Solutions to exercises

Exercise A

Consider a complex scalar field ϕ_i in the vector representation of $SU(n)$, which transforms as follows under infinitesimal transformations of $SU(n)$

$$\begin{aligned} \phi_i &\rightarrow \phi_i + i\epsilon_i^j \phi_j \\ \phi^i &\rightarrow \phi^i - i\epsilon_k^i \phi^k \end{aligned}$$

with $\phi_i^* = \phi^i$. Find an expression which is invariant under $SU(n)$ transformations and construct a renormalisable scalar potential for a general theory in 4-dimensions.

Choose a value for the vacuum of the scalar field as

$$\langle 0|\phi|0\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ v \end{pmatrix}$$

and consider the translation of this minimum of the field to study the properties of the components of the scalar field. How many Goldstone bosons remain massless in the spectrum of the theory? What is the residual group invariance of the theory?

Doing the same exercise with two complex scalar fields ϕ_{1i} and ϕ_{2i} in the vector representation of $SU(n)$, where they transform in the same way that ϕ_i previously did. Build the scalar potential and do not forget to also consider the terms which mix the two fields.

Select vacuum expectation values

$$\langle 0|\phi_1|0\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ v_1 \end{pmatrix} \qquad \langle 0|\phi_2|0\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ v_2 \\ v_3 \end{pmatrix}$$

and study the symmetry breaking.

Solutions to Exercise A

An expression invariant under the transformations of $SU(n)$ is given by the scalar product in the complex vector space

$$\phi_i \phi^i \to \left(\phi_i + i\epsilon_i^j \phi_j\right)\left(\phi^i - i\epsilon_k^i \phi^k\right) = \phi_i \phi^i \ .$$

The renormalisable invariant potential can be constructed from this invariant combination

$$V(\phi) = \mu^2 \phi_i \phi^i + \frac{\lambda}{2}\left(\phi_i \phi^i\right)^2 \ .$$

For $\mu^2 < 0$ the minimum potential is given by

$$\phi_i \phi^i = \sqrt{\frac{-\mu^2}{\lambda}} \equiv v \ .$$

The value in the vacuum for the scalar field is chosen in the direction n of the potential

$$\langle 0|\phi_i|0\rangle = \delta_{in}\, v$$

as indicated by the exercise. The symmetry is broken as follows

$$SU(n) \rightarrow SU(n-1)\ .$$

The number of Goldstone bosons is given by the number of generators broken by the theory which is in turn given by the difference between the number of generators of $SU(n)$, n^2-1 and $SU(n-1)$, $[(n-1)^2-1]$;

$$(n^2-1) - [(n-1)^2-1] = 2n-1\ .$$

To study in more detail the symmetry breaking we can place at least one translation field

$$\phi_i = +\delta_{in}\, v$$

in the theory and write the potential in terms of the new fields. The quadratic part of the potential gives mass terms

$$\mu^2\,(\phi_i'\phi'^i)+\frac{\lambda}{2}\left[v^2(\phi_n+\phi^n)^2+2v^2(\phi_i'\phi'^i)\right]=-\frac{\mu^2}{2}(\phi_n+\phi^n)^2\,.$$

The fields ϕ_i are complex (two degrees of freedom for each field) and $\phi^i = \phi_i^*$. Only the real part of ϕ_n has mass. The other $2n-1$ fields are the massless Goldstone bosons.

With two multiplets of scalar complex fields we have four invariant combinations

$$\phi_{1i}\phi^{1i}\,, \qquad \phi_{2i}\phi^{2i}\,, \qquad \phi_{1i}\phi^{2i}\,, \qquad \phi_{2i}\phi^{1i}$$

from which to build the invariant potential. The pattern of symmetry breaking is as follows

$$SU(n) \to SU(n-2)$$

with

$$(n^2-1) - [(n-2)^2 - 1] = 4n - 4$$

Goldstone bosons.

Exercise B

Consider the Lagrangian of the local $O(3)$ symmetry

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu)_{ij}\phi_j(D^\mu)_{ik}\phi_k - \frac{\mu^2}{2}\phi_i\phi_i - \frac{\lambda}{4}(\phi_i\phi_i)^2$$

with covariant derivative

$$(D_\mu)_{ij} = \delta_{ij}\partial_\mu - ig\,(T_a)_{ij}\,W^a_\mu$$

and $(T_a)_{ij} = -i\epsilon_{aij}$. Choose the solution with spontaneous symmetry breaking ($\mu^2 < 0$) and the vacuum of the theory along the 3 direction:

$$\phi_i = v\,\delta_{i3}$$

and show that both gauge fields W_1^μ and W_2^μ associated with broken generators T_1 and T_2 have weight g^2v^2. Also show that W_3^μ has zero mass.

Solutions to Exercise B

The mass term for gauge fields in the Lagrangian is

$$-\frac{1}{2}g^2v^2(T_a)_{i3}(T_b)_{i3}W_\mu^a W^{a\mu}$$

and the mass matrix is

$$(M_W^2)_{ab} = -g^2v^2(T_a)_{i3}(T_b)_{i3} \ .$$

Using $(T_a)_{ij} = -i\epsilon_{aij}$ we obtain

$$(T_a)_{i3}(T_b)_{i3} = -\epsilon_{ai3}\epsilon_{bi3} = -(\delta_{ab} - \delta_{a3}\delta_{b3})$$

and the explicit form of the mass matrix:

$$(M_W^2)_{ab} = g^2 v^2 (\delta_{ab} - \delta_{a3}\delta_{b3}) = g^2 v^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So the two fields W_1^μ and W_2^μ associated with broken generators T_1 and T_2 have weight $g^2 v^2$ whereas W_3^μ has zero mass since it is associated with the unbroken $O(2)$ symmetry.

Exercise C

Find the value of the electric charge for the leptons in the doublet Q_L.

Solutions to Exercise C

Using the formula

$$Q_{\rm em} = I_{3w} + \frac{1}{2} y \ ,$$

we find

$$Q_{\rm em} \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L = \left[\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} + \frac{-1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L = \begin{pmatrix} 0 \\ -e \end{pmatrix}_L$$

So the neutrino has no electric charge and the electron has a charge -1.

Exercise D

Defining the scalar fields on the vacuum as follows:

$$H = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} + \begin{pmatrix} \phi^+ \\ \frac{h+i\eta}{\sqrt{2}} \end{pmatrix} \equiv H_0 + H' \ ,$$

find the spectrum of masses of Goldstone bosons and the physical Higgs field in the theory.

Solutions to Exercise D

We obtain

$$|H|^2 = \frac{v^2}{2} + vh + |\phi^+|^2 \frac{h^2+\eta^2}{2}$$

and the Higgs potential becomes

$$V = -\frac{\mu^4}{4\lambda} + \lambda v^2 h^2 + 2\lambda v h \left(|\phi^+|^2 + \frac{h^2+\eta^2}{2} \right) + \lambda \left(|\phi^+|^2 + \frac{h^2+\eta^2}{2} \right)^2$$

allowing us to read the masses

$$\begin{aligned} m_\eta^2 &= m_{\phi^+} = m_{\phi^-} = 0 \\ m_h^2 &= 2\lambda v^2 = -2\mu^2 \end{aligned}$$

with $\phi^- = \phi^{+\dagger}$. We therefore have a physical scalar h with mass (the Higgs field) and three massless Goldstone bosons, which can be eliminated by a gauge transformation.