_{12}R_1)\bar{E}_{12}/|E_{12}| \\ &= -\cancel{g_{12}^2}(C_{12} - D_{12}R_1)\overline{(A_{12} - B_{12}R_1)}/\left(\cancel{g_{12}^2}|A_{12} - B_{12}R_1|\right) \\ &= -(C_{12} - D_{12}R_1)\overline{(A_{12} - B_{12}R_1)}/|A_{12} - B_{12}R_1| \end{aligned}$$

where $g_{12} = g_1 g_2^{-1}$.

Anti-block diagonal propagation

Since the eigen modes 1 & 2 in $M_{1\perp}$ and $M_{2\perp}$ are independent from each other, W_{12} is either block or **anti-block** diagonal:

$$R_{M_2} W_{12} = M_{12} R_{M_1}$$

$$g_2 \begin{pmatrix} I & \bar{R}_2 \\ -R_2 & I \end{pmatrix} \begin{pmatrix} 0 & E_{12} \\ F_{12} & 0 \end{pmatrix} = g_1 \begin{pmatrix} A_{12} & B_{12} \\ C_{12} & D_{12} \end{pmatrix} \begin{pmatrix} I & \bar{R}_1 \\ -R_1 & I \end{pmatrix}$$

Solving for E_{12}, F_{12} and R_2 gives (not in MAD8 Phys.G.):

$$\begin{aligned}
E_{12} &= \quad g_{12}(B_{12} + A_{12}\bar{R}_1) \\
F_{12} &= \quad g_{12}(C_{12} - D_{12}R_1) \\
R_2 &= -g_{12}(D_{12} + C_{12}\bar{R}_1)E_{12}^{-1} \\
&= -g_{12}(D_{12} + C_{12}\bar{R}_1)\bar{E}_{12}/|E_{12}| \\
&= -\not{g}_{12}^{\not 2}(D_{12} + C_{12}\bar{R}_1)\overline{(B_{12} + A_{12}\bar{R}_1)} / \left(\not{g}_{12}^{\not 2}|B_{12} + A_{12}\bar{R}_1|\right) \\
&= -(D_{12} + C_{12}\bar{R}_1)\overline{(B_{12} + A_{12}\bar{R}_1)}/|B_{12} + A_{12}\bar{R}_1|
\end{aligned}$$

where $g_{12} = g_1 g_2^{-1}$.

$$M_{2\perp} = W_{12} M_{1\perp} W_{12}^{-1}$$

Block diagonal:

$$E_2 = E_{12} E_1 E_{12}^{-1} = E_{12} E_1 \bar{E}_{12} / |E_{12}|$$
$$= \cancel{g_{12}^2}(A_{12} - B_{12}R_1)E_1 \overline{(A_{12} - B_{12}R_1)} / \left(\cancel{g_{12}^2}|A_{12} - B_{12}R_1|\right)$$
$$F_2 = F_{12} F_1 F_{12}^{-1} = F_{12} F_1 \bar{F}_{12} / |F_{12}|$$
$$= \cancel{g_{12}^2}(D_{12} + C_{12}\bar{R}_1)F_1 \overline{(D_{12} + C_{12}\bar{R}_1)} / \left(\cancel{g_{12}^2}|D_{12} + C_{12}\bar{R}_1|\right)$$

Anti-block diagonal (flip modes 1 & 2):

$$E_2 = E_{12} F_1 E_{12}^{-1} = E_{12} F_1 \bar{E}_{12} / |E_{12}|$$
$$= \cancel{g_{12}^2}(B_{12} + A_{12}\bar{R}_1)F_1 \overline{(B_{12} + A_{12}\bar{R}_1)} / \left(\cancel{g_{12}^2}|B_{12} + A_{12}\bar{R}_1|\right)$$
$$F_2 = F_{12} E_1 F_{12}^{-1} = F_{12} E_1 \bar{F}_{12} / |F_{12}|$$
$$= \cancel{g_{12}^2}(C_{12} - D_{12}R_1)E_1 \overline{(C_{12} - D_{12}R_1)} / \left(\cancel{g_{12}^2}|C_{12} - D_{12}R_1|\right)$$

where E_2, F_2 and R_2 depend only on M_{12}, E_1, F_1 and R_1 but not g_{12}!

Flipping modes

- Modes 1 & 2 are flipped **if** determinants become negative or zero:
  $\forall\varepsilon \in \mathbb{R}^+ \to 0, \quad |A_{12} - B_{12}R_1| < \varepsilon$ or $|D_{12} + C_{12}\bar{R}_1| < \varepsilon$.
  In this case anti-block diagonal solutions must be used.
- Modes 1 & 2 can be flipped only in elements that couple the X–Y motions! (check)
- In any case, the number of mode flips should always be **even** so that eigenaxis are the same at the start and the end of the sector map, otherwise the computed phase advance will not be correct!

Twiss parameters propagation

MAD-X propagates the Twiss parameters through M_{12} using:

$$T_2 = W_{12} T_1 W_{12}^{-1} \qquad \text{with} \qquad T_{E,F} = \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

Propagation through E_{12}:

$$\begin{aligned}
\alpha_2 &= -((E_{2,1}\beta_1 - E_{2,2}\alpha_1)(E_{1,1}\beta_1 - E_{1,2}\alpha_1) + E_{1,2}E_{2,2})/(|E_{12}|\beta_1) \\
\beta_2 &= \quad ((E_{1,1}\beta_1 - E_{1,2}\alpha_1)^2 + E_{1,2}^2)/(|E_{12}|\beta_1) \\
\mu_2 &= \quad \mu_1 + \operatorname{atan}(E_{1,2}, (E_{1,1}\beta_1 - E_{1,2}\alpha_1)) \\
\gamma_2 &= \quad (1 + \alpha_2^2)/\beta_2
\end{aligned}$$

Idem for F_{12}.

Conclusions and comments

- ☞ Propagation relies on same requirements as initialization – no more, e.g. the transport map M_{12} doesn't need to be exactly symplectic.
- ☞ Is propagation of normal form less stable than full computation?
- ☞ What about implementing the complete tracking of M?
  i.e. recompute $\Lambda_{A,D}$, R, E, and F from scratch.
  Fast tracking requires that M_{12} is symplectic,
  otherwise M_{12}^{-1} must be truly computed (still reasonable).

Summary of stable linearized coupled motion

Initialization from symplectic sector map M:

$$\Delta = (\operatorname{tr} A - \operatorname{tr} D)^2 + 4|C + \bar{B}| \geq 0$$

$$R = -\left(\frac{1}{2}(\operatorname{tr} A - \operatorname{tr} D) + \frac{1}{2}\operatorname{sign}(\operatorname{tr} A - \operatorname{tr} D)\sqrt{\Delta}\right)^{-1}(C + \bar{B})$$

$$E = A - BR$$

$$F = D + RB$$

Propagation through M_{12}:

$$E_2 = E_{12}E_1\bar{E}_{12}/|E_{12}| \quad \text{with} \quad E_{12} = A_{12} - B_{12}R_1$$

$$F_2 = F_{12}F_1\bar{F}_{12}/|F_{12}| \quad \text{with} \quad F_{12} = D_{12} + C_{12}\bar{R}_1$$

$$R_2 = -(C_{12} - D_{12}R_1)\bar{E}_{12}/|E_{12}|$$

Propagation through M_{12} with mode flip:

$$E_2 = E_{12}F_1\bar{E}_{12}/|E_{12}| \quad \text{with} \quad E_{12} = B_{12} + A_{12}\bar{R}_1$$

$$F_2 = F_{12}E_1\bar{F}_{12}/|F_{12}| \quad \text{with} \quad F_{12} = C_{12} - D_{12}R_1$$

$$R_2 = -(D_{12} + C_{12}\bar{R}_1)\bar{E}_{12}/|E_{12}|$$