An Introduction to Quantum Physics
and
Relativistic Quantum Field Theory

AIMS Lectures: 23 February – 13 March 2009
Lecture Notes: Part 2

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Special Relativity

Reference material: Available from the AIMS Library.

1. Wolfgang Rindler, *Essential Relativity, Special, General, and Cosmological* (Springer-Verlag, Berlin, 1977, Revised Second Edition):
   Chapters 1 to 6.

2. Ray D'Inverno, *Introducing Einstein's Relativity* (Clarendon Press, Oxford, 1992):
   Chapters 1 to 4.

A website of interest: `http://www.phys.unsw.edu.au/einsteinlight/`

Special Relativity

1. Introduction

Newton's Mechanics

* Space/time geometry: 3d/1d Euclidean affine spaces
  invariant line elements:

$(\vec{x}_2-\vec{x}_1)^2$	$(t_2-t_1)^2$	affine space
$d\vec{x}^2$	dt^2	↕ choice of frame
$\vec{x}^2$	t^2	vector space

symmetries: translations, rotations, P, T

* dynamics: 3 Principles

Inertia	Dynamics	Interactions
Inertial Frames	$m\ddot{\vec{x}} = \vec{F}$	Action-Reaction

Inertial Frames + $m\ddot{\vec{x}} = \vec{F}$: Principle of Relativity

Action-Reaction ↕ Conservation of momentum of a free system

* Galilei group: Changes between inertial frames

$$t' = \eta(t-t_0) \qquad \text{absolute time (time intervals)}$$

$$\vec{x}'(t') = R\cdot\vec{x}(t) - [\vec{V}_0(t-t_0)+\vec{X}_0] \qquad \text{absolute space [space intervals]}$$

$$\frac{d\vec{x}'}{dt'}(t') = \eta\left[R.\frac{d\vec{x}}{dt}(t) - \vec{V}_0\right], \quad \frac{d^2\vec{x}'}{dt'^2}(t') = R.\frac{d^2\vec{x}}{dt}(t) \qquad \vec{V}_0: \text{Galileo boost}$$

Consequence: addition theorem for velocities
$(\eta=1,\ R=\mathbb{1})$: $\vec{v}' = \vec{v} - \vec{V}_0$

⟹ Problem with electromagnetism:

speed of light $c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}$

2. The Geometric Postulates of Special Relativity

1. Principle of linearity: space + time: affine space
2. Principle of inertia: laws w.r.t. inertial frames in constant relative velocity

   transformation laws: invariance group of space + time
3. Principle of relativity: physical laws indpt of choice of inertial frame

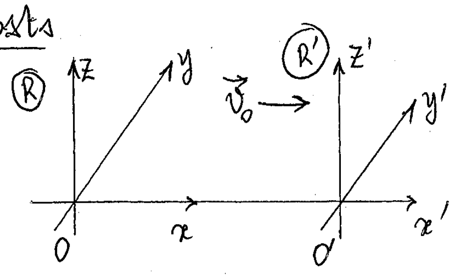
   ⇒ invariance under above group

   ⇒ symmetries, conservation laws
4. Fundamental principle of relativity:

   velocity c is identical in all inertial frames

   (new fundamental constant)

2.1. Lorentz Boosts



coincide at $t = 0 = t'$

$$\begin{cases} x' = \gamma_0 (x - v_0 t) \\ y' = y \\ z' = z \\ t' = ? \end{cases} \qquad \begin{cases} x = \gamma_0 (x' + v_0 t') \\ y = y' \\ z = z' \\ t = ? \end{cases}$$

$$x = \gamma_0 [\gamma_0 (x - v_0 t) + v_0 t'] \Rightarrow t' = \gamma_0 \left[t - \frac{\gamma_0^2 - 1}{\gamma_0^2 v_0} x \right], \quad t = \gamma_0 \left[t' + \frac{\gamma_0^2 - 1}{\gamma_0^2 v_0} x' \right]$$

a) First choice: Newton, absolute time: $t' = t \Rightarrow \gamma_0 = 1$

usual Galilei boost

b) Velocity of light: pulse in $x > 0$ direction

$$x = ct \quad , \quad x' = ct'$$

$$ct' = \gamma_0 [ct - v_0 t] = \gamma_0 (ct) \left[1 - \frac{v_0}{c} \right]$$

$$ct' = \gamma_0 (ct) \left[1 - \frac{(1 - \gamma_0^{-2})}{v_0 / c} \right]$$

$$\Rightarrow \begin{cases} \gamma_0^{-2} = 1 - v_0^2 / c^2 \\ \boxed{\gamma_0 = \dfrac{1}{\sqrt{1 - \beta_0^2}} , \quad \beta_0 = \dfrac{v_0}{c}} \end{cases}$$

Lorentz boost : 4-vectors of spacetime

$$\begin{cases} ct' = \gamma_0\left[(ct) - \beta_0 x\right] \\ x' = \gamma_0\left[-\beta_0 (ct) + x\right] \\ y' = y \\ z' = z \end{cases} \qquad \begin{cases} ct = \gamma_0\left[(ct') + \beta_0 x'\right] \\ x = \gamma_0\left[\beta_0 (ct') + x'\right] \\ y = y' \\ z = z' \end{cases}$$

Hyperbolic geometry
$$\begin{cases} \cosh w_0 = \gamma_0 , \ \sinh w_0 = \beta_0\gamma_0 \\ \tanh w_0 = \beta_0 \ ; \ w_0 : \text{rapidity} \\ \cosh^2 w_0 - \sinh^2 w_0 = +1 \\ (ct') = \cosh w_0 (ct) - \sinh w_0 \cdot x \\ x' = -\sinh w_0 (ct) + \cosh w_0 \cdot x \end{cases}$$

Euclidean geometry
$$\begin{cases} x' = \cos\theta_0\, x - \sin\theta_0\, y \\ y' = \sin\theta_0\, x + \cos\theta_0\, y \\ \cos^2\theta_0 + \sin^2\theta_0 = 1 \end{cases}$$

Remarks

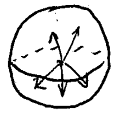
a) $$\begin{cases} ct' = \gamma_0\left[(ct) - \vec{\beta}_0 \cdot \vec{x}\right] \\ \vec{x}' = \vec{x} + \hat{\beta}_0\left[(\gamma_0 - 1)\hat{\beta}_0\cdot\vec{x} - \beta_0\gamma_0 (ct)\right] \end{cases} \quad \begin{cases} \hat{\beta}_0 = \dfrac{\vec{\beta}_0}{|\vec{\beta}_0|} \\ \vec{\beta}_0 = \dfrac{\vec{v}_0}{c} \end{cases} \quad \gamma_0 = \left(1 - \vec{\beta}_0^{\,2}\right)^{-1/2}$$

b) Nonrelativistic limit: $\beta \ll 1, \gamma \simeq 1$

$t' \simeq t , \ x' \simeq x - v_0 t , y' = y , z' = z$: Galilei boost

c) Space and time are relative to the observer

2.2. Spacetime geometry



$(ct)^2 - \vec{x}^2 = 0 = (ct')^2 - \vec{x}'^2$: light-like invariant

Spacetime invariant :

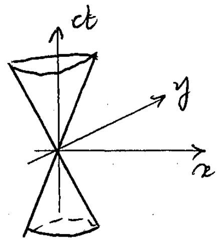
$$s^2 = (ct')^2 - \vec{x}'^2 = \left[\gamma_0 (ct - \beta_0 x)\right]^2 - \left[\gamma_0(-\beta_0 ct + x)\right]^2 - y^2 - z^2$$
$$= (ct)^2 - \vec{x}^2$$

$\Rightarrow$ $\begin{cases} \text{Spacetime : Minkowski geometry (hyperbolic geometry)} \\ \text{Minkowski metric: } \eta_{\mu\nu} = \begin{pmatrix} +1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} , \mu,\nu = 0,1,2,3 \ ; \ x^\mu = (\underset{\mu=0}{ct}, \underset{\mu=i=1,2,3}{\vec{x}}) \end{cases}$

Invariance groups :

- vector space: Lorentz group $<$ Lorentz boosts / space rotations; + P, T
- affine space: Poincaré group : Lorentz group $\ltimes$ spacetime translations

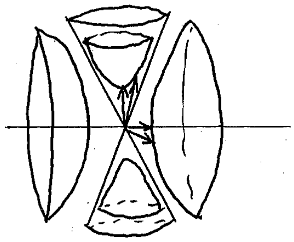
Remarks



$$\Delta s^2 = (c\Delta t)^2 - (\Delta\vec{x})^2$$

$$\begin{cases} \Delta s^2 > 0: & \text{time-like} \\ \Delta s^2 = 0: & \text{light-like} \\ \Delta s^2 < 0: & \text{space-like} \end{cases}$$

$x^\mu = (ct, \vec{x})$: 4-vector under the Lorentz group



2.3. Physical consequences

Events: $(ct_{\frac{1}{2}}, x_{\frac{1}{2}}, y_{\frac{1}{2}}, z_{\frac{1}{2}})$; $(ct'_{\frac{1}{2}}, x'_{\frac{1}{2}}, y'_{\frac{1}{2}}, z'_{\frac{1}{2}})$

a) Simultaneity: $c\Delta t = \gamma_0 [\underset{\overset{\shortparallel}{0}}{c\Delta t'} - \beta_0 \Delta x'] = -\beta_0\gamma_0 \Delta x' \neq 0$

b) Time dilation: $c\Delta t = \gamma_0 [c\Delta t' - \beta_0 \underset{\overset{\shortparallel}{0}}{\Delta x'}] = \gamma_0 (c\Delta t')$

$$\boxed{T = \gamma_0 T_0}$$ T_0: proper-time ; $ds^2 = (c\Delta\tau)^2 > 0$

[μ-decay ; twin paradox]

c) Length contraction:

$$\Delta x' = \gamma_0 [\beta_0 \underset{\overset{\shortparallel}{0}}{(c\Delta t)} + \Delta x] = \gamma_0 \Delta x$$

Longitudinal $$\boxed{L = \frac{1}{\gamma_0} L_0}$$

a) No contraction in the transverse direction

b) Parallaxe effect

d) Velocity addition theorem

$$\frac{d}{d(ct')} = \frac{d(ct)}{d(ct')}\frac{d}{d(ct)} = \frac{1}{\gamma_0[1-\beta_0\beta_x]}\frac{d}{d(ct)} \quad ; \quad \frac{d(ct')}{d(ct)} = \gamma_0[1-\beta_0\beta_x], \quad \vec{\beta} = \frac{d\vec{x}}{d(ct)}$$

$$\begin{cases} \beta'_x = \dfrac{\beta_x - \beta_0}{1-\beta_0\beta_x} \\ \beta'_y = \dfrac{1}{\gamma_0}\dfrac{\beta_y}{1-\beta_0\beta_x} \\ \beta'_z = \dfrac{1}{\gamma_0}\dfrac{\beta_z}{1-\beta_0\beta_x} \end{cases}$$

Remarks:

a) $\beta_x = 1, \beta_y = 0 = \beta_z$: $\beta'_x = 1, \beta'_y = 0 = \beta'_z$

b) Nonrelativistic limit:

$v'_x \simeq v_x - v_0$, $v'_y \simeq v_y$, $v'_z \simeq v_z$

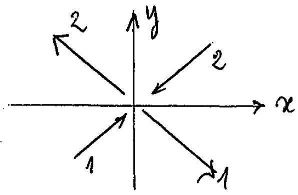
3. Relativistic Mechanics

Newton mechanics : $\left[\frac{d\vec{p}}{dt} = \vec{F} \right.$ + conservation of total momentum of free system $\left. \right]$

3.1. Relativistic momentum

$$\vec{p} = m\vec{v} = m\frac{d\vec{r}}{dt}$$

Does not work for elastic collision of 2 identical particles



Proper-time : $\begin{cases} s^2 = (ct)^2 - \vec{x}^2 = (c\tau)^2 \\ t = \gamma\tau , \ \gamma = \frac{1}{\sqrt{1-\vec{\beta}^2}} , \ \vec{\beta} = \frac{\vec{v}}{c} \end{cases}$

$$\boxed{\vec{p} = m\frac{d\vec{x}}{d\tau} = m\frac{dt}{d\tau}\frac{d\vec{x}}{dt} = m\gamma\vec{v} = mc\gamma\vec{\beta}}$$

Remarks

a) Nonrelativistic limit: $\vec{p} \simeq m\vec{v}$

b) $\lim_{|\vec{v}| \to c^-} |\vec{p}| = \infty$

3.2. Relativistic equation of motion

$$\boxed{\frac{d\vec{p}}{dt} = \vec{F} , \quad \vec{p} = m\gamma\dot{\vec{r}}}$$

Remark

$$\begin{cases} \frac{d\vec{p}}{dt} = m\gamma^3\left[\dot{\vec{v}}\left(1-\frac{\vec{v}^2}{c^2}\right) + \frac{\vec{v}\cdot\dot{\vec{v}}}{c^2}\vec{v}\right] \\ \dot{\vec{r}}\cdot\frac{d\vec{p}}{dt} = m\gamma^3\,\vec{v}\cdot\dot{\vec{v}} = \frac{d}{dt}\left[mc^2\gamma\right] \end{cases}$$

3.3. Relativistic energy

$$\frac{d\vec{p}}{dt} = \vec{F} \Rightarrow \dot{\vec{r}}\cdot\frac{d\vec{p}}{dt} = \dot{\vec{r}}\cdot\vec{F} = \mathcal{P} \ : \text{power}$$

$$\boxed{\frac{d}{dt}\left[mc^2\gamma\right] = \mathcal{P}}$$

Relativistic energy : $E = mc^2\gamma = \frac{mc^2}{\sqrt{1-\vec{v}^2/c^2}}$

Relativistic momentum : $\vec{p}c = mc^2\gamma\vec{\beta}$

Remarks:

a) $E \cong mc^2 + \frac{1}{2}m\vec{v}^2 + \dots$

- mc^2: rest mass energy
- $\frac{1}{2}m\vec{v}^2$: nonrelativistic kinetic energy

b) $\lim_{|\vec{v}| \to c^-} E = \infty$

c is a limit velocity

4. The Energy-Momentum 4-Vector

$$E = mc^2\gamma = mc^2 \frac{d(ct)}{d(c\tau)}, \quad \vec{p}c = mc^2\gamma\vec{\beta} = mc^2 \frac{d\vec{x}}{d(c\tau)}$$

$(E, \vec{p}c)$: 4-vector for Lorentz transformations

$$\begin{cases} E' = \gamma_0\left[E - \beta_0 p_x c\right] \\ p'_x c = \gamma_0\left[-\beta_0 E + p_x c\right] \\ p'_y c = p_y c \\ p'_z c = p_z c \end{cases}$$

Linear transformations

Relativistic invariant:

$$E^2 - (\vec{p}c)^2 = (mc^2)^2$$

$$E = \sqrt{(\vec{p}c)^2 + (mc^2)^2}$$

Remarks

a) $\boxed{\vec{\beta} = \frac{\vec{p}c}{E}}$ $\Longrightarrow$ Addition theorem for the velocity

b) Massless particles: light-like particles $\left.\begin{array}{l} E^2 = (\vec{p}c)^2 \\ \vec{\beta} = \frac{\vec{p}c}{E} \end{array}\right\} \Rightarrow |\vec{\beta}| = 1$: speed of light

c) Massive particles: time-like particles $|\vec{\beta}| = \frac{|\vec{p}c|}{E} = \sqrt{1 - \left(\frac{mc^2}{E}\right)^2} < 1$:

d) Space-like particles: $|\vec{\beta}| > 1$, $E^2 - (\vec{p}c)^2 = (mc^2)^2 < 0$
tachyons, imaginary mass, impossible

e) Conservation laws for a free system:

$$\boxed{\frac{dE}{dt} = 0, \quad \frac{d\vec{p}}{dt} = \vec{0}}$$

5. Contra- and Co-variant Vectors and Tensors

4-vectors: $x^\mu = (ct, \vec{x})$; $(t, \frac{\vec{x}}{c})$; $P^\mu = (E, \vec{p}c)$, $p^\mu = (\frac{E}{c}, \vec{p})$
$\mu = 0,1,2,3$

Geometrical / relativistic invariants for Minkowski geometry

$$s^2 = x^{0\,2} - \vec{x}^2, \quad E^2 - (\vec{p}c)^2 = (mc^2)^2$$

Minkowski metric: convention $(+---)$

$$\eta_{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} +1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}, \quad \eta^{\mu\nu} = \begin{pmatrix} +1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

$$s^2 = \eta_{\mu\nu} x^\mu x^\nu, \quad (mc^2)^2 = \eta_{\mu\nu} P^\mu P^\nu$$

Inner product:

dual vector space ⟷	vector space
covariant vectors	contravariant vectors
$x_\mu = \eta_{\mu\nu} x^\nu$	$x^\mu = \eta^{\mu\nu} x_\nu$

and tensor products

Examples: $x^\mu = (ct, \vec{x})$, $x_\mu = (ct, -\vec{x})$
$P^\mu = (E, \vec{p}c)$, $P_\mu = (E, -\vec{p}c)$

Invariants: $\eta_{\mu\nu} x^\mu y^\nu = x^\mu y_\mu = x_\mu y^\mu = x \cdot y$

Remarks

a) Euclidean geometry: δ_{ij}, δ^{ij}

$$x_i = \delta_{ij} x^j, \quad x^i = \delta^{ij} x_j$$

$\delta_{ij} x^i y^j = x_i y^i = x^i y_i$: invariant under rotations

Invariant tensors: $\delta_{ij}, \delta^{ij}, \epsilon^{ijk}, \epsilon_{ijk}$

b) Minkowski geometry:

Invariant tensors: $\eta_{\mu\nu}, \eta^{\mu\nu}, \epsilon^{\mu\nu\rho\sigma}, \epsilon_{\mu\nu\rho\sigma}, \epsilon^{0123} = +1, \epsilon_{0123} = -1$

c) Other examples of 4-vectors:

$\frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial(ct)}, \frac{\partial}{\partial x^i}\right) = \partial_\mu = (\partial_0, \partial_i)$: covariant vector

$\frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial(ct)}, -\frac{\partial}{\partial x^i}\right) = \partial^\mu = (\partial^0, \partial^i)$: contravariant vector

$\Box = \partial_\mu \partial^\mu = \frac{\partial^2}{\partial(ct)^2} - \vec{\nabla}^2$: d'Alembertian, invariant

6. The Lorentz and Poincaré Groups

$$s^2 = \eta_{\mu\nu} x^\mu x^\nu = x \cdot x \,, \quad (\Delta s)^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu = \Delta x \cdot \Delta x = (\Delta x)^2$$

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \qquad\qquad x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$$

$\Lambda^\mu{}_\nu$: Lorentz ; a^μ : Poincaré

Conditions: invariance of the metric

$$\boxed{\eta_{\rho\sigma} \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu = \eta_{\mu\nu}} \quad SO(3,1),\ SO(1,3)$$

7. Maxwell's Equations in Covariant Form

SI: $\begin{cases} \vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\rho \,, & \vec{\nabla}\times\vec{B} - \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J} \\ \vec{\nabla}\cdot\vec{B} = 0 \,, & \vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t} = \vec{0} \end{cases}$ $\Rightarrow \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$

$\mu_0 \epsilon_0 c^2 = 1$

[$\Rightarrow (\frac{\vec{E}}{c}, \vec{B})$: same physical dimensions]

Homogeneous equations:

$$\begin{cases} \vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t} \,, & \vec{B} = \vec{\nabla}\times\vec{A} \\ \quad \Downarrow & \\ \frac{\vec{E}}{c} = -\vec{\nabla}\left(\frac{\Phi}{c}\right) - \frac{\partial \vec{A}}{\partial (ct)} \,, & \vec{B} = \vec{\nabla}\times\vec{A} \end{cases}$$

4-vector:

$$\boxed{A^\mu = \left(\frac{\Phi}{c}, \vec{A}\right)}$$

Gauge symmetry:

$$\frac{\Phi'}{c} = \frac{\Phi}{c} - \frac{\partial \chi}{\partial (ct)} \,, \quad \vec{A}' = \vec{A} + \vec{\nabla}\chi = \vec{A} + \frac{\partial \chi}{\partial \vec{x}} \,, \quad A'^\mu = A^\mu - \partial^\mu \chi \,, \quad A'_\mu = A_\mu - \partial_\mu \chi$$

Faraday/electromagnetic field strength:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \quad F'_{\mu\nu} = F_{\mu\nu}$$

$$\begin{cases} F_{0i} = \partial_0 A_i - \partial_i A_0 = -\frac{\partial A^i}{\partial (ct)} - \frac{\partial}{\partial x^i}\left(\frac{\Phi}{c}\right) = \frac{E^i}{c} \\ F_{ij} = \partial_i A_j - \partial_j A_i = -\frac{\partial A^j}{\partial x^i} + \frac{\partial A^i}{\partial x^j} = -\epsilon^{ijk} B^k \end{cases}$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & E^1/c & E^2/c & E^3/c \\ -E^1/c & 0 & -B^3 & B^2 \\ -E^2/c & B^3 & 0 & -B^1 \\ -E^3/c & -B^2 & B^1 & 0 \end{pmatrix}$$

Dual field strength: $*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$

$$\begin{cases} *F^{0i} = \frac{1}{2}\epsilon^{0ijk} F_{jk} = -\frac{1}{2}\epsilon^{ijk}\epsilon^{jkl} B^l = -B^i \\ *F^{ij} = \frac{2}{2}\epsilon^{ij0k} F_{0k} = \epsilon^{ijk}\frac{E^k}{c} \end{cases}$$

Homogeneous equations: $\boxed{\partial^*_\nu F^{\nu\mu} = 0}$ (Bianchi identity)

Inhomogeneous equations:

$0 = \frac{\partial}{\partial(ct)}(c\rho) + \frac{\partial J^i}{\partial x^i}$: 4-vector : $J^\mu = (c\rho, \vec{J})$

$\boxed{\partial_\mu J^\mu = 0}$ Conservation equation

$$\boxed{\partial_\nu F^{\nu\mu} = \mu_0 J^\mu} \Longrightarrow$$

8. The Lorentz Force Equation in Covariant Form

$$\frac{d\vec{p}}{dt} = q[\vec{E} + \vec{v}\times\vec{B}] \Rightarrow \frac{d(\vec{p}c)}{d(c\tau)} = qc\gamma\left[\frac{\vec{E}}{c} + \vec{\beta}\times\vec{B}\right], \quad \frac{dE}{d(c\tau)} = qc\gamma\vec{\beta}\cdot\frac{\vec{E}}{c}$$

$$u^\mu = \frac{dx^\mu}{d(c\tau)} = \gamma(1, \vec{\beta}), \quad u^2 = 1, \quad u^\mu = \frac{1}{mc^2} p^\mu$$

$$\boxed{\frac{dp^\mu}{d(c\tau)} = qc F^{\mu\nu} u_\nu, \quad mc^2 \frac{du^\mu}{d(c\tau)} = qc F^{\mu\nu} u_\nu}$$

The Electromagnetic Interaction
Relativistic Classical Description

Fields		Particles
$\partial_\nu F^{\nu\mu} = \mu_0 J^\mu$ $\partial^*_\nu F^{\nu\mu} = 0$ with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ $A'_\mu = A_\mu - \partial_\mu \chi$ $A^\mu = \left(\frac{\Phi}{c}, \vec{A}\right)$ $J^\mu = (c\rho, \vec{J})$	$\longleftrightarrow$	$mc^2 \frac{du^\mu}{d(c\tau)} = qc F^{\mu\nu} u_\nu$ $u^\mu = \frac{dx^\mu}{d(c\tau)}$

Manifest spacetime covariance

[Newton's mechanics: $m\ddot{\vec{r}} = \vec{F}$]

Elements of Relativistic Quantum Field Theory

Excerpts from

J. Govaerts, *The Quantum Geometer's Universe: Particles, Interactions and Topology*, Proceedings of the Second International Workshop on Contemporary Problems in Mathematical Physics (Cotonou, Benin, October 28$^{\text{th}}$ - November 2$^{\text{nd}}$, 2001), eds. J. Govaerts, M. N. Hounkonnou and A. Z. Msezane (World Scientific, Singapore, 2002), pp. 79–212 [e-print `arXiv:hep-th/0207276`].

Available from the AIMS Library.

For the references, see the original text.

The Quantum Geometer's Universe: Particles, Interactions and Topology

With the two most profound conceptual revolutions of XX$^{\text{th}}$ century physics, quantum mechanics and relativity, which have culminated into relativistic spacetime geometry and quantum gauge field theory as the principles for gravity and the three other known fundamental interactions, the physicist of the XXI$^{\text{st}}$ century has inherited an unfinished symphony: the unification of the quantum and the continuum. As an invitation to tomorrow's quantum geometers who must design the new rulers by which to size up the Universe at those scales where the smallest meets the largest, these lectures review the basic principles of today's conceptual framework, and highlight by way of simple examples the interplay that presently exists between the quantum world of particle interactions and the classical world of geometry and topology.

1 Introduction

It is often said that the profound conceptual revolutions of XX$^{\text{th}}$ century physics may be ascribed to three fundamental physical constants, namely Newton's constant G_N characteristic of the gravitational interaction, light's velocity in vacuum c displaying the relativistic character of physical reality, and Planck's constant $\hbar = h/2\pi$ as the hallmark for the quantum character of the physical universe. All of these constants have incessantly been used much like light beacons with which to probe the as yet unexplored territories beyond the known physical laws of our material world, grasping for this ever unfulfilled dream of the ultimate unification of all of matter, radiation and their interactions.

Each of these three constants on its own has led to its separate conceptual revolution, even beyond the confines of the scientific methods of physics, in ways that shall not be recalled here. However, when considered in combination, these constants imply still further profound conceptual revisions in our understanding of the physical world, which themselves stand out as the genuine unfinished revolutions of XX$^{\text{th}}$ century physics. Indeed, even though the combinations of G_N with c on the one hand, and of c and $\hbar$ on the other hand, have each led to a profound new vision onto the material universe through the physicist's eye, the formulation of a conceptual framework in which all three constants play an equally important role is the wide open problem that confronts physics in this XXI$^{\text{st}}$ century.

As is well known, the marriage of G_N and c leads to a curved spacetime whose geometry is dynamical and is governed by the energy-matter distribution within it, a framework within which the gravitational interaction is the physical manifestation of any curvature in space and in spacetime. The most fascinating offsprings of this union are undoubtedly, on the one hand, the cosmological theory of the history of our universe from its birth to its ultimate demise if ever, and on the other hand, the prediction for regions of spacetime to be so much curled up by their energy-matter content that even light can no longer escape from such black holes. For instance, the value

$$r_0 = 2\frac{G_N M}{c^2} \tag{2}$$

for the horizon of a neutral nonrotating black hole of mass M displays the combined contribution of gravity and relativity. These examples are but two specific outcomes of classical general relativity, a relativistic invariant theory of gravity whose construction is based on a simple geometrical thus physical principle: the description of physical processes should be independent of the local spacetime observer, namely, it should be independent of the choice of local spacetime coordinate parametrisation.

The theory should be invariant under arbitrary local coordinate transformations in spacetime[1]. In other words, a gauge invariance principle is at work, leading to a description of the gravitational interaction based on a simple but powerful symmetry and thus geometry principle.

On the other hand, the marriage of c and $\hbar$ leads naturally to the quantum field theory description of the elementary particles and their interactions, at the most intimate presently accessible scales of space and energy, a fact made manifest by the value for their product,

$$\hbar c \simeq 197 \text{ MeV} \cdot \text{fm}. \tag{3}$$

In fact, one offspring of this second union is the unification of matter and radiation, namely of particles with their corpuscular propagating properties and fields with their wavelike propagating properties. Particles, characterised through their energy, momentum and spin values in correspondence with the Poincaré symmetries of Minkowski spacetime in the absence of gravity, are nothing but the relativistic energy-momentum quanta of a field, thereby implying a tremendous economy in the description of the physical universe, accounting for instance at once in terms of a single field filling all of spacetime for the indistinguishability of identical particles and their statistics. Furthermore, quantum relativistic interactions are then understood simply as couplings between the various quantum fields locally in spacetime, which translate in terms of particles as diverse exchanges of the associated quanta. Such a picture lends itself most ideally to a pertubative understanding of the fundamental interactions, which has proved to be so powerful beginning with quantum electrodynamics, up to the modern $\text{SU}(3)_C\times\text{SU}(2)_L\times\text{U}(1)_Y$ Standard Model of the strong and electroweak interactions. Such a perturbative representation of processes requires a renormalisation procedure of the basic field parameters—their normalisations, masses and couplings—, and one has had to learn how to identify theories for which this renormalisation programme is feasable. In the course of time, a general class of renormalisable field theories has been identified, all falling again under the general spell of the gauge symmetry principle as did the gravitational interaction!

Even though the physical meaning ascribed to the renormalisability criterion has evolved in such a manner that these theories are nowadays viewed rather as effective theories for some as yet unknown more fundamental description becoming manifest and relevant at still higher energies, the fact remains that the gauge symmetry principle is again at work at the most intimate level of the unification of the relativistic quantum. But this time, this invariance under local transformations in spacetime applies to some "internal" space of degrees of freedom, that fields and their quanta carry along and which are made physically manifest through the different charges and quantum numbers that particles possess. Hence, through countless experiments performed at ever increasing energies and with ever increased technical sophistication, three generations of quarks and leptons, the basic building blocks of matter, each such generation being comprised of two quarks, one charged lepton and its associated neutrino, have been identified, and their reality inscribed into the construction of the Standard Model. All interactions among these six quarks and six leptons are governed by the gauge symmetry principle, with $\text{SU}(3)_C\times\text{SU}(2)_L\times\text{U}(1)_Y$ symmetries acting within internal space and independently, though in a continuous fashion, at each point of spacetime. This local realisation of the symmetry requires the existence of gauge bosons, as the carriers of the symmetry and thus of the interactions from one spacetime point to the next. There are thus eight gluons for the strong interaction, the charged and neutral massive electroweak gauge bosons $W^\pm$ and Z_0 for the charged and neutral current weak interactions, and finally the photon for the electromagnetic interaction. Only one member of the Standard Model family has yet to be discovered experimentally, namely the so-called higgs particle which should be responsible for a mechanism at the origin of the masses for all quarks, leptons and massive gauge bosons. The higgs hunt is on at the most powerful particle accelerators in the world, the last missing offspring of the union of c and $\hbar$.

Given the fundamental role played by symmetries, hence also geometry, in the unifications of fundamental physics concepts achieved throughout the last century, it is fair to characterise XX^{th}

[1]Einstein's theory of general relativity has furthermore inscribed into it the equivalence principle between inertial and gravitational mass.

century physics as the reign supreme of the symmetry principle, this principle being pushed into its most extreme realisations possible through the gauge symmetry principle. This includes the possibility of supersymmetry, a symmetry that relates bosonic and fermionic particles which, when rendered local in spacetime, leads to theories of supergravity that must necessarily include a quantum gravitational sector. But it also appears that this symmetry principle has finally unveiled all its hidden physical secrets in the embodiement it has acquired within a field theory description of the universe, of its matter content and of its fundamental interactions. Even though the symmetry principle seems to have yielded all its potential, it proves not to be potent enough to bring order to a *ménage à trois* in which all three fundamental constants G_N, c and $\hbar$ would be living peacefully and happily together on equal terms, to bear many news fruits of their ultimate union. As is well known, there does not yet exist a commonly accepted theoretical formulation for a quantum theory of relativistic gravity which would also include the other fundamental interactions and their matter fields, all consistently expressed within a quantum framework.

Looking back at the brief and superficial highlights recalled above, one realises that the non-quantum relativistic description available for the gravitational interaction is in fact the ideal realm of the "relativistic continuum" reigning supreme, the utmost physical application as of today of the notion of differentiable structures in geometry. Likewise, the other component of the same story, namely the relativistic quantum field theory description of the elementary particles and their other fundamental interactions, is in fact the ideal realm of the "relativistic quantum" reigning supreme, the utmost physical outcome of the ideas of quantisation and its associated abstract algebraic structures. The fundamental problem that XXI$^{\rm st}$ century physics is to confront is that of the final marriage of the "continuum" and the "quantum", namely of identifying a mathematical formulation of what is referred to as "quantum geometry", the new conceptualisation of what the geometry of spacetime ought to be when explored at the most extreme and smallest scales.

In terms of the three fundamental constants G_N, c and $\hbar$, it is well known how the quantum regime for relativistic gravity is characterised by Planck's mass, length and time scales,

$$M_{\rm Pl}=\sqrt{\frac{\hbar c}{G_N}}\simeq 10^{19}\ {\rm GeV}/c^2,\qquad L_{\rm Pl}=\frac{\hbar c}{M_{\rm Pl}c^2}\simeq 2\times 10^{-35}\ {\rm m},$$
$$\tau_{\rm Pl}=\frac{L_{\rm Pl}}{c}\simeq 6\times 10^{-44}\ {\rm s}. \tag{4}$$

Even though these values lie way beyond the reach of present day accelerators, as well as of present day theories, processes at such scales must have taken place in the early universe, while from the conceptual point of view, the fundamental conflit between the classical relativistic realm of the "continuum" for gravity with the quantum relativistic realm of the "quantum" for particles and their other interactions, cries out to the XXI$^{\rm st}$ physicist for a new conceptual revolution that ought to resolve this basic mutual inconsistency of present day physics principles. From that point of view, XXI$^{\rm st}$ century physics will be the search for the Quantum Geometry Principle, the inherited unfinished physics symphony of the XX$^{\rm th}$ century composed so far according to the rules of the Symmetry Principle.

With the advent of M-theory, the nonperturbative embodiement of superstring theories, and possibly also with the loop gravity programme, we are most probably already getting the first glimpses of this quantum geometry waiting to be discovered by tomorrow's bright young minds. Such a pursuit in search of the possible ultimate unification, all at the same time, of matter and its interactions, and of geometry and the quantum, belongs to the best of scientific traditions finding its roots back in the earliest days of the human intellectual adventure. It should only be just that within all peoples of the world, as much from developing as from developed countries, those whose calling lies towards such an avenue should find an environment within which to contribute on equal terms to this ultimate understanding of our physical universe and its history. A workshop of this type is an opportunity to highlight some of the issues surrounding this unfulfilled quest, and hopefully entice bright new minds to dedicate themselves to this adventure at the frontiers of physical concepts. The education to critical and scientific thinking that such a research activity requires can only benefit any society within which it is pursued, both in its human and intellectual aspirations as well as in its educational, technological and economic development, bringing man always a little closer to the stars, the eternal yearning of his soul. Countless examples over human history bear witness to this fact, and many of us today benefit in so many ways from the fruits of this unswaying quest at the most abstract level as it has been pursued over centuries past.

These lecture notes do not, of course, have any pretence to outline what quantum geometry ought to be, which, after all, is the XXI$^{\text{st}}$ century quantum geometer's task! Rather, these lectures wish to present sort of a guided tour of the general principles of symmetry and quantum physics that have led to the relativistic quantum field theory description of the elementary particles and their fundamental interactions, aiming at the end towards illustrations of the fact that beyond the gauge symmetry principle which seems to govern all interactions, when it comes to geometry—namely the "continuum" and gravity—and the "quantum", topology is also called to play a vital role. In fact, one is very much led to suggest that the problem of quantum gravity should find a resolution only when considered together with all the other quantum matter and interacting fields, while pure quantum gravity is oblivious actually to any geometry, and would be governed only by the rules of quantum topology. Indeed, this is the programme that was launched with the discovery of topological quantum field theories. Finally, these notes concentrate on the quantum field theory side of the above story, assuming that the reader is most familiar already with the views of classical continuum geometry as applied within the physical context of the gravitational interaction and general relativity. This is thus the spirit with which these notes are offered to the aspiring quantum geometers of the XXI$^{\text{st}}$ century who are attending this Workshop.

Contents are organised as follows. Section 2 discusses the general rules of abstract canonical quantisation, based on the Hamiltonian formulation of a given dynamical system. These rules are then applied to relativistic field theories in Section 3, to establish that such quantised theories provide a natural description of quantum relativistic particles in Minkowski spacetime. Section 4 introduces then to interacting quantum field theories and, as a general class of renormalisable theories in four dimensions, to general Yang-Mills theories, possibly subjected to the Higgs mechanism of spontaneous symmetry breaking. This discussion thus also serves as a motivation for Section 5 which addresses the general problem of the quantisation of systems subjected to constraints in phase space, which include any gauge invariant system, following Dirac's general analysis of this issue. Rather than introducing then the general methods of BRST quantisation, the recent and most efficient approach towards the quantisation of constrained systems based on the physical projector is also discussed. As an example of its possible use, the quantisation of 2+1-dimensional Chern-Simons theory is briefly described in Section 6, which in fact is one of the simplest examples of a topological quantum field theory. Finally, Sections 7 and 8 introduce to bosonic string theory and its toroidal compactification. These last three sections serve as first witnesses to the necessity to develop a new mathematical framework for quantum theories of gravity, whether they include matter degrees of freedom or not, that should define the sought-for "quantum geometry" of the fundamental unification. Finally, further comments are presented in the Conclusions.

Our conventions will be stated where appropriate. Notice also that all the discussion will be confined to bosonic degrees of freedom only, but that similar developments exist of course for systems combining both bosonic and fermionic degrees of freedom. Suggestions for some exercises are also provided, some of which could in fact become PhD research topics on their own. Finally, no attempt has been made at providing an exhaustive bibliography, for which we apologise to anyone who might feel her/his work is being overlooked. Rather, we hope that references given would suffice to quickly identify further relevant sources to any particular topic of interest.

2 Abstract Canonical Quantisation

2.1 Dynamics

2.2 Hamiltonian formulation

2.3 Representations of the Heisenberg algebra

<u>The configuration space representation</u>

<u>The momentum space representation</u>

<u>The Fock space representation</u>

<u>The coherent state representation</u>

3 Relativistic Quantum Particles and Field Theories

Starting with this section, we shall explicitly work in four-dimensional Minkowski spacetime with coordinates x^μ ($\mu = 0, 1, 2, 3$) and a metric $\eta_{\mu\nu}$ of signature $(+---)$. Furthermore as is customary in quantum field theory, units such that $\hbar = 1 = c$ are also being used throughout, so that mass and energy on the one hand, as well as time and space on the other, are each measured in the same units, while energy and time, for instance, are of inverse dimensions. Hence, any mechanical quantity may always be expressed in units of mass to some power.

3.1 Motivation

It is an experimental fact that there exist particles in nature, which behave both with relativistic and quantum properties, have definite energy, momentum and thus invariant mass values, and may be created or annihilated through different physical processes. Which type of mathematical framework would be able to account for all these physical properties all at once?

As we have recalled above, the quantisation of the harmonic oscillator leads to such a framework. Indeed, the operators a and $a^\dagger$, which obey the Fock algebra $[a, a^\dagger] = \mathbb{I}$, provide for the annihilation and creation of energy quanta, each carrying an identical amount $\hbar\omega$ of energy. Furthermore, we also know that associated to these operators, there exists some configuration space operator $\hat{q}$ which in the Heisenberg picture has a time dependence defined by (from now on, the choice of reference time will be $t_0 = 0$)

$$\hat{q}(t) = \sqrt{\frac{\hbar}{2m\omega}} \left[a\, e^{-i\omega t} + a^\dagger\, e^{i\omega t} \right], \tag{5}$$

which, in the classical limit, thus defines the entire real line as the space of classical configurations of the system. Hence, the configuration space quantum operator $\hat{q}(t)$ in the Heisenberg picture obeys the following equation

$$\left[\frac{d^2}{dt^2} + \omega^2 \right] \hat{q}(t) = 0, \tag{6}$$

which also coincides with the classical equation of motion for the system, which derives from the Lagrangian action

$$S[q] = m \int dt \left[\frac{1}{2} \left(\frac{dq}{dt} \right)^2 - \frac{1}{2}\omega^2 q^2 \right]. \tag{7}$$

Let us now try to extend this mathematical framework to spinless relativistic quantum particles of definite energy-momentum $k^\mu = (k^0, \vec{k})$ and mass m such that $k^0 = (\vec{k}^{\,2} + m^2)^{1/2} = \omega(\vec{k}\,)$, and which may be created or annihilated in specific physical processes. Thus, for each of the possible momentum values $\vec{k}$, one should introduce a pair of creation and annihilation operators $a^\dagger(\vec{k}\,)$ and $a(\vec{k}\,)$ obeying the Fock space algebra

$$\left[a(\vec{k}\,), a^\dagger(\vec{k}')\right] = (2\pi)^3\, 2\omega(\vec{k}\,)\, \delta^3(\vec{k} - \vec{k}'), \tag{8}$$

where, compared to the Fock algebra for the harmonic oscillator, the normalisation of the operators has been modified for a reason to be specified presently. Thus in particular, 1-particle quantum states are obtained from the normalised Fock vacuum $|0\rangle$ as

$$|\vec{k}\rangle = a^\dagger(\vec{k}\,)\, |0\rangle, \qquad \langle 0|0\rangle = 1. \tag{9}$$

Proceeding by analogy with the harmonic oscillator case, in order to identify the configuration space for such a quantum system, let us also consider superpositions of these operators such as in (5). However, since we wish to develop a formalism which is manifestly spacetime covariant under Lorentz transformations, the product ωt appearing in the imaginary exponentials that multiply the operators and which thus corresponds to the product of the energy value of a quantum by the time interval, must be extended into the Minkowski invariant product $\omega(\vec{k}\,)t - \vec{k}\cdot\vec{x} = k\cdot x$, where the last expression denotes the inner product of four-vectors with the four-dimensional Minkowski metric. Furthermore, since in the present case we have an infinity of quantum operators labelled by the vector values $\vec{k}$ and which are all on an equal footing, one should consider a general superposition of all such linear combinations of the creation and annihilation operators with a $\vec{k}$-independent weight. Hence finally, one is led to consider the following operator, again in the Heisenberg picture, as the relativistic invariant extension of (5),

$$\hat{\phi}(x^\mu) = \int_{(\infty)} \frac{d^3\vec{k}}{(2\pi)^3 2\omega(\vec{k}\,)} \left[a(\vec{k}\,)e^{-ik\cdot x} + a^\dagger(\vec{k}\,)e^{ik\cdot x}\right]. \tag{10}$$

Note that having rescaled the creation and annihilation operators by a factor $(\omega(\vec{k}\,))^{1/2}$, the $d^3\vec{k}$ integration measure includes the same dimensionful normalisation factor as in (5) for the harmonic oscillator. The choice of numerical factor $(2\pi)^3$ is made for later convenience. As a matter of fact, the reason for the specific choice of normalisation in (8) is that the integration measure in (10), namely $d^3\vec{k}/2\omega(\vec{k}\,)$, is invariant under Lorentz transformations, as may easily be checked. In other words, this parametrisation of the operator $\hat{\phi}(x^\mu)$ is manifestly Lorentz covariant.

Hence, associated to the algebra (8), one expects that the actual configurations of the corresponding system is that of a real scalar field in spacetime! Indeed, in the classical limit, the combination (10) defines a real number $\phi(x^\mu)$ attached at each spacetime point. In other words, an arbitrary collection of identical relativistic free quantum point particles with causal and unitary propagation corresponds to quanta of a single relativistic quantum field in Minkowski spacetime. Furthermore, even though these particles display corpuscular properties by having definite energy-momentum values, their spacetime dynamical propagation also displays wavelike properties, since the field obeys the following equation of motion,

$$\left[\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + m^2\right]\hat{\phi}(x^\mu) = 0, \tag{11}$$

which is indeed a wave equation, known as the Klein-Gordon equation, and is nothing but the straightforward relativistic invariant extension of the equation of motion for the harmonic oscillator. Likewise, the corresponding classical action principle thus reads, in a manifestly Lorentz invariant form,

$$S[\phi] = \int dt \int_{(\infty)} d^3\vec{x} \left[\frac{1}{2}\left(\frac{\partial}{\partial t}\phi\right)^2 - \frac{1}{2}\left(\vec{\nabla}\phi\right)^2 - \frac{1}{2}m^2\phi^2\right]. \tag{12}$$

From this point of view, the configuration space that has been identified corresponds to an infinite set of harmonic oscillators sitting all adjacent next to one another in the three dimensions of space, and while they each oscillate away from their equilibrium position, the gradient term $\vec{\nabla}\phi$ in the action or in the equation of motion induces a coupling between adjacent oscillators, thereby leading to a propagating wave behaviour of the system in space as a function of time. This term in $\vec{\nabla}\phi$ is required by Lorentz invariance from the similar term in $\partial\phi/\partial t$ which is necessary for the time dependent dynamics of the system.

In conclusion, having considered the possibility to describe an arbitrary collection of identical relativistic free quantum spinless point particles of definite energy-momentum and mass which may be created and annihilated locally in Minkowski spacetime, we are naturally led to consider a formulation which is that of a local real relativistic scalar field in spacetime with its dynamical wave properties, whose action is real under complex conjugation (which guarantees quantum unitarity), Poincaré invariant (necessary for causality, and also leading to states of definite energy-momentum and angular momentum, which are the conserved Noether charges for the Poincaré invariance group of Minkowski spacetime), and finally local in spacetime (thus guaranteeing spacetime causality and locality of particle propagation, and later on also for their interactions). At this stage, given the algebra (8), one is only describing interactionless particles, since the complete space of energy eigenstates is the simple tensor product over all $\vec{k}$ values of a Fock space representation, without any nonvanishing matrix element of the Hamiltonian between different factors of this tensor product, which would otherwise indeed represent energy-momentum exchange, namely interactions.

3.2 The classical free relativistic real scalar field

Let us thus consider as a classical system a real scalar field $\phi(x)$ over spacetime, whose dynamics is governed by the spacetime local action

$$S[\phi] = \int d^4x^\mu \, \mathcal{L}_0(\phi, \partial_\mu\phi), \tag{13}$$

with the Lagrangian density

$$\mathcal{L}_0(\phi, \partial_\mu\phi) = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{1}{2}m^2\phi^2. \tag{14}$$

We shall apply to this system exactly the same procedure of canonical quantisation as has been described in Section 2, and establish that we have indeed a formulation of free relativistic quantum spinless particles of mass m. The infinite number of degrees of freedom is parametrised by $\phi(x^0, \vec{x}\,)$, and is thus labelled by the values of the space vector $\vec{x}$. Note that there is an abuse in our notation for the parameter m in the above Lagrangian density. At the classical level, only a length scale κ may be introduced, leading to a quadratic term of the form ϕ^2/κ^2 rather than $m^2\phi^2$ above. However, at the quantum level, it will found that the field quanta possess an invariant mass given by $m = \hbar c/\kappa$, which explains our abuse of notation at the classical level already.

In their manifestly Lorentz covariant form, the Euler-Lagrange equations read

$$\partial_\mu \frac{\partial\mathcal{L}_0}{\partial(\partial_\mu\phi)} - \frac{\partial\mathcal{L}_0}{\partial\phi} = 0, \tag{15}$$

or in the present case

$$\left[\partial_\mu\partial^\mu + m^2\right]\phi = 0, \tag{16}$$

which is the Klein-Gordon equation. Through Fourier analysis, the general solution is readily established, and may be expressed as

$$\phi(x^\mu) = \int_{(\infty)} \frac{d^3\vec{k}}{(2\pi)^3 2\omega(\vec{k}\,)} \left[a(\vec{k}\,)e^{-ik\cdot x} + a^*(\vec{k}\,)e^{ik\cdot x}\right], \tag{17}$$

$a(\vec{k}\,)$ and $a^*(\vec{k}\,)$ being complex integration constants, while in the plane wave contributions $e^{\mp ik\cdot x}$ the value $k^0 = \omega(\vec{k}\,)$ is to be used.

In order to quantise the system, let us first consider its Hamiltonian formulation. By definition, the momentum conjugate to the field $\phi(x^0,\vec{x}\,)$ at each point $\vec{x}$ in space is

$$\pi(x^0,\vec{x}\,)=\frac{\partial\mathcal{L}_0}{\partial(\partial_0\phi(x^0,\vec{x}\,))}=\partial_0\phi(x^0,\vec{x}\,), \tag{18}$$

while the phase space degrees of freedom $(\phi(x^0,\vec{x}\,),\pi(x^0,\vec{x}\,))$ possess a Poisson bracket structure defined by the canonical brackets at equal time x^0

$$\{\phi(x^0,\vec{x}\,),\pi(x^0,\vec{y}\,)\}=\delta^{(3)}(\vec{x}-\vec{y}\,). \tag{19}$$

The Hamiltonian density is

$$\mathcal{H}_0=\partial_0\phi\,\pi-\mathcal{L}_0=\frac{1}{2}\pi^2+\frac{1}{2}\left(\vec{\nabla}\phi\right)^2+\frac{1}{2}m^2\phi^2, \tag{20}$$

while the Hamiltonian equations of motion follow as usual from the Hamiltonian $H_0=\int_{(\infty)}d^3\vec{x}\,\mathcal{H}_0$ (namely the sum of $\mathcal{H}_0$ over all degrees of freedom labelled by $\vec{x}$) through the Poisson brackets. For the elementary phase space degrees of freedom, one has,

$$\partial_0\phi=\pi,\qquad \partial_0\pi=\left(\vec{\nabla}^2-m^2\right)\phi, \tag{21}$$

clearly leading back to the Klein-Gordon equation upon reduction of the conjugate momentum π. Hence, given the solution (17) for the field $\phi(x^\mu)$, that for the conjugate momentum is

$$\pi(x^\mu)=\int_{(\infty)}\frac{d^3\vec{k}}{(2\pi)^3 2\omega(\vec{k}\,)}\left(-i\omega(\vec{k})\right)\left[a(\vec{k}\,)e^{-ik\cdot x}-a^*(\vec{k}\,)e^{ik\cdot x}\right]. \tag{22}$$

On basis of these expressions, it is possible to also determine the Poisson bracket structure on the space of integration constants $a(\vec{k}\,)$ and $a^*(\vec{k}\,)$, rather than on the phase space $(\phi(x^0,\vec{x}\,),\pi(x^0,\vec{x}\,))$. A straightforward calculation finds for the only nonvanishing bracket,

$$\{a(\vec{k}\,),a^*(\vec{k}')\}=-i(2\pi)^3 2\omega(\vec{k}\,)\delta^{(3)}\left(\vec{k}-\vec{k}'\right), \tag{23}$$

while the Hamiltonian then reads

$$H_0=\int_{(\infty)}\frac{d^3\vec{k}}{(2\pi)^3 2\omega(\vec{k}\,)}\frac{1}{2}\omega(\vec{k}\,)\left[a^*(\vec{k}\,)a(\vec{k}\,)+a(\vec{k}\,)a^*(\vec{k}\,)\right], \tag{24}$$

hence leading to the Hamiltonian equations of motion

$$\dot{a}(\vec{k}\,)=-i\omega(\vec{k}\,)a(\vec{k}\,),\qquad \dot{a}^*(\vec{k}\,)=i\omega(\vec{k}\,)a^*(\vec{k}\,), \tag{25}$$

whose solutions are of course consistent with the explicit expressions already constructed above for $\phi(x^\mu)$ and $\pi(x^\mu)$.

3.3 The quantum free relativistic real scalar field

Canonical quantisation of the system in the Schrödinger picture, at the reference time $t_0=x_0^0=0$, is straightforward. The space of quantum states $|\psi\rangle$, with hermitean inner product $\langle\chi|\psi\rangle$, provides a representation of the Heisenberg algebra

$$\left[\hat{\phi}(\vec{x}\,),\hat{\pi}(\vec{y}\,)\right]=i\delta^{(3)}\left(\vec{x}-\vec{y}\,\right). \tag{26}$$

In terms of the following representation for the quantum field operators in the Schrödinger picture at $x_0^0 = 0$,

$$\begin{array}{rcl}\hat{\phi}(\vec{x}\,) & = & \int_{(\infty)} \frac{d^3\vec{k}}{(2\pi)^3 2\omega(\vec{k}\,)} \left[a(\vec{k}\,)e^{i\vec{k}\cdot\vec{x}} + a^*(\vec{k}\,)e^{-i\vec{k}\cdot\vec{x}}\right] , \\ \hat{\pi}(\vec{x}\,) & = & \int_{(\infty)} \frac{d^3\vec{k}}{(2\pi)^3 2\omega(\vec{k}\,)} \left(-i\omega(\vec{k}\,)\right) \left[a(\vec{k}\,)e^{i\vec{k}\cdot\vec{x}} - a^*(\vec{k}\,)e^{-i\vec{k}\cdot\vec{x}}\right] ,\end{array} \tag{27}$$

alternatively one has the Fock space algebra

$$\left[a(\vec{k}\,), a^\dagger(\vec{k}')\right] = (2\pi)^3 2\omega(\vec{k})\delta^{(3)}\left(\vec{k} - \vec{k}'\right) . \tag{28}$$

The Schrödinger equation for the time evolution of quantum states in the Schrödinger picture also reads

$$i\hbar\frac{d}{dt}|\psi, t\rangle = \hat{H}_0|\psi, t\rangle \ , \tag{29}$$

with the quantum Hamiltonian given by

$$\hat{H}_0 = \int_{(\infty)} d^3\vec{x} \left[\frac{1}{2}\hat{\pi}^2 + \frac{1}{2}\left(\vec{\nabla}\hat{\phi}\right)^2 + \frac{1}{2}m^2\hat{\phi}^2\right] . \tag{30}$$

Note that this operator does not suffer any operator ordering ambiguity. On the other hand, in terms of the Fock space operators, the same quantum Hamiltonian reads

$$\hat{H}_0 = \int_{(\infty)} \frac{d^3\vec{k}}{(2\pi)^3 2\omega(\vec{k}\,)} \frac{1}{2}\omega(\vec{k}\,) \left[a^\dagger(\vec{k}\,)a(\vec{k}\,) + a(\vec{k}\,)a^\dagger(\vec{k}\,)\right] , \tag{31}$$

which leads to finite matrix elements only after normal ordering of the creation and annihilation operators, a procedure which is denoted by double dots on both sides of a quantity and is defined by commuting all operators so that all creation operators are to the left of all annihilation operators, such as for example

$$: a(\vec{k}\,)a^\dagger(\vec{\ell}\,) : = a^\dagger(\vec{\ell}\,)a(\vec{k}\,), \qquad : a^\dagger(\vec{k}\,)a(\vec{\ell}\,) : = a^\dagger(\vec{k}\,)a(\vec{\ell}\,). \tag{32}$$

Applying this operator ordering prescription to the above expression for $\hat{H}_0$, one thus finds in the Fock space representation the normal ordered Hamiltonian

$$\hat{H}_0 = \int_{(\infty)} \frac{d^3\vec{k}}{(2\pi)^3 2\omega(\vec{k}\,)} \omega(\vec{k}\,)\, a^\dagger(\vec{k}\,)a(\vec{k}\,), \tag{33}$$

while an infinite normal ordering constant contribution is then subtracted away, namely

$$\int_{(\infty)} \frac{d^3\vec{k}}{(2\pi)^3 2\omega(\vec{k}\,)} \frac{1}{2}\omega(\vec{k}\,)(2\pi)^3 2\omega(\vec{0}\,)\delta^{(3)}(\vec{0}). \tag{34}$$

This contribution corresponds to the sum of all vacuum quantum fluctuations of all the $\vec{k}$-modes of the scalar field. Provided the system is not coupled to gravity, such a renormalisation of the energy eigenvalues is without physical consequence. Nonetheless, it should imply that the two representations of the quantised system, namely that achieved through the Heisenberg algebra for the fields, or that achieved through the Fock algebra for its modes, need no longer be unitarily equivalent for such a system with an infinite set of degrees of freedom, in contradistinction to the situation for a system with a finite number of degrees of freedom such as the one-dimensional harmonic oscillator.

It thus appears that one might have available two possibly physically inequivalent approaches to the quantisation of this system, the first based on the representations of the field Heisenberg algebra (26), and the second based on the representations of the field Fock space algebra (28). Let us first consider the Heisenberg algebra realisation, say in its configuration space representation. In the

Schrödinger picture, the basis of states is then spanned by states $|\phi\rangle$ which are associated to specific classical field configurations $\phi(\vec{x}\,)$ defined over space at the reference time $x_0^0 = 0$, and which are eigenstates of the quantum field operator $\hat{\phi}(\vec{x}\,)$,

$$\hat{\phi}(\vec{x}\,)\,|\phi\rangle = \phi(\vec{x}\,)\,|\phi\rangle. \tag{35}$$

The values for the vector $\vec{x}$ being the label for degrees of freedom, at least formally one has the following normalisation of these states, together with the associated spectral resolution of the unit operator,

$$\langle\phi|\phi'\rangle = \prod_{\vec{x}} \delta\left(\phi(\vec{x}\,) - \phi'(\vec{x}\,)\right), \qquad \mathbb{I} = \int_{-\infty}^{\infty} \prod_{\vec{x}} d\phi(\vec{x}\,)\,|\phi\rangle\langle\phi|, \tag{36}$$

in direct analogy with the situation for a system with a finite number of degrees of freedom. Hence, arbitrary quantum states $|\psi\rangle$ possess now a configuration space wave functional representation $\Psi[\phi]$ defined by

$$\Psi[\phi] = \langle\phi|\psi\rangle \quad , \quad |\psi\rangle = \int_{-\infty}^{\infty} \prod_{\vec{x}} d\phi(\vec{x}\,)\,|\phi\rangle\,\Psi[\phi], \tag{37}$$

which thus represents the probability amplitude for observing the given quantum state $|\psi\rangle$ in the classical field configuration $\phi(\vec{x}\,)$, again in direct analogy with the meaning of the configuration space wave function for a finite dimensional system.

Furthermore, since the field operators $\hat{\phi}(\vec{x}\,)$ and $\hat{\pi}(\vec{x}\,)$ possess the following configuration space representations,

$$\langle\phi|\hat{\phi}(\vec{x}\,)|\psi\rangle = \phi(\vec{x}\,)\,\Psi[\phi], \qquad \langle\phi|\hat{\pi}(\vec{x}\,)|\psi\rangle = -i\hbar\frac{\delta}{\delta\phi(\vec{x}\,)}\,\Psi[\phi], \tag{38}$$

the action of the quantum Hamiltonian on quantum states in their configuration space wave functional representation is

$$\langle\phi|\hat{H}_0|\psi\rangle = \int_{(\infty)} d^3\vec{x}\,\frac{1}{2}\left[-\hbar^2\left(\frac{\delta}{\delta\phi(\vec{x}\,)}\right)^2 + \left(\vec{\nabla}\phi(\vec{x}\,)\right)^2 + m^2\phi^2(\vec{x}\,)\right]\,\Psi[\phi]. \tag{39}$$

This Schrödinger functional representation of a quantum field theory could prove to be an appropriate framework in which to attempt a nonperturbative quantisation. Even though it may well be that for a noninteracting field, which is the above situation, this approach would be unitarily equivalent to the Fock space one to be discussed presently, it is far from clear that such an equivalence should survive the introduction of nonlinear interactions. Given the wide success of the perturbative treatment of particle interactions, based on the Fock space quantisation of a field theory briefly described hereafter, such nonperturbative functional quantisations have not been developed to the same extent, making this issue a worthwhile topic of further investigation, especially when it comes to nonlinear field theories whose space of classical solutions includes topological configurations such as solitons and higher dimensional monopole-like configurations.

Turning now to the field Fock space algebra (28) and its representations, it is clear that the space of states is spanned by all possible n-particle states ($n = 0, 1, 2, \cdots$) of arbitrary momentum values $\vec{k}_i$ ($i = 1, 2, \cdots, n$), which are built through the action of the creation operators $a^\dagger(\vec{k}\,)$ from the normalised Fock vacuum $|0\rangle$, itself annihilated by the $a(\vec{k}\,)$ operators, $a(\vec{k}\,)|0\rangle = 0$,

$$|\vec{k}_1, \vec{k}_2, \cdots, \vec{k}_n\rangle = N(\vec{k}_1, \vec{k}_2, \cdots, \vec{k}_n)\,a^\dagger(\vec{k}_1)\,a^\dagger(\vec{k}_2)\cdots a^\dagger(\vec{k}_n)|0\rangle, \tag{40}$$

where $N(\vec{k}_1, \vec{k}_2, \cdots, \vec{k}_n)$ denotes some normalisation factor. In particular, the 1-particle quantum states correspond to

$$|\vec{k}\rangle = a^\dagger(\vec{k}\,)|0\rangle, \qquad \langle\vec{k}|\vec{k}'\rangle = (2\pi)^3 2\omega(\vec{k}\,)\,\delta^{(3)}\left(\vec{k} - \vec{k}'\right). \tag{41}$$

In addition, given the manifest spacetime invariance of the system under the Poincaré group, the quantum operators $\hat{P}^\mu$ and $\hat{M}^{\mu\nu}$ associated to the conserved Poincaré Noether charges generate the

Poincaré algebra on the space of quantum states, the latter thus getting organised into irreducible representations of that symmetry. The eigenstates of these operators, thus of definite energy-momentum, angular-momentum and invariant mass, define the 1-particle states of the quantised field. Clearly, these eigenstates must correspond to the 1-particle quantum states $|\vec{k}\rangle$ constructed above, which is indeed the case. For instance, the energy-momentum operator in Fock space is given by

$$\hat{P}^\mu = \int_{(\infty)} \frac{d^3\vec{k}}{(2\pi)^3 2\omega(\vec{k}\,)}\, k^\mu\, a^\dagger(\vec{k}\,)a(\vec{k}\,), \tag{42}$$

so that the 1-particle states $|\vec{k}\rangle$ are eigenstates of this operator, namely $\hat{P}^\mu|\vec{k}\rangle = k^\mu|\vec{k}\rangle$, with the eigenvalues

$$\hat{P}^0 \;:\; k^0 = \omega(\vec{k}\,); \qquad \hat{\vec{P}} \;:\; \vec{k}. \tag{43}$$

In particular, the relativistic invariant mass eigenvalue of these states is m^2, showing that the parameter m indeed measures the mass of the quanta of the quantised field. Likewise for the generalised angular-momentum operator $\hat{M}^{\mu\nu}$, the 1-particle states $|\vec{k}\rangle$ possess an eigenvalue which measures their orbital angular-momentum, thus expressing the fact that the quanta associated to the scalar field $\phi(x^\mu)$ are indeed spinless particles. In order to obtain 1-particle states with a nontrivial spin value, one has to use fields which transform nontrivially under the Lorentz group SO(3,1), such as a vector field leading then to particles of unit spin or helicity (the latter in the massless case), or a spinor field (whether a Weyl, a Dirac or a Majorana spinor) leading to particles of 1/2 spin or helicity values (Grassmann odd variables must be used to parametrise spinor field degrees of freedom, leading, at the classical level, to Grassmann graded Poisson bracket structures and, at the quantum level, to anticommutation rather than commutation rules for fermionic quantum operators).

Hence, as expected on basis of the heuristic construction of Section 3.1, the Fock space representation of a relativistic quantum field theory (whose action is quadratic in the field) shows that the physical content of such a system is that of an arbitrary ensemble of identical free relativistic quantum point particles of definite mass, energy- and angular-momentum. The interpretation of the field quanta as being such relativistic particles is made consistent by the manifest Poincaré invariance of the action principle.

The above Fock space construction of the quantised field is performed within the Schrödinger picture at the reference time $x^0_0 = 0$. Within the corresponding Heisenberg picture, states are time independent whilst the quantum operators, among which the basic field $\hat{\phi}(\vec{x}\,)$, are rather now explicitly time dependent and carry the whole dynamics of the system. Given the quantum Hamiltonian (33), it is straightforward to show, based exactly on the definition of the time dependence of operators in that picture, that in the Heisenberg picture the relativistic quantum scalar field is given precisely by the expression (10) which was constructed heuristically in Section 3.1. Hence, it is precisely the ordinary rules of canonical quantisation, and only these, which, when applied to the classical system describing the dynamics of a relativistic field theory, lead to a framework which readily accounts for all the observed physical spacetime properties of relativistic quantum particles including the possibility of their creation and annihilation, which is possible only within a formalism which includes both special relativity and quantum mechanics.

In particular, acting with the quantum field $\hat{\phi}(x^\mu)$ in the Heisenberg picture on the Fock vacuum, one obtains a plane wave superposition of 1-particle states of definite momentum,

$$\hat{\phi}(x^\mu)|0\rangle = \int_{(\infty)} \frac{d^3\vec{k}}{(2\pi)^3 2\omega(\vec{k}\,)}\, e^{ik\cdot x}\, |\vec{k}\,\rangle. \tag{44}$$

Such a state may thus be viewed as the quantum configuration of the field such that one particle has been created exactly at the spacetime point x^μ, which, as a consequence of Heisenberg's uncertainty principle, thus possesses a totally undertermined energy-momentum value with its characteristic plane wave probability amplitude. More generally, this interpretation also enables one to construct the probability amplitude for the process in which one particle is created at a given initial spacetime point x_i^μ and then annihilated at the final point x_f, while it propagates in a causal manner between these two positions. This quantity is thus defined by the time-ordered two-point function of the field operator,

$$\begin{aligned}\langle 0|T\left(\hat{\phi}(x_f)\hat{\phi}(x_i)\right)|0\rangle = &\ \theta(x_f^0 - x_i^0)\,\langle 0|\hat{\phi}(x_f)\hat{\phi}(x_i)|0\rangle \\ &+\ \theta(x_i^0 - x_f^0)\,\langle 0|\hat{\phi}(x_i)\hat{\phi}(x_f)|0\rangle,\end{aligned} \tag{45}$$

($\theta(x)$ is the usual step function such that $\theta(x>0)=1$ and $\theta(x<0)=0$) and corresponds to what is called the Feynman propagator for single field quanta. Using the explicit expansion (10) of the field operator in the Heisenberg picture in terms of the creation and annihilation operators, it is straightforward to establish that the Feynman propagator is given by the manifestly spacetime invariant expression

$$\langle 0|T\left(\hat{\phi}(x_f)\hat{\phi}(x_i)\right)|0\rangle = \int_{(\infty)} \frac{d^4k^\mu}{(2\pi)^4}\, e^{-ik\cdot(x_f-x_i)}\, \frac{i}{k^2-m^2+i\epsilon}, \tag{46}$$

where the infinitesimal parameter $\epsilon > 0$ is introduced in order to specify the contour integration in the complex plane for the energy contribution k^0, so that the correct causal structure of this propagator is recovered. This quantity is also one of the Green functions for the Klein-Gordon operator $[\partial_\mu\partial^\mu + m^2]$.

Hence, the marriage of special relativity and of quantum mechanics, namely of the constants c and $\hbar$, leads in a most natural way to a fundamental convergence and unification of concepts: relativistic quantum particles are nothing but the quanta of relativistic quantum fields, displaying at the same time the corpuscular properties of particles and the wavelike properties of the spacetime dynamics of fields. This is indeed a most powerful and all encompassing outcome of the unification of relativity and quantum mechanics. Among other consequences, it explains at once why identical particles are necessarily indistinguishable, since they simply correspond to actual physical quantum fluctuations of a single physical entity filling all of spacetime, namely the corresponding relativistic quantum field, and which may be excited or absorbed, namely created or annihilated, by acting on the system through some interaction with another field. In fact, and as shall become clear in Section 4, even interactions, namely changes in the total energy-momentum content of given quantum field states, are understood in terms of exchanges of such 1-particle quanta between given quantum states. The notion of a force acting on a relativistic particle, or of a potential energy contributing to the Hamiltonian of a quantum system, is also superseded by that of fields filling all of spacetime, and interacting with one another through local spacetime couplings, thereby leading to the exchanges of 1-particle quanta. Other profound consequences of the relativistic quantum field picture of physical reality are the spin-statistics connection (namely the fact that integer spin particles obey the Bose-Einstein statistics while half-integer spin particles the Fermi-Dirac statistics), the invariance of any relativistic quantum field theory under the combined product of the parity, time reversal and charge conjugation transformations (the so-called CPT theorem), and the particle/antiparticle duality (only this latter point is discussed explicitly hereafter).

It is clear that the Fock space quantisation of field theory is ideally suited for a perturbative description of interactions, namely by starting with a situation with only free quanta, corresponding to an action which is quadratic in the fields, and then adding as perturbations to be summed through a series expansion further corrections involving locally in spacetime higher order products of the fields and their couplings, thus leading to successive perturbative corrections to quantum matrix elements of specific observables which may be viewed in terms of specific 1-particle exchanges among quantum states. This procedure will briefly be outlined in Section 4. On the other hand, the Schrödinger functional quantisation of a field theory is from the outset nonperturbative in character, and may thus be better suited to study nonperturbative issues in quantum field theory, in ways that have not been explored to the same extent as the perturbative picture of quantum field theory.

A final remark may also be in order concerning some vocabulary. Note that exactly the same methods of canonical quantisation are applied whether for a finite or an infinite dimensional dynamical system. Often in the literature, one finds written that the first situation is that of "first quantisation", while the second that of "second quantisation". Furthermore, there is also quite often mention of "negative energy states" and "negative probabilities", which must be circumvented through "second quantisation". The fact of the matter is that this vocabulary is due to an historical accident. Initially, one wished to develop a relativistic extension of the nonrelativistic Schrödinger equation for, say, the harmonic oscillator and its configuration space wave function. Doing so, one unavoidably encounters diverse problems of negative energy and/or probability states, which defy a consistent physical interpretation. Considering then that the "relativistic wave function" itself needs to be quantised, one discovered that these issues are evaded altogether, leading in fact to the quantum field theory representations that were described above. In other words, the correct physical point of view is that, rather than quantising some relativistic wave function, from the outset one is in fact (first!) quantising a classical field theory which obeys some relativistic invariant wave equation, and at no point whatsoever do issues of "negative energy or probability states" arise. In the same way that quantum mechanics, whether relativistic or not, is the quantisation of finite dimensional systems whose configurations represent as a function of time, say, the positions in space of a finite collection of particles, quantum field theory is the quantisation of infinite dimensional systems whose configurations are, say, the values taken by a finite collection of fields in space as a function of time, all in a spacetime invariant manner in the case of a relativistic field theory.

4 Interactions and the Gauge Symmetry Principle

Having understood how the dynamics of a relativistic quantum field whose Lagrangian density is quadratic in the field in fact describes a system whose quantum states correspond to an arbitrary number of identical free relativistic quantum particles of definite energy-momentum, spin and invariant mass, it becomes possible to envisage an extension of this formalism in order to account for interactions among such particles, namely the exchange of energy and momentum between such quantum states through the creation and annihilation of the associated quanta. Clearly, such a formulation is perturbative in character, since the free particle picture provides the starting point for a perturbative expansion in which an increasing number of interaction points are included for a given physical process. The purpose of the present section is to briefly outline how this point of view, which has proved to be so powerful and relevant to high energy particle physics and their fundamental interactions except for the gravitational one, has led, on the one hand, to the local gauge symmetry principle as an essential requirement for any theory of the fundamental interactions, and on the other hand, to the Feynman diagrammatic representation of physical processes through a perturbative expansion of the associated probability amplitudes order by order in the exchanges of interacting particles.

4.1 Field coupling and interactions

For definitiness, the discussion to be presented uses the simplest of examples, namely that of an interacting real scalar field $\phi(x^\mu)$ whose Lagrangian density now includes also a quartic term in the potential contribution, in addition to the quadratic contribution considered so far,

$$\mathcal{L}(\phi,\partial_\mu\phi)=\frac{1}{2}\left(\partial_\mu\phi\right)^2-\frac{1}{2}m^2\phi^2-\frac{1}{4!}\lambda\phi^4, \tag{47}$$

$\lambda>0$ being a real positive parameter which turns out to correspond to a coupling constant measuring the strength of a spacetime local interaction in which four quanta of the field $\phi(x)$ are involved in a perturbative expansion. Compared to the free field case, we thus have

$$\mathcal{L}=\mathcal{L}_0\ +\ \mathcal{L}_{\rm int},\qquad \mathcal{L}_{\rm int}=-\frac{1}{4!}\lambda\phi^4, \tag{48}$$

$\mathcal{L}_0$ being the free field Lagrangien density whose quantisation has been discussed above, while $\mathcal{L}_{\rm int}$ corresponds to an additional contribution associated to some specific interaction. The canonical quantisation of such a system follows the same rules as those applied in the free field case, with in particular the fundamental Poisson brackets

$$\{\phi(x^0, \vec{x}\,), \pi(x^0, \vec{y}\,)\} = \delta^{(3)}\left(\vec{x} - \vec{y}\,\right), \tag{49}$$

which remain those of the free field case. Note that the conjugate momentum is still given by the relation $\pi(x) = \partial_0\phi(x)$ (had the interacting Lagrangian $\mathcal{L}_{\rm int}$ included some derivative coupling of the field ϕ, the conjugate momentum would have been different). However, the canonical Hamiltonian density acquires an additional contribution directly related to and determined by $\mathcal{L}_{\rm int}$, namely

$$\begin{aligned} \mathcal{H} &= \tfrac{1}{2}\pi^2 + \tfrac{1}{2}\left(\vec{\nabla}\phi\right)^2 + \tfrac{1}{2}m^2\phi^2 + \tfrac{1}{4!}\lambda\phi^4 \\ &= \mathcal{H}_0 \;+\; \mathcal{H}_{\rm int}, \qquad \mathcal{H}_{\rm int} = -\mathcal{L}_{\rm int} = \tfrac{1}{4!}\lambda\phi^4. \end{aligned} \tag{50}$$

The restriction on the coupling constant $\lambda > 0$ stems from the requirement that the energy spectrum of the system be bounded below, since otherwise no stable ground state may exist. The same requirement also explains why a purely ϕ^3 term, without the quartic contribution in $\mathcal{L}_{\rm int}$, is not considered in the above discussion, even though the perturbative expansion to be described presently is then somewhat simpler to implement in actual calculations.

Consequently, the canonical quantisation of the system, even in the presence of the interaction contribution, may still be performed, say, in the Fock space representation in terms of the creation and annihilation operators of free particle quanta, with a specific definition of a self-adjoint Hamiltonian operator $\hat{H} = \int_{(\infty)} d^3\vec{x}\,\hat{\mathcal{H}}$ through normal ordering in these operators. However, what then becomes a nontrivial issue is the actual diagonalisation of this Hamiltonian, namely the identification of the actual spectrum of the quantised interacting field theory. A perturbative approach in the parameter λ enables an order by order identification of the quantum physical content of such a system and of its physical properties, starting from the free field quanta.

The scattering matrix

In practical terms, an extremely important method for the experimental investigation of the quantum relativistic properties of physical systems is that of scattering measurements. Different components of a given system are prepared in a given initial configuration in causally separated regions of space, and are then made to scatter within a given local neighbourdhood of an interaction point, from where interaction products emerge whose properties are then measured and analysed, in order to infer the specific characteristics of the interactions at work and responsible for the observed process. In other words, all the physical information related to these interactions is encoded into the corresponding scattering probability amplitude.

Given such a general scheme, the basic implicit idea is that the interaction takes place over a region of space whose extent is so small that for all practical purposes the interactions are only short-ranged, so that beyond that interaction region the separated components of the system are free from interactions. In a classical picture, such components may be viewed as independent free particles each following asymptotically a straight trajectory. When the interactions are "turned off", these trajectories are not modified as they pass one another, and are thus not scattered. However, when the interaction is "turned on", the more the particles approach one another, the more their trajectories deviate from a straight path, leading in the asymptotic final state to a scattered configuration of straight trajectories as the final state components which emerge from the spatial interaction region. In other words, the characterisation of a nontrivial scattering process proceeds by extrapolating to both the infinite past and the infinite future the time dependent dynamics of a given configuration of the system, and by comparing the asymptotic states to what they would have been had there not been any interaction.

Clearly, the same heuristic understanding of the characterisation of the scattering process applies at the quantum level, by comparing the time dependence of given in- and out-states in the presence or absence of some given interaction, provided the initial asymptotic states are identical. The characterisation of the scattering process, and of the interaction responsible for it, is then obtained by identifying the operator in Hilbert space which leads to this transition between the in- and out-asymptotic states. This is the scattering operator S whose matrix elements are thus the quantities of interest, which represent the probability amplitude for a given physical scattering process to occur.

Let us translate this reasoning in mathematical terms. Concentrating first on the initial state, let us represent the free Hamiltonian by $\hat{H}_0$, the total Hamiltonian including interactions by $\hat{H}$, and assume to be working in the Schrödinger picture at some reference time t_0. A given state $|\psi_{\rm in}, t_0\rangle$ of the free theory is then evolved backwards in time into the asymptotic in-state

$$|\psi_{\rm in}, -\infty\rangle = \lim_{t\rightarrow-\infty} e^{-i(t-t_0)\hat{H}_0}\,|\psi_{\rm in}, t_0\rangle = \lim_{t\rightarrow-\infty} |\psi_{\rm in}, t\rangle, \tag{51}$$

while a given state $|\psi, t_0\rangle$ of the interacting theory is likewise propagated back in the infinite past according to

$$|\psi, -\infty\rangle = \lim_{t\rightarrow-\infty} e^{-i(t-t_0)\hat{H}}\,|\psi, t_0\rangle = \lim_{t\rightarrow-\infty} |\psi, t\rangle. \tag{52}$$

However, these two asymptotic states should correspond to an identical asymptotic quantum in-state, so that the asymptotic correspondence is defined by the relation

$$|\psi, -\infty\rangle = |\psi_{\rm in}, -\infty\rangle. \tag{53}$$

Likewise for the asymptotic quantum out-state, one has the identification

$$|\chi, +\infty\rangle = |\chi_{\rm out}, +\infty\rangle, \tag{54}$$

where

$$|\chi_{\rm out}, +\infty\rangle = \lim_{t\rightarrow+\infty} e^{-i(t-t_0)\hat{H}_0}\,|\chi_{\rm out}, t_0\rangle = \lim_{t\rightarrow+\infty} |\chi_{\rm out}, t\rangle, \tag{55}$$

$$|\chi, +\infty\rangle = \lim_{t\rightarrow+\infty} e^{-i(t-t_0)\hat{H}}\,|\chi, t_0\rangle = \lim_{t\rightarrow+\infty} |\chi, t\rangle. \tag{56}$$

Note that behind this construction lies the fact that the quantum theories based on $\hat{H}_0$ and $\hat{H}$ share a common space of quantum states, namely an identical representation space of a common algebraic structure of commutation relations for the fundamental degrees of freedom. The scattering operator, whose matrix elements we are about to characterise, is thus an operator acting withing this common space of quantum states, which must reduce to the identity operator in the absence of any interaction, $\hat{H} = \hat{H}_0$.

Given the above formulation, it is clear that the transition probability amplitude between the asymptotic in- and out-states of the interacting theory is simply given by

$$\langle\chi, t|\psi, t\rangle = \langle\chi, t_0|\psi, t_0\rangle, \tag{57}$$

the value of this matrix element being independent of the time t at which it is evaluated since the evolution operator $e^{-i(t-t_0)\hat{H}}$ for the interacting theory defines a unitary isomorphism between all Schrödinger pictures for all values of t. However, this matrix element may also be expressed in terms of the in- and out-states of the free theory, since the asymptotic in-states for either theory are identical. A direct substitution of the above relations then finds

$$\langle\chi, t|\psi, t\rangle = \langle\chi, t_0|\psi, t_0\rangle = \langle\chi_{\rm out}, t_0|S|\psi_{\rm in}, t_0\rangle, \tag{58}$$

where the scattering operator S is defined by the asymptotic limits

$$S = \lim_{t_\mp\rightarrow\mp\infty} M(t_+, t_0)\, M^\dagger(t_-, t_0), \tag{59}$$

with

$$M(t, t_0) = e^{i(t-t_0)\hat{H}_0}\, e^{-i(t-t_0)\hat{H}}. \tag{60}$$

Note that in the absence of any interaction, $\hat{H} = \hat{H}_0$, the scattering operator S indeeds reduces to the identity operator. Since the operator $M(t, t_0)$ plays such a central role in the construction of the scattering operator S, it is important to obtain alternative expressions for it. In particular, one readily establishes the differential equation

$$\begin{aligned} i\partial_t M(t,t_0) &= e^{i(t-t_0)\hat{H}_0} \left[\hat{H} - \hat{H}_0\right] e^{-i(t-t_0)\hat{H}} \\ &= e^{i(t-t_0)\hat{H}_0} \hat{H}_{\rm int}(t_0) e^{-i(t-t_0)\hat{H}_0} M(t,t_0) \\ &= \hat{H}^{(I)}_{\rm int}(t) M(t,t_0), \end{aligned} \tag{61}$$

having introduced

$$\hat{H}^{(I)}_{\rm int}(t) = e^{i(t-t_0)\hat{H}_0} \hat{H}_{\rm int}(t_0) e^{-i(t-t_0)\hat{H}_0} \quad , \quad \hat{H}_{\rm int}(t_0) = \hat{H} - \hat{H}_0. \tag{62}$$

Note that this latter definition coincides with that of the Heisenberg picture associated to the free Hamiltonian $\hat{H}_0$. Since in the interacting theory the Heisenberg picture should be defined in a likewise manner but in terms of the full Hamiltonian $\hat{H}$ rather than the free Hamiltonian $\hat{H}_0$, one refers to the "interaction picture" as being associated to the general definition of time dependent operators $\mathcal{O}_{(I)}$ given by

$$\mathcal{O}_{(I)}(t) = e^{i(t-t_0)\hat{H}_0} \mathcal{O}(t_0) e^{-i(t-t_0)\hat{H}_0}, \tag{63}$$

where $\mathcal{O}(t_0)$ is the operator as constructed through canonical quantisation of the interacting theory in its Schrödinger picture.

In other words, in the interaction picture, quantum states as well as operators carry a split time dependence, such that the one carried by the quantum states is solely induced by the interactions and the interacting Hamiltonian $\hat{H}_{\rm int}$, while the one carried by the quantum operators is solely induced by the time dependence related to the free field dynamics and the free Hamiltonian $\hat{H}_0$. In the interaction picture, any time dependence in the quantum states is totally ascribed to the interactions only.

Returning to the equation (61) characterising the operator $M(t, t_0)$, one sees that its solution may also be expressed in the form

$$M(t,t_0) = T\, e^{-i\int_{t_0}^{t} dt'\, \hat{H}^{(I)}_{\rm int}(t')}, \tag{64}$$

where the symbol T in front of the exponential in the r.h.s. of this expression stands for the time-ordered product and exponential in which products of time-dependent operators are integrated from left to right in decreasing order of their time arguments (this is indeed required given that the operator $M(t, t_0)$ is to the right of $\hat{H}^{(I)}_{\rm int}(t)$ in (61)).

Hence, using this solution for the operator $M(t, t_0)$, the scattering operator acquires the expression

$$S = T\, e^{-i\int_{t_0}^{\infty} dt \hat{H}^{(I)}_{\rm int}(t)}\, T\, e^{-i\int_{-\infty}^{t_0} dt \hat{H}^{(I)}_{\rm int}(t)} = T\, e^{-i\int_{-\infty}^{\infty} dt \int_{(\infty)} d^3\vec{x}\, \hat{\mathcal{H}}^{(I)}_{\rm int}}, \tag{65}$$

which, in the absence of any derivative coupling in the interacting Lagrangian density, so that $\mathcal{L}_{\rm int} = -\mathcal{H}_{\rm int}$, finally reduces to

$$S = T\, e^{-i\int_{(\infty)} d^4x^\mu\, \hat{\mathcal{H}}^{(I)}_{\rm int}} = T\, e^{i\int_{(\infty)} d^4x^\mu\, \hat{\mathcal{L}}^{(I)}_{\rm int}}. \tag{66}$$

In this form, it should be clear why this formulation of any scattering process is ideally suited for a perturbative treatment. Since scattering matrix elements are given by matrix elements of the operator S for free field external states, see (58), it suffices to consider the creation and annihilation mode expansions of the field and its conjugate momentum in the interaction picture, and substitute these in the expressions for the interacting Lagrangian and Hamiltonian densities in the interaction picture. In particular, these fields in the interaction picture retain their expressions valid for the Heisenberg picture of the free field theory. One has

$$\begin{aligned}\hat{\phi}_{(I)}(t,\vec{x}\,) &= e^{i(t-t_0)\hat{H}_0}\,\hat{\phi}(t_0,\vec{x}\,)\,e^{-i(t-t_0)\hat{H}_0}\ , \\ \hat{\pi}_{(I)}(t,\vec{x}\,) &= e^{i(t-t_0)\hat{H}_0}\,\hat{\pi}(t_0,\vec{x}\,)\,e^{-i(t-t_0)\hat{H}_0},\end{aligned} \tag{67}$$

with the mode expansions

$$\hat{\phi}_{(I)}(x) = \int_{(\infty)} \frac{d^3\vec{k}}{(2\pi)^3 2\omega(\vec{k}\,)} \left[a(\vec{k}\,)e^{-ik\cdot x} \,+\, a^\dagger(\vec{k}\,)e^{ik\cdot x}\right], \tag{68}$$

$$\hat{\pi}_{(I)}(x) = \int_{(\infty)} \frac{d^3\vec{k}}{(2\pi)^3 2\omega(\vec{k}\,)} \left(-i\omega(\vec{k}\,)\right) \left[a(\vec{k}\,)e^{-ik\cdot x} \,-\, a^\dagger(\vec{k}\,)e^{ik\cdot x}\right], \tag{69}$$

while the creation and annihilation operators still obey the usual algebra

$$\left[a(\vec{k}\,), a^\dagger(\vec{\ell}\,)\right] = (2\pi)^3 2\omega(\vec{k}\,)\delta^{(3)}\left(\vec{k}-\vec{\ell}\,\right), \tag{70}$$

since canonical quantisation in the Schrödinger picture of the interacting theory still requires the commutation relations

$$\left[\hat{\phi}(t_0,\vec{x}\,), \hat{\pi}(t_0,\vec{y}\,)\right] = i\delta^{(3)}\left(\vec{x}-\vec{y}\,\right). \tag{71}$$

Furthermore, once such a substitution has been effected, a straightforward expansion of the time-ordered exponential (65) defining the scattering operator in terms of the interacting Hamiltonian in the interaction picture leads to an expansion in powers of the coupling coefficient λ, namely a perturbative representation of the probability amplitude associated to a given set of external states in terms of successive exchanges of free particle quanta being created and annihilated through the interaction couplings of the fields as they contribute to the interacting Hamiltonian.

In particular, it should be clear that successive contractions of these creation and annihilation operators as they are commuted past one another in the evaluation of the matrix elements, all in a manner consistent with the causal time ordering implied by the solution (65), always lead precisely to the time-ordered two-point function of the field operator in the interaction picture, namely the Feynman propagator computed previously for the free field theory,

$$\langle 0|T\left(\hat{\phi}_{(I)}(x)\hat{\phi}_{(I)}(y)\right)|0\rangle = \int_{(\infty)} \frac{d^4k^\mu}{(2\pi)^4}\,\frac{i}{k^2-m^2+i\epsilon}\,e^{-ik\cdot(x-y)}, \tag{72}$$

where $|0\rangle$ still denotes the perturbative normalised Fock vacuum annihilated by the operators $a(\vec{k}\,)$, $a(\vec{k}\,)|0\rangle = 0$.

Even though we cannot consider here a discussion of perturbation theory in any detail whatsoever, once put within such a framework, it takes little effort of imagination to understand how a systematic set of rules for such a perturbative expansion and evaluation of scattering matrix elements may be identified, thus providing an efficient approach towards the determination of scattering cross sections of direct relevance to experimental results. Such a discussion would consist in a whole set of lectures on their own, which is not the purpose of the present notes and may be found exposed in great detail in any quantum field theory textbook. Nonetheless, from the above description, it should be clear that Fock space quantisation of relativistic quantum field theory is ideally suited for a perturbative representation of interacting relativistic quantum particles, and that this perturbation theory approach is directly based on the interacting Hamiltonian and Lagrangian contribution to the total Lagrangian density, namely all those contributions which are not purely quadratic in the fields, the latter on their own being relevant to the description of free relativistic quantum particles.

Perturbation theory

In spite of the fact that this is not the place for a detailed presentation of perturbative quantum field theory, let us nevertheless highlight some points of relevance to the discussion hereafter, particularising again to the simplest $\mathcal{L}_{\rm int} = -\lambda\phi^4/4!$ interacting Lagrangian. As far as scattering processes are concerned, all possible results are encoded into the scattering operator

$$S = \mathbb{I} \, + \, \sum_{n=1}^{\infty} \frac{1}{n!} T \left(-i \int_{(\infty)} d^4x^\mu \, \hat{\mathcal{H}}^{(I)}_{\rm int}(x) \right)^n , \tag{73}$$

where now the interacting Hamiltonian density in the interaction picture is defined according to the usual normal ordering prescription for the creation and annihilation operators $a^\dagger(\vec{k}\,)$ and $a(\vec{k}\,)$,

$$\hat{\mathcal{H}}^{(I)}_{\rm int}(x) = \frac{1}{4!}\lambda \, : \hat{\phi}^4_{(I)}(x) : . \tag{74}$$

Clearly, when considering the scattering operator in this series expanded form and the evaluation of its matrix elements for external states associated to definite numbers of incoming and outgoing particles, time ordering of operator products commuted with one another implies the contribution of the Feynman propagator which, in momentum space, simply leads to the following contribution for any internal propagating line connecting two interaction vertices at which particle quanta are created or annihilated,

$$\frac{i}{k^2 - m^2 + i\epsilon}. \tag{75}$$

Likewise, whenever the operator $\hat{\mathcal{H}}^{(I)}_{\rm int}(x)$ contributes at a given order of the perturbative expansion, it implies a spacetime local interaction in which four particle quanta are either created or annihilated, with an amplitude given by the factor

$$-i\lambda, \tag{76}$$

up to some combinatorics factor depending on the topology of the associated diagram.

In other words, it is possible to translate the mathematical expression for the relevant matrix element evaluation into a diagrammatic representation in which internal lines are connected to interaction vertices, and for which the above contributions are then multiplied with one another, and integrated over internal momenta in a manner such as to obey the rules of energy-momentum conservation at each vertex, in order to determine the associated probablity amplitude. These rules relating such Feynman diagrams to the required mathematical quantity are the Feynman rules of perturbative quantum field theory. In the specific case of the $\lambda\phi^4/4!$ scalar field theory, the above discussion thus establishes that these rules consist only of the single interaction vertex accompanying the scalar Feynman propagator. In principle, given such rules, any scattering amplitude for whatever physical process may be computed to an arbitrary order in the perturbative expansion in the coupling constant λ.

As far as we are concerned, the main conclusion to be drawn from the above is that once relativistic quantum fields are coupled to one another through local spacetime couplings, such as $\mathcal{L}_{\rm int} = -\lambda\phi^4/4!$, one in facts has made available within a perturbative picture a formalism in which local and causal quantum interactions are directly understood in terms of exchanges of quantum particles free to propagate between interaction vertices that occur locally in spacetime but at arbitrary positions which are integrated over when they are not observed. The marriage of $\hbar$ and c leading to quantum field theory as the natural framework for the description of relativistic quantum point particles also implies a physical understanding of the physical origin of forces and interactions simply as following from the spacetime local couplings of fields, which also translate in the dual corpuscular picture into a process in which particles are being created, annihilated and exchanged, thereby leading to changes in their energy-momentum, hence to their interactions. The mysterious action at a distance of classical mechanics is forever gone, superseded by relativistic quantum fields which provide a natural framework not only for a unified description both of the corpuscular properties of matter and of the wavelike properties of their spacetime dynamics, but also a unified understanding of the fundamental quantum interactions in terms of both spacetime local couplings of fields and causal exchanges of particle quanta, all in a manner consistent with the principles of special relativity, of unitary quantum mechanics, and of causality.

However, this amazing convergence of physical concepts based on a few general basic principles comes with a price. When considering the perturbative expansion of scattering matrix elements, one soon comes across loop diagram contributions in which one must integrate over the internal momenta running around closed loops. For instance when considering the propagation of a single particle quantum, the first order correction to the propagator is obtained by inserting into it the four-point vertex $\lambda\phi^4/4$ and then contracting two of its four external lines with one another, leading to a 1-loop contribution with the factor

$$(-i\lambda)\int_{(\infty)}\frac{d^4p^\mu}{(2\pi)^4}\left(\frac{i}{p^2-m^2+i\epsilon}\right), \tag{77}$$

the origin of each of the factors in parentheses being obvious, while the closed loop propagator must be integrated over the associated energy-momentum. Likewise, when considering a $2\rightarrow 2$ scattering process with two initial and two final particles, beyond the nonscattering and one interaction vertex contributions, there appears a 1-loop correction in which two 4-point vertices are inserted with two lines of each being contracted in pairs with two lines of the other. The corresponding contribution is given by

$$(-i\lambda)^2\int_{(\infty)}\frac{d^4p^\mu}{(2\pi)^4}\left(\frac{i}{p^2-m^2+i\epsilon}\right)\left(\frac{i}{(p+k)^2-m^2+i\epsilon}\right), \tag{78}$$

where p^μ is again the energy-momentum running around the closed loop (say, that running through one of the two internal contracted lines), while k^μ is the total external energy-momentum of the two initial or final particles ($k+p$ being then the energy-momentum running through the other internal line).

The characteristic feature of such contributions, which arise whenever closed loops appear in a diagram, is their divergence for large values of the internal momentum, namely in the ultra-violet at small distances. The fundamental reason for this feature is that interactions occur locally in spacetime at given points where the fields are multiplied with one another. In order to perform calculations nonetheless, one has to introduce some regularisation procedure to tame such divergencies, and hope that at the very end, when all contributions are summed up again, all the divergent contributions would combine is such a manner that physical observables remain nevertheless finite, even if affected by finite renormalisation. Many different regularisation procedures have been developed, and this is not the place to discuss such issues. The most straightforward one is to introduce an upper cut-off value Λ_c in the momentum integration, to keep track of the different types of divergencies that may arise. For instance, the 1-loop correction to the scalar field propagator given above leads to a quadratic divergence proportional to Λ_c^2, while that to the $2\rightarrow 2$ scattering process is only logarithmically divergent and proportional to $\ln\Lambda_c$, as may easily be seen through simple power counting and dimensional analysis of the relevant expressions.

The crucial issue thus arises as to which are the interacting quantum field theories which, in a perturbative quantisation, lead to physically meaningful and thus finite predictions for scattering processes, in spite of the existence of these ultraviolet short distance higher-loop divergencies. In practical terms, and to put it into just a few words, here is how the procedure works. Given any specific regularisation procedure, order after order in perturbation theory, one needs to add further and further corrections ("counterterms") to the initial Lagrangian density, in order to introduce additional contributions to scattering amplitudes such that the perturbative series summed up to the given order remains finite when the regulator is removed, thereby leading to a finite physical result, even though the basic quantities appearing separately in the renormalised Lagrangian may be divergent. However, if the number of the required countertems grows with the order in perturbation theory, no specific prediction remains possible, since each new counterterm requires the specification of a new coupling constant whose value may be inferred only from experiment. Hence, a quantum field theory possesses any predictive power provided only a finite number of counterterms is required to render the renormalised scattering amplitudes to whatever order of perturbation theory finite and thus physical. Field theories for which this programme is feasable are called renormalisable. In fact, all such renormalisable field theories are such that all counterterms belong to a finite class of local quantum operators such that the renormalisation of the theory amounts to a redefinition of the field normalisations, masses and couplings (the "bare" quantities of the classical Lagrangian density) in terms of renormalised and finite physical observables directly related to the physical external states, their masses and couplings. The "bare" quantities are obtained in terms of the renormalised ones through factors multiplying the latter, these factors being given as power series expansions in the coupling constants whose coefficient are divergent as the regulator is removed. Theories for which finite renormalisation is achieved in this manner are called "multiplicatively renormalisable". These are the only perturbative quantum field theories of possible relevance to relativistic quantum particle physics and their fundamental quantum interactions. Under such circumstances, one thus obtains a predictive framework for the representation and evaluation of these processes.

The above $\lambda\phi^4/4$ scalar field theory is the simplest example of such a renormalisable quantum field theory. All the required counterterms to all orders of perturbation theory simply amount to a redefinition, through a multiplicative factor, of the field normalisation, its mass m^2 and its self-coupling λ, each of these renormalisation factors being given as power series expansions in the coupling λ whose coefficients include both finite and infinite contributions as the regulator is removed. Nevertheless, all physical quantities remain finite in that same limit, and may be predicted in terms only of the renormalised mass and coupling of asymptotic quanta.

Renormalisable relativistic quantum field theories

Among all possible Lagrangian densities for collections of fields of a variety of spin values, how does one characterise those that define a renormalisable quantum field theory? Through power counting and dimensional analysis of loop amplitudes, a necessary condition, though not a sufficient one, for renormalisability may be established. Namely, when working in units such that $\hbar = 1 = c$ so that all dimensionful quantities may be measured in units of mass, whenever the Lagrangian density contains a specific contribution whose coupling coefficient, say λ, has a mass dimension to some strictly negative power, $\lambda = \alpha_0/\Lambda^\kappa$ with α_0 dimensionless, Λ some mass scale and $\kappa > 0$, then the associated interactions are not renormalisable.

For example, let us consider a real scalar field ϕ whose dynamics derives from the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{1}{2}m^2\phi^2 - V(\phi). \tag{79}$$

Since in units such that $\hbar = 1 = c$ the quantum action must be dimensionless, in a four-dimensional spacetime the scalar field must have a mass dimension of unity, as well as the mass parameter m. Consequently, any trilinear coupling $g\phi^3$ contribution to the potential density $V(\phi)$ must have a coupling strength g of mass dimension unity, while a quartic interaction $\lambda\phi^4$ a dimensionless strength coupling λ. In other words, in four-dimensional spacetime, any quartic potential $V(\phi)$ leads to a renormalisable quantum scalar field theory (in the absence of a quartic coupling, a cubic coupling is excluded on physical grounds, since otherwise the energy is not bounded below). However, any coupling of higher order, $\lambda\phi^n$ with $n > 4$, requires a strength coupling of mass dimension $[\lambda] = 4 - n$, and thus represents a nonrenormalisable interaction in a four-dimensional spacetime.

A similar analysis may be developed for any other field theory of higher spin content. Incidentally, in the case of general relativity, the fact that Newton's constant, which then defines the coupling strength for gravity, has a strictly negative mass dimension is one of the reasons why the perturbative quantisation of that classical metric field theory of spacetime geometry is nonrenormalisable.

Historically, the requirement of renormalisability was viewed as defining, albeit for not thoroughly convincing physical arguments, a basic restriction on the construction of realistic quantum field theories for the fundamental interactions of the elementary particles. Nowadays, this point of view has considerably shifted, and renormalisable quantum field theories are rather considered to define effective low energy approximations to some more fundamental underlying description of the basic physical phenomena, which need not be given even in terms of a quantum field theory. By integrating out from a given theory its high energy modes above its characteristic energy scale Λ, one recovers a low energy effective description in terms of a field theory in which the effects of the underlying theory relevant to the higher energy scales contribute only through nonrenormalisable effective coupling coefficients of the form $\lambda = \alpha_0/\Lambda^\kappa$. Hence, as the energy scale of the underlying theory becomes arbitrary large, only renormalisable couplings survive in its low energy effective field theory approximation, thereby leading to a decoupling of energy scales as one passes from one level of effective description to the next. From that point of view, the principle of renormalisability for the construction of physical quantum field theories is nothing but a principle for the decoupling of energy scales when formulating a theory capable of describing phenomena up to some characteristic energy scale, without the knowledge and independently of the physics lying beyond that energy scale. The procedure of renormalisation described above is then also seen to correspond to a renormalisation of the low energy observables through the resummation of all the known contributions up to some cut-off energy scale, beyond which there may lie some unknown territory, and then at the same time make sure that the low energy observables remain independent of this unknown physics, and thus remain finite as well, as this cut-off scale possibly characteristic of some unknown interactions and particles is pushed to arbitrary large values. In effect, this indeed corresponds to a decoupling of scales for the effective low energy approximate quantum field theory description.

Nonetheless, this rationale for the decoupling of scales translated into the requirement of renormalisability still leaves us with the general issue of the construction of such theories. The necessary condition mentioned above in terms of the mass dimension of interaction coupling constants, even when met, is not sufficient to ensure renormalisability of the corresponding coupling. The answer to this issue has been given above in the case of scalar fields, but not yet for spinor nor vector fields in interactions, which are certainly required for a description for the fundamental interactions of quarks and leptons. It turns out this is far from a trivial matter, and throughout the 1960's and early 1970's, it has been established that the only renormalisable interactions of vector fields, massive or not, with matter are those governed by the general gauge symmetry principle of Yang-Mills theories based on some internal symmetry whose algebraic group is a compact Lie group. The stringent and elegant symmetry constraints brought about by the local gauge symmetry principle on the structure of such interactions are just powerful enough to guarantee renormalisability.

Hence, in conclusion, the general principles of special relativity, quantum mechanics and decoupling of scales for effective field theory descriptions of the fundamental interactions and particles has led to the general gauge symmetry principle, and its actual realisation in terms of internal symmetries, as the guiding principle for the construction of renormalisable interacting relativistic quantum unitary local field theories as the appropriate framework for the description of the causal interactions of relativistic quantum point particles and their wavelike spacetime dynamics. Quite an achievement for the marriage of $\hbar$ and c, the genuine third conceptual revolution of XX$^{\rm th}$ century physics following general relativity and quantum mechanics!

4.2 Global internal symmetries

Hence, it is time now to turn to the meaning of internal symmetries, namely symmetries acting on a system but which are not associated to transformations in spacetime. In technical terms, a symmetry is a transformation of a system such that it leaves its equations of motion form invariant. Or in other words, a symmetry transforms a given solution to the dynamics of a system into another solution to the same dynamics. Note that a symmetry is not necessarily an invariance property of configurations of the system, but rather it is an invariance property of the set of its dynamical configurations. In particular, it may be that even for the lowest energy configuration of a system, this solution may or may not be invariant under the action of a symmetry of the equations of motion. As we shall see, this possibility has profound consequences in the context of field theory, especially when it comes to symmetries that are realised locally at each point in spacetime, so-called local gauge symmetries.

Given the character of these notes, only the simplest examples of these different issues are presented here. However, the reader should be aware that many generalisations have been developed, that are available in the literature as well as standard quantum field theory textbooks.

The simplest example

So far, we have considered only the case of a single real scalar field of mass m. Let us now extend the discussion to a system composed of two such fields $\phi_1(x)$ and $\phi_2(x)$ sharing identical masses m and interaction couplings. Consequently, such a system possesses a continuous symmetry whose transformations mix these two fields by an arbitrary amount while preserving their normalisation, namely a rotation of arbitrary angle in the two dimensional space (ϕ_1, ϕ_2) in which they take their values. Specifically, combining the two fields into a single complex valued scalar field,

$$\phi(x) = \frac{1}{\sqrt{2}}\left[\phi_1(x) + i\phi_2(x)\right], \tag{80}$$

the corresponding total Lagrangian density, which then reads

$$\begin{aligned}\mathcal{L} &= \partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi - V(|\phi|) \\ &= \tfrac{1}{2}(\partial_\mu\phi_1)^2 - \tfrac{1}{2}m^2\phi_1^2 + \tfrac{1}{2}(\partial_\mu\phi_2)^2 - \tfrac{1}{2}m^2\phi_2^2 - V\left(\sqrt{\phi_1^2+\phi_2^2}\right),\end{aligned} \tag{81}$$

$V(|\phi|)$ being an arbitrary renormalisable interaction potential, is clearly invariant under the class of continuous transformations

$$\phi'(x) = e^{i\alpha}\,\phi(x), \tag{82}$$

α being an arbitrary constant real parameter representing the rotation angle of this SO(2)=U(1) symmetry.

This symmetry, which leaves the action and thus also the equations of motion invariant, is a global symmetry, since it acts in an identical fashion on the field $\phi(x)$ irrespective of the spacetime point labelled by x^μ. The symmetry shifts the phase of the complex field by an identical amount globally throughout the whole of spacetime, namely not only instantaneously through all of space but also identically throughout the whole time history of the system. Furthermore, the action of the symmetry is not on the spacetime points at which the field is evaluated, but rather within the "internal two dimensional space" in which the complex field takes its values. From that point of view, these values for $\phi(x)$ define a two dimensional space associated to each of the spacetime points, the "internal" space of the system. Consequently, one says that the symmetry is a global internal one.

By virtue of Noether's theorem, associated to such a continuous symmetry, there exists a current and its charge which are locally conserved for solutions to the equations of motion. In the present instance, these conserved Noether current and charge are given by

$$J^\mu = -i\left[\phi^\dagger\partial^\mu\phi - \partial^\mu\phi^\dagger\phi\right], \qquad Q = \int_{(\infty)} d^3\vec{x}\, J^0(x^0,\vec{x}), \tag{83}$$

while for solutions to the dynamics of the system, these quantities obey the conservation conditions,

$$\partial_\mu J^\mu = 0, \qquad \frac{dQ}{dt} = 0. \tag{84}$$

These Noether current and charge thus characterise the specific properties of the system that follow from its U(1) continuous global internal symmetry. In particular, in its Hamiltonian formulation, the charge Q generates the algebra of the symmetry group, in the present case that of the abelian group U(1), through the Poisson bracket structure. Acting on phase space through these brackets, the Noether charge also generates, in linearised form, the associated symmetry transformations of the phase space degrees of freedom. Through the correspondence principle, the same properties should remain valid at the quantum level in terms of commutation relations. However, because of possible operator ordering ambiguities for composite quantities such as Noether charges and currents, it may be that the quantum consistency requirements for the definition of quantum physical observables clash with the symmetry properties, namely that the symmetry algebra is no longer realised in terms of the commutation relations of the Noether charges. In such a case, the symmetry is said to be anomalous, by which is meant in fact that the symmetry is explicitly broken for the quantised system.

In the present case, it may be checked by straightforward construction that the U(1) internal symmetry is not anomalous. Associated to the creation and annihilation mode expansions of the real fields ϕ_1 and ϕ_2, the complex field $\phi(x)$ acquires of course also such an expansion, but in terms of creation and annihilation operators which are superpositions of those of the initial fields. Having initially two independent fields, one still obtains two independent sets of creation and annihilations operators, given by

$$\begin{array}{rclcrcl} a(\vec{k}\,) &=& \frac{1}{\sqrt{2}}\left[a_1(\vec{k}\,)+ia_2(\vec{k}\,)\right] & , & b^\dagger(\vec{k}\,) &=& \frac{1}{\sqrt{2}}\left[a_1^\dagger(\vec{k}\,)+ia_2^\dagger(\vec{k}\,)\right], \\ b(\vec{k}\,) &=& \frac{1}{\sqrt{2}}\left[a_1(\vec{k}\,)-ia_2(\vec{k}\,)\right] & , & a^\dagger(\vec{k}\,) &=& \frac{1}{\sqrt{2}}\left[a_1^\dagger(\vec{k}\,)-ia_2^\dagger(\vec{k}\,)\right], \end{array} \tag{85}$$

and obeying the appropriate Fock space algebras

$$[a(\vec{k}\,),a^\dagger(\vec{\ell}\,)] = (2\pi)^3 2\omega(\vec{k}\,)\delta^{(3)}\left(\vec{k}-\vec{\ell}\,\right) = [b(\vec{k}\,),b^\dagger(\vec{\ell}\,)]. \tag{86}$$

The mode expansion of the complex field in the interacting picture is then

$$\hat{\phi}_{(I)}(x) = \int_{(\infty)} \frac{d^3\vec{k}}{(2\pi)^3 2\omega(\vec{k}\,)} \left[a(\vec{k}\,)e^{-ik\cdot x} + b^\dagger(\vec{k}\,)e^{ik\cdot x}\right], \tag{87}$$

while a direct substitution in the normal ordered expression for the quantum Noether charge $\hat{Q}$ finds

$$\hat{Q} = -\int_{(\infty)} \frac{d^3\vec{k}}{(2\pi)^3 2\omega(\vec{k}\,)} \left[a^\dagger(\vec{k}\,)a(\vec{k}\,) - b^\dagger(\vec{k}\,)b(\vec{k}\,)\right]. \tag{88}$$

In comparison with the mode expansion for a real scalar field, one notices a common structure with, however, the role played by the creation operator component now taken over by that of the independent mode $b^\dagger(\vec{k}\,)$ rather than $a^\dagger(\vec{k}\,)$, since the field need no longer be real under complex conjugation. Furthermore, it is precisely this complex character of the field which makes possible the existence of the U(1) symmetry, whose Noether charge should thus distinguish the two types of modes present in the system. Indeed, a direct calculation finds, for instance for the creation operators,

$$[\hat{Q},a^\dagger(\vec{k}\,)] = -a^\dagger(\vec{k}\,), \qquad [\hat{Q},b^\dagger(\vec{k}\,)] = +b^\dagger(\vec{k}\,), \tag{89}$$

with in particular

$$\hat{Q}\,a^\dagger(\vec{k}\,)\,|0\rangle = -a^\dagger(\vec{k}\,)\,|0\rangle, \qquad \hat{Q}\,b^\dagger(\vec{k}\,)\,|0\rangle = +b^\dagger(\vec{k}\,)\,|0\rangle. \tag{90}$$

In other words, the conserved quantum number Q associated to this Noether quantum charge which generates the U(1) symmetry of the system, is an additive quantum number for quantum states, and takes opposite values for the field quanta created by either the operators $a^\dagger(\vec{k}\,)$ or $b^\dagger(\vec{k}\,)$. To put it still differently, these two types of field quanta are distinguished by an opposite U(1) charge under the U(1) global internal symmetry. Fields neutral under complex conjugation are associated to neutral particles under some given continuous symmetry, while fields complex under complex conjugation lead to charged particles for the associated U(1) global internal symmetry. Hence, these two types of quanta correspond to particles and their antiparticles, since except for the opposite values for the

U(1) conserved charge, they otherwise share identical physical properties under the spacetime Lorentz symmetry, namely their mass and spin values.

Consequently, this is yet one more outcome of the marriage of $\hbar$ and c: the existence of particles and antiparticles of identical mass and spin, but opposite charge under internal continuous symmetries, such as their electric charge. Even for electrically neutral particles, it could be that the particle and antiparticle species are still distinct due to some other conserved quantum number than the electric charge taking opposite values. Of course, a particle which coincides with its antiparticle, and whose field is thus necessarily real under complex conjugation, is necessarily electrically neutral.

The Noether charge operator $\hat{Q}$ being the generator of the U(1) global symmetry, finite transformations of parameter α are induced through the exponentiated form

$$e^{i\alpha\hat{Q}} \tag{91}$$

acting on the space of quantum states of the system. In particular, note that the perturbative vacuum $|0\rangle$ carries a vanishing U(1) charge, $\hat{Q}|0\rangle = 0$, hence is also invariant under the action of the symmetry group,

$$e^{i\alpha\hat{Q}}\,|0\rangle \;=\; |0\rangle. \tag{92}$$

When the ground state or vacuum of the system is left invariant under the action of the symmetry, one says that the symmetry is realised in its Wigner mode.

It is straightforward to extend the above considerations to any internal compact Lie symmetry group. Assume that a given system of fields is invariant under a continuous group G whose algebra is spanned by a set of generators T^a such that

$$[T^a, T^b] = i f^{abc}\, T^c, \tag{93}$$

f^{abc} being its structure constants, and for which the collection of fields spans some linear representation of that algebra. Hence, if $\phi(x)$ denotes this collection of fields (with the representation index suppressed), and T^a now stand for the G-generators in that specific representation, the action of the symmetry on the fields may be represented as

$$\phi'(x) = e^{i\theta^a T^a}\,\phi(x), \tag{94}$$

θ^a being arbitrary constant but continuous parameters for G-transformations. These quantities being constant and acting independently of the value of x^μ, such transformations define a global internal symmetry, assuming of course that the Lagrangian density $\mathcal{L}(\phi, \partial_\mu\phi)$ is invariant under these transformations. Consequently, because of Noether's theorem, there exists conserved currents $J^a_\mu(x)$ and charges $Q^a = \int_{(\infty)} d^3\vec{x}\, J^{a0}$ generating the symmetry algebra and its transformations on the space of classical as well as quantum states of the system. In particular, if the ground state of the system is invariant under all G-transformations, namely if the symmetry is realised in the Wigner mode, the quantum space of states gets organised into irreducible representations of G, with in particular the one-particle states falling into the same G-representations as the original fields $\phi(x)$, since the creation and annihilation operators also carry that same representation index. All the latter properties are clearly met in the simple U(1) example above, and it should be straightforward to understand why they should remain valid for an arbitrary nonabelian symmetry group as well.

Spontaneous global symmetry breaking

The above results still leave open the case of a symmetry which is not realised in the Wigner mode, namely when the vacuum or ground state of the system is not invariant under the action of the symmetry. It is well known that specific physical systems may possess such a property, as is the case for instance for spontaneous magnetisation in a ferromagnetic material below the transition temperature. Let us recall the point made already previously, namely that what is meant by a symmetry is not the invariance of any of its configurations in particular, but rather the invariance of its equations of motion, hence also of the set of its configurations solving these equations viewed as a whole. If a given solution is not invariant, the existence of the continuous symmetry simply implies that there exists an infinite degeneracy of distinct solutions of identical energy all related through the action of the symmetry transformations. For example, imagine a simple linear stick standing along the vertical direction, onto which a certain pressure is applied along that axis. This system is obviously invariant for all rotations around the vertical axis. As long as the applied pressure is mild enough, the stick does not bend, and the lowest energy configuration of the system is indeed invariant under the axial symmetry. However, as soon as the applied pressure exceeds a specific critical value, the stick does bend until it reaches some equilibrium configuration. The horizontal direction in which this bending occurs is arbitrary, but it clearly spontaneously breaks the axial symmetry. Nevertheless, all the configurations of the system associated to all possible horizontal bending directions are degenerate in energy, and are related to one another precisely by the action of the axial symmetry group. The specific solution to the equations of motion singled out by the bending process is no longer invariant, but the set of all these solutions remains invariant, all the degenerate solutions being related through the axial symmetry group. When a symmetry is realised in such a manner, namely when the ground state of the system is not invariant under the symmetry, one says that the symmetry is spontaneously broken, or that it is realised in the Goldstone mode.

Whether a symmetry is realised in the Wigner or in the Goldstone mode is governed by the details of the dynamics of the system, whether in a perturbative or a nonperturbative regime. Once again for the purpose of simplicity, here we only discuss the simplest example, namely that of the spontaneous symmetry breaking already at the level of the classical theory of a single complex scalar field $\phi(x)$ possessing the U(1) global symmetry

$$\phi'(x) = e^{i\alpha}\,\phi(x), \tag{95}$$

with the real constant angular parameter α.

Let us consider again the Lagrangian density

$$\mathcal{L}\left(\phi, \partial_\mu\phi\right) = |\partial_\mu\phi|^2 - V(|\phi|), \tag{96}$$

where the potential contribution is given by

$$V(|\phi|) = \mu^2|\phi|^2 + \lambda|\phi|^4, \tag{97}$$

with $\lambda > 0$. In our previous considerations, the quantity μ^2 was taken to be positive, in which case it defined the mass-squared of the particle quanta associated to the field, describing the quantum excitations of this field above its ground state, namely the perturbative vacuum $|0\rangle$ associated to the classical value $\phi = 0$ up to the vacuum quantum fluctuations subtracted away through normal ordering, which is invariant under the U(1) symmetry.

Presently however, we shall consider the situation when $\mu^2 < 0$, corresponding to the so-called mexican hat potential, which very much looks like the bottom of a wine bottle. In such a case, the configuration $\phi = 0$ no longer defines the lowest energy configuration of the system, since the potential $V(|\phi|)$ now reaches its lowest value for

$$|\phi(x)| = \frac{1}{\sqrt{2}}v, \qquad v = \sqrt{\frac{-\mu^2}{\lambda}}. \tag{98}$$

Such a configuration also defines the lowest energy state of the field, since all field gradient contributions to the energy then vanish identically, the field being constant throughout spacetime. Such a configuration however, is no longer invariant under the U(1) symmetry, which is thus realised in the Goldstone mode. What are then the physical consequences of this spontaneous symmetry breaking in the vacuum?

In order to properly identify the physical quanta of the field, it is necessary to consider the field fluctuations about its vacuum configuration. Note that the two independent degrees of freedom per spacetime point defined by the complex scalar field may also be represented through a polar decomposition around a given choice of vacuum configuration,

$$\phi(x) = \frac{1}{\sqrt{2}} e^{i\xi(x)/v} \left[\rho(x) + v\right], \tag{99}$$

where $\xi(x)$ and $\rho(x)$ are two real scalar fields with a mass dimension of unity. Note that the vacuum about which this expansion is performed is

$$\phi_0 = \frac{1}{\sqrt{2}}\, v, \tag{100}$$

but that choice may easily be modified by adding to the mode $\xi(x)$ an arbitrary real constant quantity. This remark also shows that the U(1) symmetry now leaves the radial field $\rho(x)$ invariant, while it simply shifts the field $\xi(x)$ by the product αv. All the minimal energy configurations correspond the constant field ϕ lying at the bottom of the potential, with the norm $|\phi| = v/\sqrt{2}$ but an arbitrary phase. The U(1) symmetry simply induces a transformation of any such vacuum into any another such vacuum, the difference in their phases being set by the value of the U(1) angle α (note the perfect analogy with the above example of a bent stick). Hence, one should expect that the fluctations associated to the field $\xi(x)$ are massless, since they may be excited at zero-momentum at no extra energy cost. On the other hand, the radial fluctuation $\rho(x)$, moving the field out from its lowest energy configuration, must correspond to massive quanta of the field. Furthermore, this physical conclusion does not depend on the choice of complex phase for the reference constant vacuum configuration ϕ_0, since this amounts to a simple constant shift in the massless field $\xi(x)$.

More explicitly, a direct substitution of the mode expansion (99) gives

$$\mathcal{L} = \frac{1}{2}\partial_\mu\rho\partial^\mu\rho + \frac{1}{2}\left(1 + \frac{1}{v}\rho\right)^2 \partial_\mu\xi\partial^\mu\xi - \frac{1}{2}\mu^2(\rho+v)^2 - \frac{1}{4}\lambda(\rho+v)^4. \tag{101}$$

Isolating then the terms quadratic in $\xi(x)$ and $\rho(x)$ indeed confirms that the mode $\xi(x)$ is massless, while the $\rho(x)$ field is massive, with the values

$$m^2_\rho = -2\mu^2 > 0, \qquad m^2_\xi = 0. \tag{102}$$

Hence, we reach the conclusion that since the vacuum is not invariant under the action of transformations which nevertheless define a symmetry of the system and its equations of motion, necessarily in the Goldstone mode realisation of the symmetry there exist massless modes, namely massless quanta for a quantised field, which in the zero momentum limit correspond to the excitation of one vacuum state into another one, all these vacuum states being degenerate in energy and infinite in number. Hence, rather than being explicitly realised in the space of states as is the case for the Wigner mode, the symmetry is now hidden through the existence of Golstone bosons. Nonetheless, the symmetry is still active within the system, even though it is no longer realised in a linear fashion. Indeed, within the field basis which diagonalises its fluctuations, the symmetry acts as

$$\rho'(x) = \rho(x), \qquad \xi'(x) = \xi(x) + \alpha\, v, \tag{103}$$

which, among other consequences, implies that the Goldstone modes may only possess derivative or gradient couplings with other fields. The symmetry thus restricts to some extent the form of interactions of Goldstone fields.

In fact, it should be quite clear that this is a conclusion valid in full generality, which is known as Goldstone's theorem. Whenever a continuous global symmetry is spontaneously broken in the vacuum, associated to each of its broken generators, there exist massless quanta carrying the corresponding quantum numbers, known as the Goldstone bosons of the symmetry. This conclusion is valid whether the spontaneous symmetry breaking mechanism is perturbative or nonperturbative, and whether the symmetry is abelian or nonabelian. The only specific requirement is that the symmetry be a continuous one (in the case of a fermionic or spacetime symmetry, the Goldstone mode need not be bosonic, as is the case for instance for spontaneous supersymmetry breaking leading to a spin 1/2 goldstino massless mode).

4.3 Local or gauged internal symmetries

So far, we have briefly discussed the meaning of a global internal symmetry, and described some of its physical consequences, whether in the Wigner or the Goldstone mode. However, the existence of a global symmetry is not very appealing, at least from some theoretical aesthetic point of view. Indeed, any global internal symmetry defines transformations on the set of fields which act in an identical manner irrespective of the spacetime point at which the field values are being considered. For instance in the case of the U(1) symmetry associated to the electric charge and the electromagnetic interactions, this would mean that in order to render the transformation unobservable, one is required to change the phases of all the electrons of the Universe by exactly the same amount instantaneously throughout all of infinite space and troughout the whole of spacetime history! Although there is no technical or mathematical inconsistency that arises with such a relativistic quantum field theory, certainly it is a property of such symmetry transformations which runs counter to our belief that causality ought to be a stringent requirement on the construction of any physical theory.

Hence, one should rather prefer to develop a formalism in which internal symmetries are still possible, but such that now transformations may be realised locally in spacetime, though in a continuous fashion as to their spacetime dependence, while they would remain nevertheless unobservable to any conceivable experiment. Namely, is it possible to locally change the quantum phase of some electron while not at the same time by the same amount that of all the other electrons of the Universe, and nevertheless keep such a change hidden from any experimentalist? Clearly, this would require some information to be sent to all the other electrons in the Universe to tell them how to adjust their quantum phases accordingly, and this at the speed of light so that no experimentalist may catch up with this signal and measure the phase of some electron before it would have had the opportunity to adjust itself to the action of the symmetry transformation. In other words, by making the symmetry local, or by gauging the symmetry, one must introduce some additional propagating field coupling with equal strength to all other matter carrying the same symmetry charge, and whose quanta are necessarily massless.

This is the heuristic idea of the local gauge symmetry principle. As we shall explicitly see through the simplest examples, such a principle in fact provides a unifying principle for the existence of fundamental interactions, whose quantum carriers are massless and couple with identical strength to all other quanta with which they interact. These gauge bosons are necessarily vector fields for internal symmetries, and as stated previously, such Yang-Mills gauge theories based on compact Lie groups are the only possible renormalisable field theories including spin 0 and 1/2 matter fields interacting with vector fields.

The simplest example

As the simplest illustration of the above description, let us consider once again the theory of a single complex scalar field ϕ whose Lagrangian density is U(1) invariant under global phase transformations of the field, see (81) and (82). Clearly, if one wishes to gauge this symmetry, namely to construct a system which remains invariant under the local phase transformations

$$\phi'(x) = e^{i\alpha(x)}\,\phi(x), \tag{104}$$

$\alpha(x)$ now being an arbitrary spacetime dependent parameter rather than a constant angle as in the case of a global symmetry, a problem arises with the original Lagrangian. Indeed, this Lagrangian is no longer invariant, since the gradient contribution does not transform in the same covariant manner as the original field does,

$$\partial_\mu\phi'(x) = e^{i\alpha(x)}\left[\partial_\mu\phi(x) + i\partial_\mu\alpha(x)\,\phi(x)\right]. \tag{105}$$

However, this expression suggests a modification of the ordinary derivative or gradient of the field of the form

$$\partial_\mu \longrightarrow D_\mu(x) = \partial_\mu + igA_\mu(x), \tag{106}$$

where g is some dimensionless real quantity, which turns out to represent the coupling strength of the U(1) gauge interaction, and $A_\mu(x)$ the vector field for the gauge boson associated to the gauging of the U(1) symmetry. Indeed, it now suffices to assume that this vector field transforms under the local U(1) symmetry according to

$$A'_\mu(x) = A_\mu(x) - \frac{1}{g}\partial_\mu\,\alpha(x)\ , \tag{107}$$

to check that the modified gradient does possess the same covariant transformation as the field does under the symmetry,

$$D'_\mu(x)\phi'(x) = \left[\partial_\mu + igA'_\mu(x)\right] e^{i\alpha(x)}\phi(x) = e^{i\alpha(x)}\,D_\mu(x)\phi(x), \tag{108}$$

hence the name "covariant derivative" for the differential operator $D_\mu(x)$. Clearly, a simple substitution of the ordinary derivative by the covariant one in the original Lagrangian density invariant under the global U(1) symmetry leads to an expression invariant now under any local U(1) symmetry transformation. The U(1) symmetry has been gauged.

However, we still need to provide the vector field $A_\mu(x)$ with some dynamics, which is done by adding the pure gauge Lagrangian density to that of the matter field,

$$\mathcal{L}_A = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{109}$$

$F_{\mu\nu}$ being the gauge field strength, indeed the sole gauge invariant quantity that may be constructed out of the gauge field A_μ and its first-order gradients, in order to obtain a Lagrangian density which is of second-order in spacetime gradients, and thus represents a causal propagation of the gauge field (for the same reason, the absolute sign and normalisation of this Lagrangian density are fixed as given). This field being real under complex conjugation, its mode expansion is of the form, in the interacting picture,

$$A_\mu(x) = \int_{(\infty)} \frac{d^3\vec{k}}{(2\pi)^3 2|\vec{k}\,|} \sum_{\lambda=\pm} \left[e^{-ik\cdot x}\epsilon_\mu(\vec{k},\lambda)a(\vec{k},\lambda) \ + \ e^{ik\cdot x}\epsilon^*_\mu(\vec{k},\lambda)a^\dagger(\vec{k},\lambda)\right], \tag{110}$$

$a(\vec{k},\lambda)$ and $a^\dagger(\vec{k},\lambda)$ being annihilation and creation operators with the Fock space algebra normalised in the usual manner for massless quanta, and λ denotes the different polarisation states possible associated to the polarisation vectors $\epsilon_\mu(\vec{k},\lambda)$. These polarisation tensors are subjected to some restrictions which stem from the gauge invariance properties of the field, and shall not be discussed here (even though the issue of the quantisation of gauge invariant systems is discussed hereafter, but not explicitly for such abelian and nonabelian Yang-Mills theories). Note that the mass dimension of the gauge field indeed needs to be unity, hence leading to a dimensionless gauge coupling constant g.

In conclusion, the gauging of the simplest U(1) invariant scalar field theory is defined by the total Lagrangian density

$$\mathcal{L}_{\rm total} = \mathcal{L}_A + \mathcal{L}_\phi, \tag{111}$$

with the pure gauge Lagrangian $\mathcal{L}_A$ given above, and the matter one by

$$\begin{aligned}\mathcal{L}_\phi &= \mathcal{L}\left(\phi, D_\mu\phi\right) = |(\partial_\mu + igA_\mu)\phi|^2 - m^2|\phi|^2 - V(|\phi|) \\ &= |\partial_\mu\phi|^2 - m^2|\phi|^2 - V(|\phi|) - igA_\mu\left[\phi^\dagger\partial^\mu\phi - \partial^\mu\phi^\dagger\phi\right] + g^2 A_\mu A^\mu.\end{aligned} \tag{112}$$

In the case that the U(1) symmetry is that associated to the electromagnetic interaction, this system is simply that of scalar electrodynamics, namely that describing the interactions of a massive charged spin 0 particle with the photon.

From the latter expression, we immediately read off the different interaction terms coupling the matter and gauge fields. The term linear in A_μ is in fact $gA_\mu J^\mu$, namely the coupling of gauge field to the U(1) Noether current, and represents the coupling of one gauge quantum to two scalar field quanta of opposite U(1) charges. Such a feature is generic for all Yang-Mills theories: gauge fields always couple linearly to the associated Noether currents. The term quadratic in A_μ describes the coupling of two gauge quanta to two scalar quanta, also of opposite U(1) charges, in order for the total U(1) charge to be conserved in the interactions. Note that the single gauge boson interaction is proportional to ig, while the quadratic interaction is proportional to ig^2. In other words, the gauge symmetry principle not only explains, on the basis of a given internal symmetry, the appearance of local interactions, but it also sets specific restrictions on the properties of these interactions by predicting particular relations between the coupling strengths of different interactions, such restrictions being a consequence of the symmetry.

Among the interactions, the gauge boson $A_\mu(x)$ does not couple to itself, but only to the charged matter field with the universal coupling strength g. The reason for the fact that the gauge boson lacks such a self-coupling is that it is neutral under the U(1) symmetry, and does not carry any U(1) charge. Indeed, under a global symmetry transformation $\alpha(x) = \alpha$, we simply have for the transformed field $A'_\mu = A_\mu$. Furthermore, it is also the U(1) symmetry, but this time in its gauged embodiement, which explains why the gauge boson quanta are massless particles. Indeed, any mass term of the form $M_A^2 A_\mu A^\mu$ is clearly not gauge invariant under the local gauge transformations of the vector field. Hence, it is the local gauge symmetry which protects the gauge boson from acquiring any mass. In particular, this implies that physical (gauge invariant) quanta of that field may possess only two transverse polarisation states, such that $k^\mu \epsilon_\mu(\vec{k}, \lambda) = 0$, $\lambda = \pm$, a fact related to the issue of the quantisation of such Yang-Mills fields.

All the above considerations are readily extended to other matter fields, including fermionic ones not addressed in these notes. Furthermore, even though our discussion concentrates on the abelian U(1) case, the same developments apply to a nonabelian internal symmetry group G, leading then to Yang-Mills gauges theories. In such a case, for a collection of fields transforming in a G-representation whose generators are T^a, the covariant derivative, which now is Lie-algebra valued, reads

$$D_\mu = \partial_\mu + igA^a_\mu T^a, \tag{113}$$

g being the real gauge coupling constant, and A^a_μ the real gauge vector fields, which, for infinitesimal local gauge transformations of parameters $\theta^a(x)$, transform according to

$$A'^a_\mu = A^a_\mu - \frac{1}{g}\partial_\mu \theta^a - f^{abc}\theta^b A^c_\mu, \tag{114}$$

f^{abc} being the structure constants of the Lie algebra of G (it is also straightforward to establish the transformations of the gauge bosons for finite gauge transformations). The total Lagrangian of such a system is again given by the sum of the original G-invariant Lagrangian of the matter fields in which the ordinary derivative is substituted by the covariant derivative D_μ, to which one simply adds the pure Yang-Mills Lagrangian density

$$\mathcal{L}_A = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}, \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc}A^b_\mu A^c_\nu. \tag{115}$$

Once again, any mass term for the gauge bosons A^a_μ is forbidden by local gauge invariance, while gauged matter interactions are directly read off from the matter Lagrangian, leading again to linear and quadratic interactions of scalar fields with the gauge bosons. However, for a nonabelian symmetry, given the nonvanishing structure constants f^{abc}, the gauge bosons themselves possess now G-charges, actually those of the adjoint representation as may be seen from their gauge transformations for constant parameters $\theta^a(x) = \theta^a$. Consequently, from the expansion of the pure Yang-Mills Lagrangien, one identifies cubic and quartic terms representing gauge boson trilinear and quadrilinear couplings, whose strengths are directly proportional to g and g^2, respectively. Hence once again, the symmetry governs the details of all the gauge interactions, namely their strengths and their symmetry properties as well. Such predictions are specific to Yang-Mills theories, and provide important signatures for high energy experiments as to the relevance of the gauge symmetry principle for the physics of the fundamental interactions and the elementary particles. Note also that it is precisely these nonlinear gauge boson self-couplings which must be, in ways still to be thoroughly understood, at the origin of the specific nonperturbative phenomena of nonabelian theories, such as the property of confinement for the theory of the strong interactions among quarks, namely quantum chromodynamics (QCD) based on the local gauge symmetry $SU(3)_C$ for colour degree of freedom of quarks.

Spontaneous gauge symmetry breaking

The above discussion of the construction of abelian and nonabelian internal gauge symmetries implicitly assumed the symmetry to be realised in the Wigner mode. Hence, it is also important to consider the situation when the symmetry is rather realised in the Goldstone mode. For the purpose of illustration in the simplest case, let consider once again the U(1) gauged single scalar field theory, but this time with a potential leading to spontaneous symmetry breaking. The associated Lagrangian density is thus

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |(\partial_\mu + igA_\mu)\phi|^2 - V(|\phi|), \tag{116}$$

with

$$V(|\phi|) = \mu^2|\phi|^2 + \lambda|\phi|^4, \qquad \mu^2 < 0 \ \ , \ \ \lambda > 0. \tag{117}$$

This time however, because of the U(1) local symmetry transformation properties of the fields,

$$\phi'(x) = e^{i\alpha(x)}\,\phi(x), \qquad A'_\mu(x) = A_\mu(x) - \frac{1}{g}\partial_\mu\alpha(x), \tag{118}$$

when expanding any scalar field configuration about one of its vacuum configurations,

$$\phi(x) = \frac{1}{\sqrt{2}}e^{i\xi(x)/v}\,\left[\rho(x) + v\right], \qquad \phi_0(x) = \frac{1}{\sqrt{2}}v, \qquad v = \sqrt{\frac{-\mu^2}{\lambda}}, \tag{119}$$

it is always possible to effect a local U(1) gauge transformation, with parameter

$$\alpha(x) = -\frac{1}{v}\xi(x), \tag{120}$$

(note that since in general $\xi(x)$ is spacetime dependent, such a procedure is possible only when the internal symmetry is gauged), such that the Goldstone mode is completely gauged away from the scalar field, but lies hidden now in the transformed gauge field A'_μ,

$$\phi'(x) = \frac{1}{\sqrt{2}}\left[\rho(x) + v\right], \qquad A'_\mu(x) = A_\mu(x) + \frac{1}{gv}\partial_\mu\xi(x). \tag{121}$$

Upon substitution of the transformed fields in the Lagrangian density, which is physically equivalent to the original expression for the Lagrangian on account of local gauge invariance, one then finds

$$\mathcal{L} = -\frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} + \frac{1}{2}\left[\partial_\mu\rho + igA'_\mu(\rho + v)\right]^2 - \frac{1}{2}\mu^2(\rho+v)^2 - \frac{1}{4}\lambda(\rho+v)^4. \tag{122}$$

Isolating now all quadratic terms in the fields, one immediately notices that the radial field ρ still possesses the mass $m_\rho^2 = -2\mu^2 > 0$, but that in place of a massless Goldstone mode $\xi(x)$ which no longer appears in this Lagrangian by having been gauged away, there now appears an explicit mass term for the gauge boson field, with value

$$m_A^2 = g^2v^2. \tag{123}$$

Hence, even though a local symmetry when realised in the Wigner mode forbids any mass for its gauge bosons, when spontaneously broken in the vacuum and realised in the Goldstone mode, gauge bosons do acquire a mass! Nevertheless, their mass is then not just any parameter in the Lagrangian, but is in fact governed by the symmetry properties and takes a very specific value proportional both to the gauge coupling constant g and the scalar field vacuum expectation value v which spontaneously breaks those symmetry generators whose gauge bosons are massive. The counting of degrees of freedom is also in order. In the Wigner phase, one has two real scalar modes (one massless and one massive, the Goldstone and the radial ones, ξ and ρ) and two massless gauge modes (the two transverse modes of the gauge field). In the Goldstone phase, one has one real massive scalar mode (the radial field ρ) and three massive gauge boson polarisation modes. Note that the longitudinal massive gauge boson component is nothing but the would-be Goldstone mode ξ which has been gauged away and turned into the longitudinal component of the gauge field A'_μ, see (121).

These general features of the spontaneous symmetry breaking of a local gauge symmetry remain valid in general, and characterise the so-called Higgs mechanism. Whenever a local internal symmetry is spontaneously broken in the vacuum, those gauge bosons associated to the generators which do not leave invariant the vacuum acquire a mass proportional to the product of the gauge coupling and the scalar vacuum expectation value. Moreover, the Goldstone modes in the case of a global symmetry then provide the longitudinal polarisation states of the massive gauge bosons, leaving over the massive scalar modes, referred to as higgs scalars, as the only remnants of the spontaneously broken scalar matter sector. The gauge transformation which gauges away the Goldstone modes from the scalar sector to hide them in the gauge fields is known as the unitary gauge. It is in the unitary gauge that the physical content of such a theory is most readily identified. In the simplest example above, we thus conclude that the physical field content is that of a neutral massive spin 0 particle of mass $\sqrt{-2\mu^2}$, the higgs particle, interacting with itself and with a neutral massive spin 1 particle of mass $|gv|$.

Remarks

As already mentioned, it turns out that the gauge symmetry principle uniquely singles out among all possible quantum field theories of interacting spin 0, 1/2 and 1 particles, all those that are renormalisable, whether the gauge bosons are massive or not, provided however that in the former case their mass arises through the Higgs mechanism. This is quite a remarkable result, since such a local internal symmetry principle also implies the existence of specific interactions between matter particles and gauge bosons, whose detailed properties are totally governed by the underlying symmetry, whether abelian or nonabelian. In other words, all the relativistic and quantum dynamics of fundamental interactions among elementary point particles, through the marriage of $\hbar$ and c, appears to follow simply from the very elegant and powerful idea of a fundamental symmetry based on a compact Lie group.

Thus in order to describe all the known strong, electromagnetic and weak interactions observed to act between all known quarks and leptons, a gauge group as simple as $\mathrm{SU}(3)_c\times\mathrm{SU}(2)_L\times\mathrm{U}(1)_Y$ suffices, with a specific choice of representations for the quark and leptons fermionic fields, as well as for the scalar sector required for the Higgs mechanism leading to massive electroweak gauge bosons but nonetheless a massless photon. If not yet totally unified within this Standard Model of these interactions, at least all these interactions are brought within the unified framework of relativistic quantum Yang-Mills theories, leading to predictions whose precision is without precedent and which are confirmed through remarkable particle physics experiments. Nevertheless, this raises the issue of the rationale behind such a principle, as well as for the choice of internal symmetry and matter content.

From another perspective, with such Yang-Mills theories we are encountering dynamical theories whose quantisation requires an approach more general than that which was briefly reviewed in Sect.2. Indeed, considering the issue for example from within the Hamiltonian approach, when identifying the momentum conjugate to the U(1) gauge field A_μ coupled to the single scalar field through the Lagrangian density discussed above, one finds

$$\pi^\mu = \frac{\partial\mathcal{L}_{\rm total}}{\partial(\partial_0 A_\mu)} = -F^{0\mu}, \tag{124}$$

thus leading to the following constraint for its time component

$$\pi^0 = 0. \tag{125}$$

In other words, all phase space degrees of freedom of the system are not independent. Some are in fact constrained, and as we shall see in the forthcoming section, this is a generic feature for any system possessing a local symmetry whose parameters are not constant. How is one then to quantise such systems, since their physical dynamics is not contained within all of phase space, but only within some subspace of it? Clearly, gauge invariance implies that all degrees of freedom are not physical and relevant to the dynamics. How does one then account consistenly for such redundant features of a gauge invariant system in its quantisation? In the above example, it would be possible to solve for these gauge degrees of freedom, but at the cost of loosing a manifestly spacetime covariant description of such systems, which is also not welcome in itself. Hence, it is time now to turn to the discussion of the quantisation of constrained dynamics.

5 Dirac's Quantisation of Constrained Dynamics

5.1 Classical Hamiltonian formulation of singular systems

The system of constraints

First- and second-class quantities and constraints

Second-class constraints and Dirac brackets

First-class constraints and gauge invariance

5.2 The relativistic scalar particle

The action principle

The Hamiltonian formulation

5.3 Gauge fixing, reduced phase space and Gribov problems

Faddeev's reduced phase space

Admissible gauge fixing and Gribov problems

5.4 Dirac's quantisation

5.5 Klauder's physical projector: gauge invariant quantum dynamics without gauge fixing

6 Chern-Simons Quantum Field Theory

7 The Closed Bosonic String

7.1 The nonlinear Nambu-Goto action

7.2 Conformal gauge fixing

7.3 Dirac's conformal quantisation

Fundamental operator algebra

Physical states

Poincaré and conformal algebras

The no-ghost theorem

7.4 Light-cone quantisation

8 Toroidal Compactification of the Closed Bosonic String

8.1 Toroidal compactification in field theory

8.2 Toroidal compactification in string theory

9 Conclusions

The principle aim of these notes has been to provide a brief outline, restricted to bosonic degrees of freedom only, of the relativistic and quantum concepts that are at the basis of our present understanding of all fundamental quantum interactions and elementary particles. The general considerations that have led during the $\mathrm{XX}^{\mathrm{th}}$ century to the identification of relativistic quantum Yang-Mills gauge field theories as the appropriate framework for a consistent causal and quantum unitary description of relativistic quantum point particles and their interactions have been recalled. The same convergence of ideas centered onto the fundamental concept of the local gauge symmetry principle applies to the gravitational interaction, which, when described within general relativity and its extensions all based on the dynamics of the geometry of spacetime, has been successful so far only at the classical level, while a full-fledged theory for quantum gravity is still eluding us. It appears that the physicist of the $\mathrm{XXI}^{\mathrm{st}}$ century has arrived at the cross-roads of the three fundamental paths that have guided him during the previous one, and which may be characterised in terms of the three fundamental constants

c, $\hbar$ and G_N. It seems that in spite of the amazing successes of the marriage of c with $\hbar$, it is close to impossible to force these sets of ideas to happily live within a *ménage à trois*. Some new paradigm of geometrical and topological concepts is most probably called for within the realm of the quantum gravitational interaction coupled to all other quantum interactions and particles.

As another but complementary aim of these notes, the general issues surrounding the quantisation of constrained systems, which include all possible gauge invariant theories based on a field theory formulation, have been described, providing the basic tools necessary for such a study in general. In particular, having shown that the potential difficulties which follow from gauge fixing procedures for such theories are often unavoidable, an alternative and recent approach based on a physical projector onto the gauge invariant quantum configurations of such systems and free of the necessity of gauge fixing, has been advocated as a powerful new tool with which to address these difficult issues, especially with regards to nonperturbative aspects of strongly interacting Yang-Mills theories.

Yang-Mills, and more generally local gauge invariant theories have also shown that topological features, either of spacetime or of the field configuration space, do play a fundamental role in the proper understanding of such interactions. With the discovery of topological quantum field theories, void of any genuine dynamics but not of any quantum physics nonetheless, it is conceivable that pure quantum gravity could be the physics of quantum topology rather than of spacetime geometry, and that it is by coupling quantum topology to matter and interactions that the quantum geometric properties of spacetime should arise, local relativistic quantum field theories with gauge invariances being their appropriate low energy effective description.

As one illustration among possibly many others that have not been discussed at the Workshop, some of these issues have briefly been touched on within the context of bosonic string theory. Specific fascinating new features having to do with the gravitational sector of such systems and its interplay with the geometry and topology of spacetime have been described in the simplest terms available. Many more such issues have arising within that context, such as for example the possible noncommutative character of spacetime itself within string theory.

It is equipped with this understanding of the world of the fundamental quantum interactions and particles, and the role played by topology within the relativistic gauge invariant quantum field theoretic framework describing this world today, that the physicist of the XXI$^{\rm st}$ century in quest of the ultimate unification is to set out into the unchartered territory towards a truly genuine formulation and understanding of what quantum geometry will turn out to be, the final unification of the relativistic quantum and the relativistic continuum, the completed symphony of the three constants c, $\hbar$ and G_N which have guided us already through the three fundamental conceptual revolutions of XX$^{\rm th}$ century physics.

Acknowledgements

Prof. John R. Klauder is gratefully acknowledged for his many relevant comments concerning the contents of these lectures, and for his supportive interest over the years in applications of the physical projector. Thanks also go out to Prof. M. Norbert Hounkonnou for his insistence to give these lectures, and to all participants to the Workshop for their active participation, questions and discussions throughout the presentation of this material.

It is hoped that this write-up will entice them to raise ever more inquisitive issues, embark onto their own adventures into the quantum geometer's world of XXI$^{\rm st}$ century physics, and contribute with the mathematics and physics world community to the completion of this unfinished symphony by uncovering with a definite african beat some of its music scores, the full colours of its harmonies being Nature's own.