Lagrangian of the Standard Model

$$\mathcal{L}_{\mathrm{SM}} = \mathcal{L}_{\mathrm{YM}} + \mathcal{L}_{\mathrm{D}} + \mathcal{L}_{\mathrm{Yukawa}} + \mathcal{L}_{\mathrm{Higgs}}$$

The kinetic part of the gauge fields

The Dirac fermions

The Yukawa sector (interactions between the Higgs doublet and fermions)

Higgs dynamics and EWSB

In this course we shall not consider possible gauge-fixing and ghost field contributions (which may result from other choices of gauge)

The Gauge Sector

The first part of the SM Lagrangian is the kinetic part of the gauge fields:

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4g_1^2} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4g_2^2} W^a_{\mu\nu} W^{a\,\mu\nu} - \frac{1}{4g_3^2} G^A_{\mu\nu} G^{A\,\mu\nu}$$

where g_1, g_2, g_3 are the couplings respectively of the hypercharge, of isotopic spin (isospin), and colour. The tensors in the above equation are

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

for hypercharge, with B_μ being the boson vector field of the hypercharge $U(1)$.

For the isospin:

$$W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - \epsilon^{abc} W^b_\mu W^c_\nu$$

with W^a_μ $(a = 1, 2, 3)$ being the vector bosons of the $SU(2)$ weak isospin and ϵ^{abc} the antisymmetric structure constant of $SU(2)$. For the $SU(3)$ colour group

$$G^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu - f^{ABC} A^B_\mu A^C_\nu$$

where the A^A_μ $(A = 1, \ldots 8)$ are the gluon fields, and f^{ABC} the antisymmetric structure constants of $SU(3)$.

Note that hypercharge and isospin for the $SU(2) \otimes U(1)$ gauge theory will be broken by the Higgs mechanism to the give the unbroken $U(1)$ EM theory (QED), and the broken generators give the three massive vector bosons of the weak interactions.

But more on that later.

The Dirac Sector

The term $\mathcal{L}_{\mathrm{D}}$ is the Lagrangian for the Dirac fermions, describing the freely moving fermions and the fermionic interactions with gauge bosons.

Recall that weak interactions violate parity, as such, we'll describe the Dirac fermions in terms of Weyl spinors with two-components

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

to highlight that fact. To intuitively understand this notation is in terms of the algebra of Lorentz transformations, in four-dimensions the Lorentz group is generated by two $SU(2)$ factors, $\vec{J} + i\vec{K}$ and $\vec{J} - i\vec{K}$, where $\vec{J}$ is the angular momentum and $\vec{K}$ the vector for *boosts*.

It is easy to see that both these $SU(2)$'s are related by a conjugation C or a parity transformation P ($\vec{J} \to \vec{J}$ and $\vec{K} \to -\vec{K}$). They are therefore invariant under a CP transformation. We can exploit the conjugation C to write two types of fermions with two components in terms of one type:

$$\bar{\psi}_L \equiv \sigma_2 \psi_R^* \qquad \bar{\psi}_R \equiv \sigma_2 \psi_L^*$$

(σ_2 is the Pauli matrix). We also have

$$\begin{array}{lll} C: & \psi_L \to \sigma_2 \psi_R^* & \psi_R \to \sigma_2 \psi_L^* \\ P: & \psi_L \to \psi_R & \psi_R \to \psi_L \end{array}$$

where quarks and leptons of the SM are written in terms of multiplets $(SU(3)_c,\ SU(2)_w,\ U(1)_y)$ and using only two component spinors of the type L:

$$
\begin{aligned}
L_i &= \begin{pmatrix} \nu_i \\ e_i \end{pmatrix}_L \sim (1,\ 2,\ y_1) \\
\bar{e}_{i\,L} &\sim (1,\ 1,\ y_2) \\
Q_i &= \begin{pmatrix} u_i \\ d_i \end{pmatrix}_L \sim (3,\ 2,\ y_3) \\
\bar{u}_{i\,L} &\sim (\bar{3},\ 1,\ y_4) \\
\bar{d}_{i\,L} &\sim (\bar{3},\ 1,\ y_5)
\end{aligned}
$$

where i is the index which indicates the family.

For now the values $y_1 \ldots y_5$ of the hypercharge shall remain undetermined.

The coupling of fermions to gauge fields is done with covariant derivatives. For gauge fields we will use a notation in terms of matrices

$$\tilde{W}_\mu = \frac{1}{2} W_\mu^a \tau^a \qquad \tilde{A}_\mu = \frac{1}{2} A_\mu^A \lambda^A$$

with τ^a being $SU(2)_w$ (Pauli) matrices and λ^A those of $SU(3)_c$ (Gell-Mann matrices). In the following we will indicate the Pauli matrices with τ^i when done in reference to $SU(2)_w$ matrices and with σ^i for spin.

The covariant derivatives are defined by

$$
\begin{aligned}
D_\mu L_i &= \left(\partial_\mu + i\tilde{W}_\mu + i\frac{y_1}{2}B_\mu\right) L_i \\
D_\mu \bar{e}_i &= \left(\partial_\mu + \frac{i}{2}y_2 B_\mu\right) \bar{e}_i \\
D_\mu Q_i &= \left(\partial_\mu + i\tilde{A}_\mu + i\tilde{W}_\mu + \frac{i}{2}y_3 B_\mu\right) Q_i \\
D_\mu \bar{u}_i &= \left(\partial_\mu - i\tilde{A}^*_\mu + \frac{i}{2}y_4 B_\mu\right) \bar{u}_i \\
D_\mu \bar{d}_i &= \left(\partial_\mu - i\tilde{A}^*_\mu + \frac{i}{2}y_5 B_\mu\right) \bar{d}_i \ .
\end{aligned}
$$

The Dirac part of the SM Lagrangian is

$$
\begin{aligned}
\mathcal{L}_{\mathrm{D}} &= \sum_{i=1}^{3}\Big(L_i^\dagger \sigma^\mu D_\mu L_i + \bar{e}_i^\dagger \sigma^\mu D_\mu \bar{e}_i + Q_i^\dagger \sigma^\mu D_\mu Q_i \\
&\quad + \bar{u}_i^\dagger \sigma^\mu D_\mu \bar{u}_i + \bar{d}_i^\dagger \sigma^\mu D_\mu \bar{d}_i\Big) \ .
\end{aligned}
$$

Note that the Lagrangian $\mathcal{L}_{\mathrm{Y}M} + \mathcal{L}_{\mathrm{D}}$ has a symmetry larger than the full Lagrangian of the SM. For the multiplets used above

$$M_i \to M_i' = U_{ij} M_j$$

leaves $\mathcal{L}_{\mathrm{Y}M} + \mathcal{L}_{\mathrm{D}}$ invariant. Given that we have 5 types of fermions the overall symmetry appears to be

$$[U(3)]^5 \ .$$

In reality this symmetry will not be respected in the other parts of the Lagrangian.

In particular, when we try writing mass terms for the fermion fields (through interactions with a scalar field).

Hypercharge and anomalies

In order to determine the hypercharges we will require a choice that is consistent with the constraints of symmetry and the renormalisability of the theory.

The symmetry relations between the Green's functions are called Ward identities.

The renormalisability of a theory depends critically on the differences between different sectors of the theory, then the Ward identities. Quantum corrections do not necessarily respect the symmetries and in this case we speak of anomalies for the Ward identities.

At the quantum level we must ensure the absence of anomalies for Ward identities because they prevent the retention of a gauge current (the renormalisability of the theory would be destroyed).

To verify the absence of anomalies we must calculate loop diagrams. To do this we will limit ourselves to a consideration of triangle graphs, since higher order contributions or those with more external lines are zero if the triangle diagram is zero.

Where it is possible to see in general that the symmetric structure constants of the group involved, in the terms of abnormalities arising when calculating the triangle diagrams, can limit the number of diagrams to be considered.

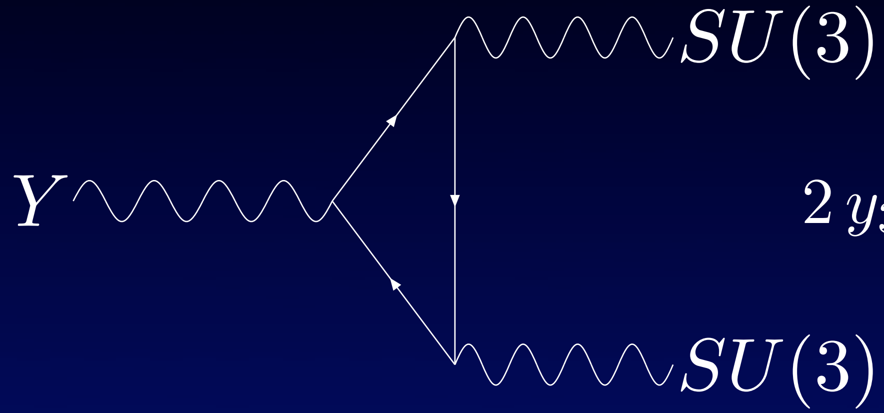
The $SU(3)$ colour group has symmetric structure constants but no abnormalities because the number of quarks and anti-quarks is the same and this guarantees that each contribution from a fermion in the loop is negated by the contribution from a corresponding anti-fermion.

The $SU(2)$ group has only anti-symmetric structure constants, and so has no contributions to the anomaly.

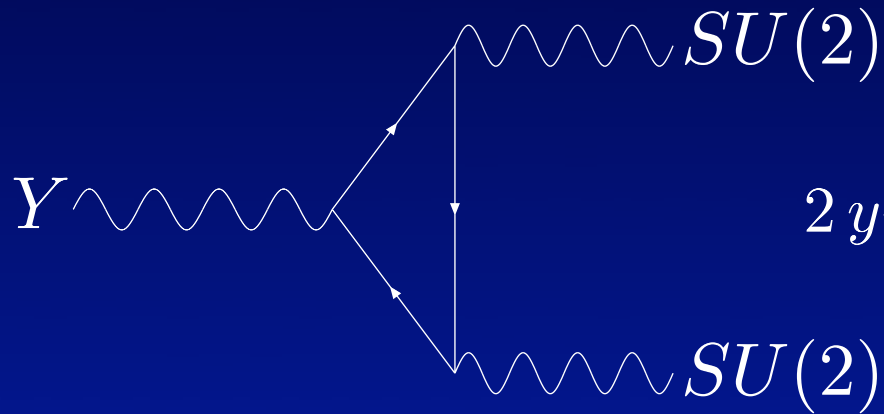
The group $U(1)$ of hypercharges can give rise to anomalies. We will therefore consider the diagrams when we have at least one boson, Y, of $U(1)_y$.

Note that the diagrams which contain a single line with a boson of $SU(3)_c$ or $SU(2)_w$ are invalid because the traces on a single matrix of these groups is zero.

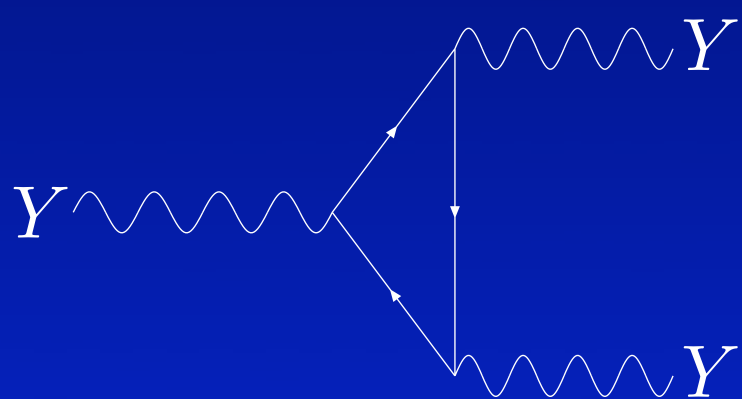
We therefore have the following constraints on the hypercharge to eliminate the triangle diagrams:



$$2\,y_3 + y_4 + y_5 = 0$$



$$2\,y_1 + 2 \cdot 3\,y_3 = 0$$



$$2\,y_1^3 + y_2^3 + 3\,(2\,y_3^3 + y_4^3 + y_5^3) = 0$$

Yukawa interactions

The need to introduce Yukawa terms (terms of dimension 4 with two spinors and a scalar field) is due to the impossibility of writing mass terms which are invariant and renormalisable, such as:

$$L^T \sigma_2 \bar{e}_L \ , \qquad Q^T \sigma_2 \bar{u}_L \ , \qquad Q^T \sigma_2 \bar{d}_L \ ,$$

which are not invariant with respect to the weak isospin. One possible way to construct mass terms which are invariant is to introduce a scalar field that is an isospin doublet, like the Higgs field:

$$H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \sim (1,\, 2,\, y_h) \ ,$$

and construct interaction terms (scalar-fermion-fermion), the Yukawa terms:

$$\begin{aligned}\mathcal{L}_{\text{Yukawa}} &= iY^e_{ij}L_i^T\sigma_2\bar{e}_{jL}H^* + iY^u_{ij}Q_i^T\sigma_2\bar{u}_{jL}\tau_2 H \\ &\quad + iY^d_{ij}Q_i^T\sigma_2\bar{d}_{jL}H^* + \text{h.}c.\end{aligned}$$

where the Y_{ij} are complex 3×3 matrices of Yukawa couplings.

Recall that τ_i denotes the Pauli matrices for the $SU(2)_w$ group and σ_i the same Pauli matrices for spin.

Note that after spontaneous symmetry breaking these terms give rise to fermionic mass terms.

The hypercharge conservation imposes the following relations:

$$y_h = y_1 + y_2 = -(y_3 + y_4) = y_3 + y_5$$

and if we fix $y_h = 1$, a choice consistent with the equations from the last section are:

$$y_1 = -1 \,, \qquad y_2 = +2 \,, \qquad y_3 = +1/3 \,, \qquad y_4 = -4/3 \,, \\ y_5 = 2/3 \,.$$

The Yukawa couplings Y_{ij} are not all independent because redefinitions of the fields are possible using the global symmetries of $\mathcal{L}_{\rm YM} + \mathcal{L}_{\rm D}$.

Note that any complex matrix can be written as:

$$Y^e = U^{e\,T} M^e V^e$$

with $U^e U^{e\dagger} = V^e V^{e\dagger} = 1$ (U^e and V^e being unitary matrices) and M^e a real diagonal matrix. The unitary matrices can be absorbed by a redefinition of fields:

$$L' = U^e L \qquad\qquad \bar{e}'_L = V^e \bar{e}_L$$

without changing the $\mathcal{L}_{\mathrm{YM}} + \mathcal{L}_{\mathrm{D}}$. For leptons this redefinition makes $\mathcal{L}_{\mathrm{Yukawa}}$ diagonal:

$$i y^e_{ii} L_i^T \sigma_2 \bar{e}_i H^* + \mathrm{h.}c.$$

with

$$M^e = \begin{pmatrix} y^e_{11} & 0 & 0 \\ 0 & y^e_{22} & 0 \\ 0 & 0 & y^e_{33} \end{pmatrix} .$$

These Yukawa terms break the global $U(3) \times U(3)$ symmetry and only retain the $U(1)$ invariance with phase

$$L_i \to e^{i\alpha_i} L_i \ , \qquad \bar{e}_i \to e^{-i\alpha_i} \bar{e}_i \ ,$$

the α_i being interpreted as the three leptonic numbers.

The redefinition of quark fields of type *up* and *down* can not be done independently, because the two types of Yukawa interactions for quarks always contain Q_i.

Note that if we wanted to have massive neutrinos, we would have the same problem with L_i, which would lead to the V_{PMNS} in the same way that this section will lead to V_{CKM}.

Writing for leptons:

$$Y^u = U^{u\,T} M^u V^u \,, \qquad Y^d = U^{d\,T} M^d V^d \,,$$

and for quarks

$$\bar{u} \to V^u \bar{u} \,, \qquad \bar{d} \to V^d \bar{d} \,,$$

the doublet Q_i can be redefined to eliminate the matrix U and the remaining two Yukawa interaction terms leave matrices in the other couplings:

$$i y^d_{ii} Q^T_i \sigma_2 \bar{d}_i H^* + i y^u_{jj} Q^T_i \sigma_2 \mathcal{V}_{ji} \bar{u}_j \tau_2 H$$

with $\mathcal{V} = U^u U^{d\dagger}$. The matrix $\mathcal{V}$ is unitary and therefore there are now 9 independent parameters instead of 18, by the relationship $\mathcal{V}^\dagger \mathcal{V} = 1$.

Note that $\mathcal{V}_{ji}$ will give rise to V_{CKM}, but more of that after we break the EW symmetry.

A further simplification is possible using the Euler decomposition

$$\mathcal{V} = P^T \mathcal{U} P'$$

where P and P' are the diagonal phase matrices and the $\mathcal{U}$ matrix contains the remaining parameters.

With a redefinition of phase the three u_i and three d_i can be expected to completely eliminate the 6 parameters in the matrices P, P', but in reality we can not determine more than 5 (normally $n \times n$ matrices can be used to eliminate $n^2 - 1$ phases). But a common phase factor can always be added

$$\bar{u}'_L = e^{i\alpha} \begin{pmatrix} e^{i\alpha_1} & 0 & 0 \\ 0 & e^{i\alpha_2} & 0 \\ 0 & 0 & e^{-i(\alpha_1+\alpha_2)} \end{pmatrix} \bar{u}_L \ ,$$

$$\bar{d}'_L = e^{i\beta} \begin{pmatrix} e^{i\beta_1} & 0 & 0 \\ 0 & e^{i\beta_2} & 0 \\ 0 & 0 & e^{-i(\beta_1+\beta_2)} \end{pmatrix} \bar{d}_L \ .$$

The matrix $\mathcal{V}$ becomes

$$\mathcal{V}' = e^{-i\alpha} \begin{pmatrix} e^{-i\alpha_1} & 0 & 0 \\ 0 & e^{-i\alpha_2} & 0 \\ 0 & 0 & e^{i(\alpha_1+\alpha_2)} \end{pmatrix} \mathcal{V} \begin{pmatrix} e^{i\beta_1} & 0 & 0 \\ 0 & e^{i\beta_2} & 0 \\ 0 & 0 & e^{-i(\beta_1+\beta_2)} \end{pmatrix} e^{i\beta}$$

thus redefining the phases of the quark fields can eliminate the 4 phases in both matrices; the above equation and the combination $\alpha - \beta$.

Note that the phase $\alpha + \beta$ does not exist.

Therefore, we are left with a matrix containing three parameters (angles) of the matrix $\mathcal{U}$, and a phase, all of which are physical!

The only remaining symmetry in the quark sector of the global symmetry is a $U(1)$ phase common to all quarks

$$Q_i \to e^{i\delta} Q_i \qquad \bar{u}_i \to e^{-i\delta} \bar{u}_i \qquad \bar{d}_i \to e^{-i\delta} \bar{d}_i$$

which corresponds to a conserved quantum number, baryon number.

The Higgs sector

Before we proceed to EWSB, we need to study the final piece of $\mathcal{L}_{SM}$. This will also be the sector responsible for the symmetry breaking $SU(2)_w \otimes U(1)_y \to U(1)_{\text{em}}$.

In the previous section we introduced a complex scalar doublet of $SU(2)_w$

$$H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \sim (1,\, 2,\, 1)\ ,$$

where we now write its Lagrangian as

$$\mathcal{L}_{\text{Higgs}} = (D_\mu H)^\dagger\ (D^\mu H) - V(H)$$

with

$$
\begin{aligned}
D_\mu H &= \left(\partial_\mu + i\tilde{W}_\mu + \frac{i}{2} y_h B_\mu\right) H \text{ with } y_h = 1 \\
V(H) &= -\mu^2 H^\dagger H + \lambda (H^\dagger H)^2 \ .
\end{aligned}
$$

The potential V is the broadest possible renormalisable and invariant potential under the $SU(2)_w \otimes U(1)_y$ symmetry.

The invariance of the vacuum is that of the $U(1)_{\mathrm{em}}$, so one component of this pair must be a neutral scalar field of the electric charge.

One can check that our choice of the previous section, $y_h = 1$, is in agreement with this observation.

The relationship between the electric charge, hypercharge and isospin is

$$Q_{\rm em} = I_{3w} + \frac{1}{2} y \ .$$

For both components of the doublet Higgs has

$$\begin{aligned} Q_{\rm em}(\phi^+) &= \frac{1}{2} + \frac{1}{2} y_h = 1 \\ Q_{\rm em}(\phi^0) &= -\frac{1}{2} + \frac{1}{2} y_h = 0 \end{aligned}$$

Thus we find the component ϕ^0 with zero electric charge.

Exercise C

Find the value of the electric charge for the leptons in the doublet Q_L.

Spontaneous electroweak symmetry breaking

With the choice of parameters $\mu^2 < 0$ and $\lambda > 0$ the Higgs potential has its minimum on the surface

$$|H|^2_{\rm min} = -\frac{\mu^2}{2\lambda} = \frac{v^2}{2}$$

with $v^2 = -\mu^2/\lambda$. We will choose the vacuum

$$\langle 0|H|0\rangle = \left(\begin{array}{c} 0 \\ \frac{v}{\sqrt{2}} \end{array} \right)$$

and set the fields around this vacuum as

$$H = \exp\left(\frac{i}{v}\xi_i(x)\sigma_i\right)\left(\begin{array}{c} 0 \\ \frac{v+h(x)}{\sqrt{2}} \end{array}\right) \equiv U(x)H_0 \ ,$$

where we have introduced the fields $\xi_i(x)$ ($i = 1,\ 2,\ 3$) and $h(x)$ which "vanish into the void".

The unitary phase matrix $U(x)$ is a gauge transformation of $SU(2)$ and is a direct result of the unitary gauge. The corresponding gauge transformation on the $SU(2)$ gauge fields lies in studying the covariant derivative

$$D_\mu H \quad = \quad \left(\partial_\mu + i\tilde{W}_\mu + \frac{i}{2}B_\mu\right) U(x) \left(\begin{array}{c} 0 \\ \frac{v+h(x)}{\sqrt{2}} \end{array}\right)$$

$$
\begin{aligned}
D_\mu H &= U(x)\,U(x)^\dagger\left(\partial_\mu + i\tilde{W}_\mu + \frac{i}{2}B_\mu\right)U(x)\begin{pmatrix} 0 \\ \frac{v+h(x)}{\sqrt{2}}\end{pmatrix} \\
&= U(x)\left(\partial_\mu + i\tilde{W}'_\mu + \frac{i}{2}B_\mu\right)\begin{pmatrix} 0 \\ \frac{v+h(x)}{\sqrt{2}}\end{pmatrix},
\end{aligned}
$$

where the last equality is obtained by taking

$$
\tilde{W}'_\mu = -iU(x)^\dagger\partial_\mu U(x) + U(x)^\dagger\tilde{W}_\mu U(x)\ .
$$

In this way the matrix $U(x)$ vanishes completely from the Lagrangian:

$$
\begin{aligned}
\mathcal{L}_{\text{Higgs}} &= \frac{1}{2}\partial_\mu h\,\partial^\mu h + \frac{1}{8}\left(B_\mu - W_{3\mu}\right)\left(B^\mu - W_3^\mu\right)(v+h)^2 \\
&+ \frac{1}{8}\left(W_{1\mu} - iW_{2\mu}\right)\left(W_1^\mu + iW_2^\mu\right)(v+h)^2 \\
&+ \lambda v^2 h^2 + \lambda v h^3 + \frac{\lambda}{4}h^4 - \lambda\frac{v^2}{4}\ .
\end{aligned}
$$

You can read the mass term for the Higgs boson as:

$$\lambda v^2 h^2 = \frac{1}{2} 2\lambda v^2 h^2 = \frac{1}{2} m_h^2 h^2$$

where we define:

$$m_h^2 = 2\lambda v^2 \ .$$

However, it is difficult to read the mass terms of the gauge bosons due to the mixing of terms.

We must therefore define linear combinations appropriate to the fields in order to eliminate mixing terms between gauge bosons.

Before this we will reintroduce the coupling constants which were hidden in the fields earlier

$$B_\mu \to g_1 B_\mu \ , \qquad \tilde{W}_\mu \to g_2 \tilde{W}_\mu \ , \qquad \tilde{A}_\mu^A \to g_3 \tilde{A}_\mu^A \ .$$

This will make the $\mathcal{L}_{YM}$ I gave earlier look like the more usual kinetic terms of gauge fields.

To find the diagonal form of the masses we will impose in the appropriate area:

$$m_W^2 W_\mu^+ W^{-\mu} \equiv \frac{g_2^2 v^2}{8} \left(W_{1\mu} - iW_{2\mu}\right) \left(W_1^\mu + iW_2^\mu\right)$$

where

$$W_\mu^\pm = \frac{1}{\sqrt{2}} \left(W_{1\mu} \mp iW_{2\mu}\right)$$

and the mass of the two gauge bosons is

$$m_W^2 = \frac{g_2^2 v^2}{4} \ .$$

For the neutral gauge bosons the electrical charge imposes a massless linear combination corresponding to the photon as

$$\frac{1}{2} m_Z^2 Z_\mu Z^\mu + \frac{1}{2}\, 0\, A_\mu A^\mu \equiv \frac{v^2}{8} \left(g_1 B_\mu - g_2 W_{3\mu}\right) \left(g_1 B^\mu - g_2 W_3^\mu\right) \, .$$

The above equation can be written in terms of a mass matrix

$$\begin{aligned} \frac{1}{2}(Z_\mu, \, A_\mu) \begin{pmatrix} m_Z^2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Z^\mu \\ A^\mu \end{pmatrix} &\equiv \frac{v^2}{8}(W_{3\mu}, \, B_\mu) \\ &\times \begin{pmatrix} g_2^2 & -g_1 g_2 \\ -g_1 g_2 & g_1^2 \end{pmatrix} \begin{pmatrix} W_3^\mu \\ B^\mu \end{pmatrix} \end{aligned}$$

and the link between the two descriptions is an orthogonal transformation

$$\begin{pmatrix} Z^\mu \\ A^\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_w & -\sin\theta_w \\ \sin\theta_w & \cos\theta_w \end{pmatrix} \begin{pmatrix} W_3^\mu \\ B^\mu \end{pmatrix}$$

with

$$\cos\theta_w \equiv \frac{g_2}{\sqrt{g_1^2 + g_2^2}} \qquad \sin\theta_w \equiv \frac{g_1}{\sqrt{g_1^2 + g_2^2}} \ ,$$

where θ_w is the Weinberg angle. The mass of the photon is zero and that of the boson Z^0 equals

$$m_Z^2 = \frac{v^2}{4}(g_1^2 + g_2^2) \ .$$

Recall that the couplings of fermions to gauge fields were given by covariant derivatives in $\mathcal{L}_D$. So if we expand $\mathcal{L}_D$ in terms of our "new" physical gauge fields we get[1]

$$\mathcal{L}_{\rm em} = -ieA_\mu \left(e_L^\dagger \sigma^\mu e_L + e_R^\dagger \bar{\sigma}^\mu e_R\right) ,$$

with the interactions of charged weak currents

$$\mathcal{L}_{\rm cc} = i\frac{g_2}{\sqrt{2}} \left(W_\mu^- \nu_{eL}^\dagger \sigma^\mu e_L + W_\mu^+ e_L^\dagger \sigma^\mu \nu_{eL}\right) ,$$

and neutral weak currents

$$\begin{aligned}\mathcal{L}_{\rm cn} \quad &= \quad i\frac{g_2}{\cos\theta_w} Z_\mu \left[\frac{1}{2}\nu_{eL}^\dagger \sigma^\mu \nu_{eL} - \frac{1}{2} e_L^\dagger \sigma^\mu e_L \right. \\ &\qquad \left. + \sin^2\theta_w \left(e_L^\dagger \sigma^\mu e_L + e_R^\dagger \bar{\sigma}^\mu e_R\right)\right] .\end{aligned}$$

---

[1] Show that $\frac{1}{e^2} = \frac{1}{g_1^2} + \frac{1}{g_2^2}$

From the previous equations we can see the interactions of charged and neutral currents have the same interaction force.

We shall discuss quarks a little later.

Exercise D

Defining the scalar fields on the vacuum as follows:

$$H = \left(\begin{array}{c} 0 \\ \frac{v}{\sqrt{2}} \end{array} \right) + \left(\begin{array}{c} \phi^{+} \\ \frac{h+i\eta}{\sqrt{2}} \end{array} \right) \equiv H_0 + H' \ ,$$

find the spectrum of masses of Goldstone bosons and the physical Higgs field in the theory.

Couplings of the Higgs boson

The Higgs Lagrangian can be written in terms of the new fields

$$\begin{aligned}\mathcal{L}_{\mathrm{Higgs}} &= \frac{1}{2}\partial_\mu h \partial^\mu h + \frac{1}{2} m_Z^2 Z_\mu Z^\mu + m_W^2 W_\mu^+ W^{-\mu} \\ &+ \left(\frac{2h}{v} + \frac{h^2}{v^2}\right)\left(\frac{1}{2} m_Z^2 Z_\mu Z^\mu + m_W^2 W_\mu^+ W^{-\mu}\right) + V \, .\end{aligned}$$

This shows that the scalar field, the Higgs, does not couple to the photon and the coupling to the gauge boson masses is proportional to the square of the masses of these bosons.

The couplings to fermions are obtained in the unitary gauge from the Yukawa terms. For example in the case of leptons

$$\mathcal{L}^e_{\mathrm{Yukawa}} = i y^e_{ii} L_i^T \sigma_2 \bar{e}_i H^* + \mathrm{h.c.} = i y^e_{ii} L_i^T \sigma_2 \bar{e}_i U^*(x) H_0 + \mathrm{h.c.}$$

Using the Hermiticity of the matrix $U^* = (U^\dagger)^T$ we can define the new field $L'_i = U^\dagger L_i$ where the U matrix disappears from the Lagrangian (we can verify that this is also valid for the kinetic part of the fermions)

$$\mathcal{L}^e_{\text{Yukawa}} = i y^e_{ii} {L'}^T_i \sigma_2 \bar{e}_i H_0 + \text{h.c.} = i y^e_{ii} {L'}^T_{2i} \sigma_2 \bar{e}_i \left(\frac{v+h}{\sqrt{2}} \right) + \text{h.c.}\ .$$

Using $\bar{e}_{Li} = -\sigma_2 e^*_{Ri}$ and the definitions

$$e_{Ri} = (e_R,\, \mu_R,\, \tau_R)\,, \qquad L'_{2iL} = (e_L,\, \mu_L,\, \tau_L)$$

for the names of leptons

$$\mathcal{L}^e_{\text{Yukawa}} = \frac{i}{\sqrt{2}}(v+h) \left(y^e_{11} e^\dagger_R e_L + y^e_{22} \mu^\dagger_R \mu_L + y^e_{33} \tau^\dagger_R \tau_L \right) + \text{h.c.}\ .$$

You can then read off the masses of the leptons as

$$\frac{v}{\sqrt{2}} y^e_{ii} = (m_e,\, m_\mu,\, m_\tau)\ .$$

In the notation of four-dimensional Dirac spinors

$$\mathcal{L}^e_{\text{Yukawa}} = i\frac{v+h}{v}\left(m_e \bar{e} e + m_\mu \bar{\mu}\mu + m_\tau \bar{\tau}\tau\right)\ .$$

Doing the same thing for up-type quarks

$$\begin{aligned} \mathcal{L}^u_{\text{Yukawa}} &= i y^u_{jj} Q^T_i \sigma_2 \mathcal{V}_{ji} \bar{u}'_j \tau_2 H + \text{h.c.} \\ &= i y^u_{jj} Q^T_i \sigma_2 \mathcal{V}_{ji} \bar{u}_j \tau_2 U H_0 + \text{h.c.} \end{aligned}$$

and with $\tau_2 \tau_i \tau_2 = -\tau_i^*$

$$\tau_2 U = U^* \tau_2 = (U^\dagger)^T \tau_2$$

we can remove U in a redefinition of the field $Q'_i = U^\dagger Q_i$:

$$\begin{aligned} \mathcal{L}^u_{\text{Yukawa}} &= i y^u_{jj} {Q'}^T_i \sigma_2 \mathcal{V}_{ji} \bar{u}'_j \tau_2 H_0 + \text{h.c.} \\ &= \frac{i}{\sqrt{2}} y^u_{jj} {Q'}^T_{1i} \sigma_2 \mathcal{V}_{ji} \bar{u}'_j (v + h) + \text{h.c.} \end{aligned}$$

with $\mathcal{V}$ the unitary matrix of mixing, the Cabibbo-Kobayashi-Maskawa (CKM) matrix:

$$\mathcal{V} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} .$$

If we define the mass eigenstates

$$u_L = \mathcal{V}_{1i} Q'_{1i}\,, \quad c_L = \mathcal{V}_{2i} Q'_{1i}\,, \quad t_L = \mathcal{V}_{3i} Q'_{1i}\,,$$

$$\bar{u}_1 = i\sigma_2 u_R^*\,, \quad \bar{u}_2 = i\sigma_2 c_R^*\,, \quad \bar{u}_3 = i\sigma_2 t_R^*\,,$$

we can write the Yukawa couplings with the notation of the 4-component Dirac spinors:

$$\mathcal{L}^u_{\text{Yukawa}} = i\frac{v+h}{v}\left(m_u \bar{u} u + m_c \bar{c} c + m_t \bar{t} t\right)$$

with the masses

$$y^u_{jj}\,\frac{v}{\sqrt{2}} = (m_u,\, m_c,\, m_t)\ .$$

The CKM matrix has been eliminated in terms of mass by a redefinition of the fields, but it can not be completely eliminated from the Lagrangian and will therefore intervene in the interaction terms.

For quarks of the *down*-type this is not a complication of the matrix $\mathcal{V}$ since it was associated with quarks of *up*-type:

$$\mathcal{L}^{d}_{\text{Yukawa}} = i y^{u}_{jj} {Q'}^{T}_{i} \sigma_2 \bar{d}_i H^* + \text{h.c.}$$

In the four-component Dirac notation for spinors one obtains

$$\mathcal{L}^{d}_{\mathrm{Yukawa}} = i\frac{v+h}{v}\left(m_d\bar{d}d + m_s\bar{s}s + m_b\bar{b}b\right)$$

with the masses

$$y^{d}_{jj}\,\frac{v}{\sqrt{2}} = \left(m_d,\, m_s,\, m_b\right)\,.$$

The results of this analysis shows that the couplings of the Higgs boson is a universal coupling which is proportional to masses.