FEYNMAN´S TREE THEOREM & LOOP-TREE DUALITIES

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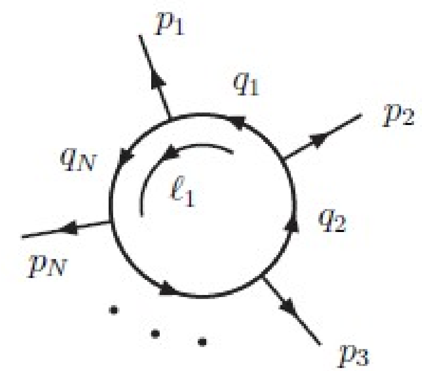
Catani, Gleisberg, Krauss, Rodrigo, Winter, JHEP 09 (2008) 065

Feynman´s Tree Theorem



A general one loop N-leg diagram

$$L^{(1)}(p_1, p_2, \ldots, p_N) = \int_{l_1} \prod_{i=1}^{N} G_F(q_i) \qquad q_i = l_1 + p_{1,i}, \quad i \in \alpha_1 = \{1,2,\ldots,N\}$$

with the Feynman propagator

$$G_F(q_i) = \frac{1}{q_i^2 - m_i^2 + i0} \qquad q_{i,0} = \pm\sqrt{\boldsymbol{q}_i^2 - m_i - i0}$$

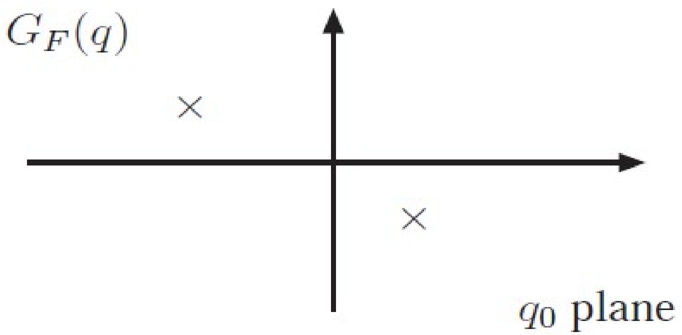
Notation

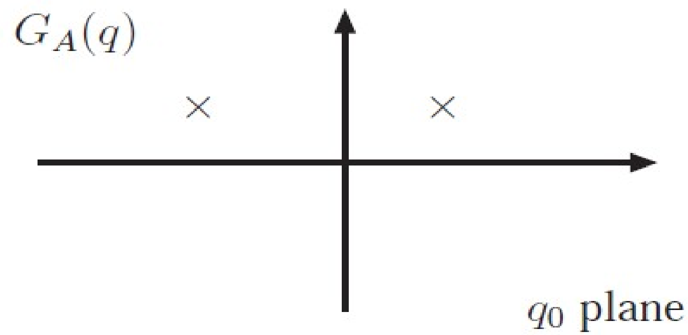
$$\int_{l_i} \cdots = -i \int \frac{d^d l_i}{(2\pi)^d} \cdots$$

$$\tilde{\delta}(q_i) \equiv 2\pi i \, \delta_+(q_i^2 - m_i^2)$$

$$\int \frac{d^d q_i}{(2\pi)^{d-1}} \delta_+(q_i^2 - m_i^2) \cdots = \int_{q_i} \tilde{\delta}(q_i) \cdots$$

Feynman´s Tree Theorem





We also introduce the advanced propagator

$$G_A(q_i) = \frac{1}{q_i^2 - m_i^2 - i\, q_{i,0}} \qquad q_{i,0} \simeq \pm\sqrt{\boldsymbol{q}_i^2 - m_i} + i0$$

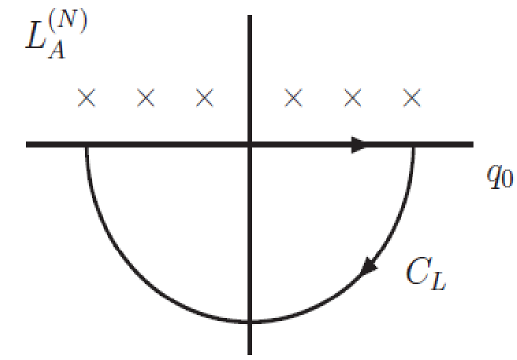
Relation between the two: $G_A(q_i) = G_F(q_i) + \tilde{\delta}(q_i)$

Catani, Gleisberg, Krauss, Rodrigo, JHEP 09 (2008) 65

Feynman´s Tree Theorem

We start with a modified scalar integral

$$L_A^{(1)}(p_1, p_2, \ldots, p_N) = \int_{l_1} \prod_{i=1}^{N} G_A(q_i) = 0$$



Feynman´s Tree Theorem

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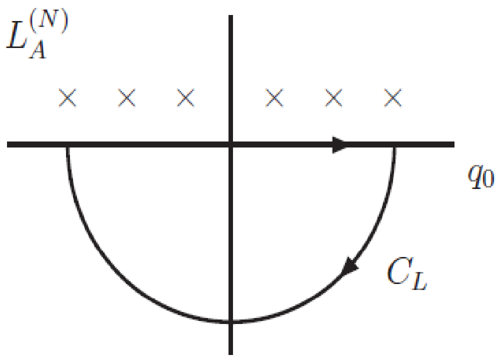

$$G_A(q_i) = G_F(q_i) + \tilde{\delta}(q_i)$$

Feynman´s Tree Theorem

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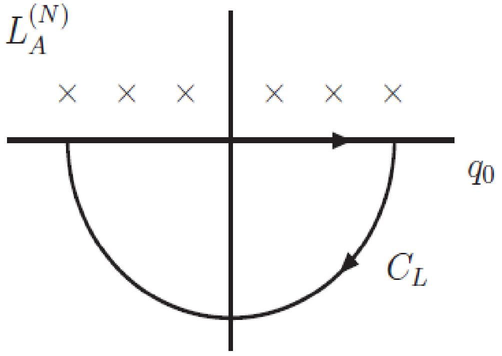


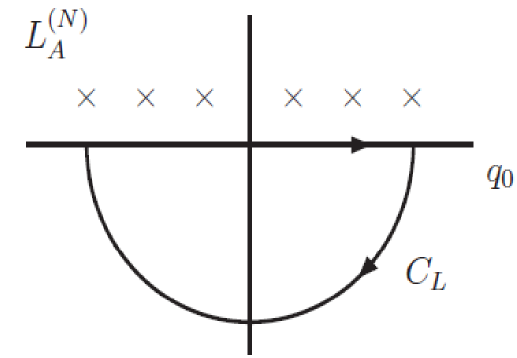
$$L^{(1)}(p_1, p_2, \ldots, p_N) + L^{(1)}_{1-cut}(p_1, p_2, \ldots, p_N) + \cdots + L^{(1)}_{N-cut}(p_1, p_2, \ldots, p_N) = 0$$

$$L^{(N)}_{m-cut}(p_1, p_2, \ldots, p_N) = \int_{l_i} \{\tilde{\delta}(q_1) \cdots \tilde{\delta}(q_m) G_F(q_{m+1}) \cdots G_F(q_N) + perms.\}$$

Feynman´s Tree Theorem

We start with a modified scalar integral

$$L_A^{(1)}(p_1, p_2, \ldots, p_N) = \int_{l_1} \prod_{i=1}^{N} G_A(q_i) = 0$$



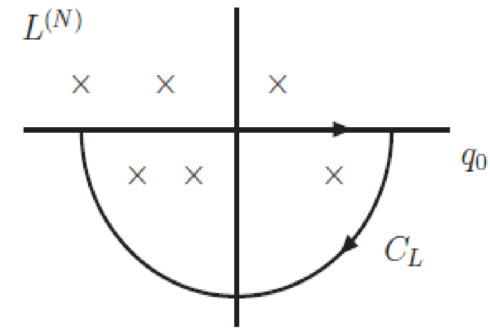
$$L^{(1)}(p_1, p_2, \ldots, p_N) + L^{(1)}_{1-cut}(p_1, p_2, \ldots, p_N) + \cdots + L^{(1)}_{N-cut}(p_1, p_2, \ldots, p_N) = 0$$

$$L^{(1)}(p_1, p_2, \ldots, p_N) = -L^{(1)}_{1-cut}(p_1, p_2, \ldots, p_N) - \cdots - L^{(1)}_{N-cut}(p_1, p_2, \ldots, p_N)$$

FEYNMAN´S TREE THEOREM

Duality relation at one loop

We apply directly the residue theorem



$$L^{(1)}(\{p_m\}) = -2\pi i \int_{l_1} \sum Res_{\Im q_{i,0}<0} \left[\prod_{j=1}^{N} G_F(q_j)\right]$$

$$Res_{\Im q_{i,0}<0} \frac{1}{q_i^2 - m_i^2 + i0} = \int dl_{1,0}\, \delta_+(q_i^2 - m_i^2) \qquad \left[\prod_{j \neq i} G_F(q_j)\right]_{(i-th\ pole)} = \prod_{j \neq i} \frac{1}{q_j^2 - m_j^2 - i0\,\eta(q_j - q_i)}$$

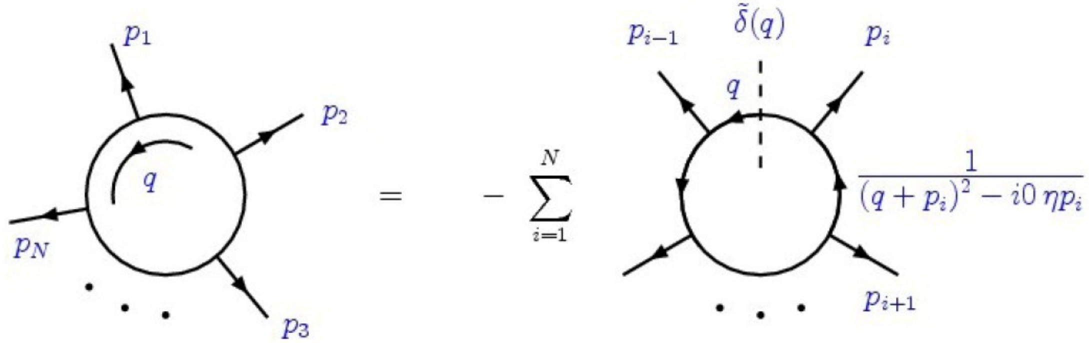
DUAL PROPAGATOR

$$G_D(q_i; q_j) = \frac{1}{q_j^2 - m_j^2 - i0\,\eta(q_j - q_i)}$$

η is a future-like vector

$$\eta_0 \geqslant 0 \quad , \quad \eta^2 \geqslant 0$$

Duality relation at one loop



$$L^{(1)}(p_1, p_2, \ldots, p_N) = -\sum \int_{l_1} \tilde{\delta}(q_i) \prod_{\substack{j=1 \\ j\neq i}}^{N} G_D(q_i; q_j)$$

TREE-LOOP DUALITY THEOREM

Duality relation at one loop

Recasts virtual corrections to a form similar to real radiative corrections

$$\int_{m+1} d\sigma^R = \int d\Phi^{(m+1)}(\{p_i\}, q) \;\; M^{(m+1)}(\{p_i\}) \;\; F^{(m+1)}(\{p_i\})$$

$$\int_{m+q} d\sigma^{DUAL} = \int d\Phi^{(m+1)}(\{p_i\}, q) \;\; \tilde{M}^{(m+1)}(\{p_i\}, q) \;\; F^{(m+1)}(\{p_i\})$$

Single cut terms

Duality relation at one loop

- Unlike FTT it contains only single cuts at the price of introducing a modified i0 prescription, the dual prescription
- The singularities of the loop diagram appear as singularities of the Dual Integrals
- Feynman graphs are treated in the same way since the duality operation works only on propagators
- The form of the duality theorem closely resembles the form of real radiative corrections

Bierenbaum, Catani, P.D., Rodrigo, JHEP 10 (2010) 073

Towards the extension to two loops

Extension to sets of momenta

$$G_F(\alpha_k) = \prod_{i\in\alpha_k} G_F(q_i)$$

$$G_D(\alpha_k) = \sum_{i\in\alpha_k} \tilde{\delta}(q_i) \prod_{\substack{j\in\alpha_k \\ j\neq i}} G_D(q_i; q_j)$$

$$G_D(\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_N) = \sum_{\beta_N^{(1)} \cup \beta_N^{(2)} = \beta_N} \prod_{i_1 \in \beta_N^{(1)}} G_D(\alpha_{i_1}) \prod_{i_2 \in \beta_N^{(2)}} G_F(\alpha_{i_2})$$

Reverse flow of momenta

$$G_D(-\alpha_k) = \sum_{i\in\alpha_k} \tilde{\delta}(-q_i) \prod_{\substack{j\in\alpha_k \\ j\neq i}} G_D(-q_i; -q_j)$$

$$G_F(-\alpha_k) = G_F(\alpha_k)$$

$$G_A(-\alpha_k) = G_R(\alpha_k)$$

For example: $G_D(\alpha_1 \cup \alpha_2) = G_D(\alpha_1) G_D(\alpha_2) + G_D(\alpha_1) G_F(\alpha_2) + G_F(\alpha_1) G_D(\alpha_2)$

Relation between props: $G_A(\alpha_k) = G_F(\alpha_k) + G_D(\alpha_k)$

Duality in compact form: $L^{(1)}(p_1, p_2, \ldots, p_N) = -\int_{l_1} G_D(\alpha_1)$

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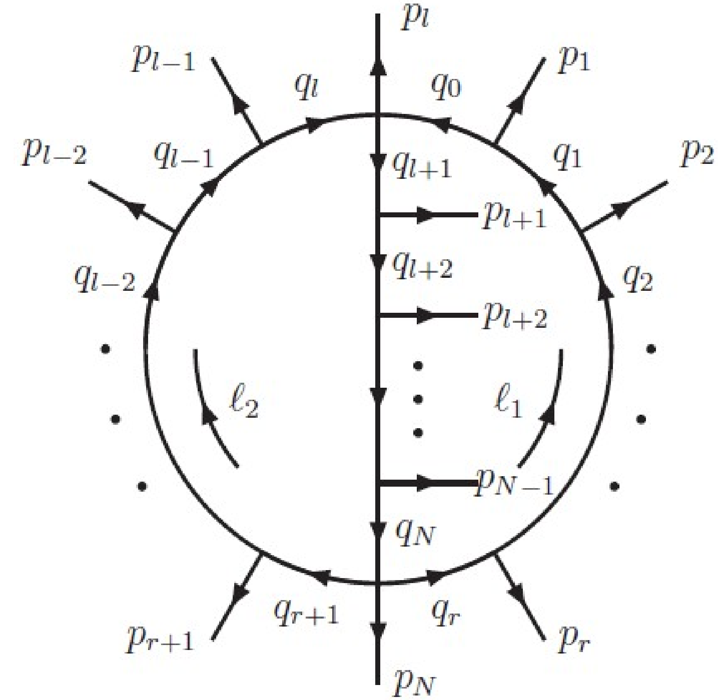
Bierenbaum, Catani, P.D., Rodrigo, JHEP 10 (2010) 073

Duality relation at two loops

A general two-loop master diagram

$$q_i = \begin{cases} l_1 + p_{1,i} & , i \in \alpha_1 \\ l_2 + p_{i,l-1} & , i \in \alpha_2 \\ l_1 + l_2 + p_{i,l-1} & , i \in \alpha_3 \end{cases}$$

$$\alpha_1 = \{0, 1, \ldots, r\} \qquad \alpha_2 = \{r+1, r+2, \ldots, l\}$$
$$\alpha_3 = \{l+1, l+2, \ldots, N\}$$



Application of duality to one of the integration momenta

$$\int_{l_i} G_F(\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_N) = -\int_{l_i} G_D(\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_N)$$

Duality relation at two loops

Starting point

$$L^{(2)}(p_1, p_2, \ldots, p_N) = \int_{l_1} \int_{l_2} G_F(\alpha_1 \cup \alpha_2 \cup \alpha_3)$$

Duality relation at two loops

Starting point

$$L^{(2)}(p_1, p_2, \dots, p_N) = \int_{l_1} \int_{l_2} G_F(\alpha_1 \cup \alpha_2 \cup \alpha_3)$$

$$G_F(\alpha_1 \cup \alpha_2 \cup \alpha_3) = G_F(\alpha_1)\, G_F(\alpha_2)\, G_F(\alpha_3)$$

Duality relation at two loops

$$\int_{l_1} G_F(\alpha_1 \cup \alpha_3) = -\int_{l_1} G_D(\alpha_1 \cup \alpha_3)$$

Starting point

$$L^{(2)}(p_1, p_2, \ldots, p_N) = \int_{l_1} \int_{l_2} G_F(\alpha_1 \cup \alpha_2 \cup \alpha_3)$$

Apply to all sets which depend on loop momentum l_1

Assignment to sets

$$q_i = \begin{cases} l_1 + p_{1,i} & , i \in \alpha_1 \\ l_2 + p_{i,l-1} & , i \in \alpha_2 \\ l_1 + l_2 + p_{i,l-1} & , i \in \alpha_3 \end{cases}$$

Duality relation at two loops

Starting point

$$L^{(2)}(p_1, p_2, \ldots, p_N) = \int_{l_1} \int_{l_2} G_F(\alpha_1 \cup \alpha_2 \cup \alpha_3)$$

Apply to all sets with depend on loop momentum l_1

$$L^{(2)}(p_1, p_2, \ldots, p_N) = -\int_{l_1} \int_{l_2} G_D(\alpha_1 \cup \alpha_3) G_F(\alpha_2)$$

Duality relation at two loops

Starting point

$$L^{(2)}(p_1, p_2, \ldots, p_N) = \int_{l_1} \int_{l_2} G_F(\alpha_1 \cup \alpha_2 \cup \alpha_3)$$

Apply to all sets with depend on loop momentum l_1

$$L^{(2)}(p_1, p_2, \ldots, p_N) = -\int_{l_1} \int_{l_2} G_D(\alpha_1 \cup \alpha_3) G_F(\alpha_2)$$

Expand the dual propagator of the union

$$G_D(\alpha_1 \cup \alpha_3) = G_D(\alpha_1) G_D(\alpha_3) + G_D(\alpha_1) G_F(\alpha_3) + G_F(\alpha_1) G_D(\alpha_3)$$

Duality relation at two loops

Starting point

$$L^{(2)}(p_1, p_2, \ldots, p_N) = \int_{l_1} \int_{l_2} G_F(\alpha_1 \cup \alpha_2 \cup \alpha_3)$$

Apply to all sets with depend on loop momentum l_1

$$L^{(2)}(p_1, p_2, \ldots, p_N) = -\int_{l_1} \int_{l_2} G_D(\alpha_1 \cup \alpha_3) G_F(\alpha_2)$$

Expand the dual propagator of the union

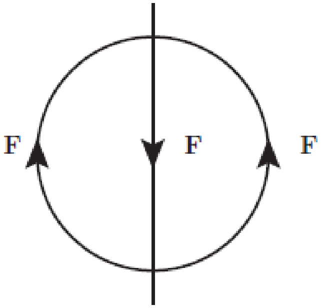
$$L^{(2)}(p_1, p_2, \ldots, p_N) = -\int_{l_1} \int_{l_2} \{ G_D(\alpha_1) G_D(\alpha_3) + G_D(\alpha_1) G_F(\alpha_3) + G_F(\alpha_1) G_D(\alpha_3) \} G_F(\alpha_2)$$

Duality relation at two loops

Starting point

$$L^{(2)}(p_1, p_2, \ldots, p_N) = \int_{l_1} \int_{l_2} G_F(\alpha_1 \cup \alpha_2 \cup \alpha_3)$$

Apply to all sets with depend on loop momentum l_1



$$L^{(2)}(p_1, p_2, \ldots, p_N) = -\int_{l_1} \int_{l_2} G_D(\alpha_1 \cup \alpha_3) G_F(\alpha_2)$$

Expand the dual propagator of the union

$$L^{(2)}(p_1, p_2, \ldots, p_N) = -\int_{l_1} \int_{l_2} \{G_D(\alpha_1) G_D(\alpha_3) + G_D(\alpha_1) G_F(\alpha_3) + G_F(\alpha_1) G_D(\alpha_3)\} G_F(\alpha_2)$$

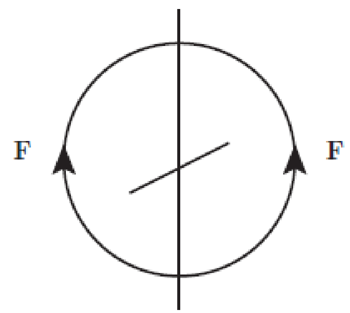
$$\int G_F(\alpha_2) G_F(\alpha_3) = -\int G_D(\alpha_2 \cup \alpha_3)$$

Duality relation at two loops

Starting point

$$L^{(2)}(p_1, p_2, \ldots, p_N) = \int_{l_1} \int_{l_2} G_F(\alpha_1 \cup \alpha_2 \cup \alpha_3)$$

Apply to all sets with depend on loop momentum l_1



$$L^{(2)}(p_1, p_2, \ldots, p_N) = -\int_{l_1} \int_{l_2} G_D(\alpha_1 \cup \alpha_3) G_F(\alpha_2)$$

Expand the dual propagator of the union

$$L^{(2)}(p_1, p_2, \ldots, p_N) = -\int_{l_1} \int_{l_2} \{ G_D(\alpha_1) G_D(\alpha_3) + G_D(\alpha_1) G_F(\alpha_3) + G_F(\alpha_1) G_D(\alpha_3) \} G_F(\alpha_2)$$

Combine to a dual propagator

$$L^{(2)}(p_1, p_2, \ldots, p_N) = \int_{l_1} \int_{l_2} \{ -G_D(\alpha_1) G_F(\alpha_2) G_D(\alpha_3) + G_D(\alpha_1) G_D(\alpha_1 \cup \alpha_3) + G_D(\alpha_3) G_D(-\alpha_1 \cup \alpha_2) \}$$

Duality relation at two loops

Expand the remaining dual propagators of the union of sets

$$L^{(2)}(p_1, p_2, \dots, p_N) = \int_{l_1} \int_{l_2} \Big\{ \; G_D(\alpha_1) G_D(\alpha_2) G_F(\alpha_3) + G_D(-\alpha_1) G_F(\alpha_2) G_D(\alpha_3)$$

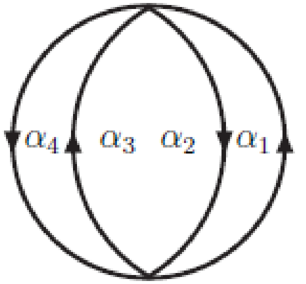
$$+ G^*(\alpha_1) G_D(\alpha_2) G_D(\alpha_3) \Big\}$$

$$G^*(\alpha_k) = G_F(\alpha_k) + G_D(\alpha_k) + G_D(-\alpha_k)$$

- The i0 prescription depends on external momenta only
- The duality relation contains triple cuts as well

Three Loops and beyond

Duality can be readily applied to three loops



$$L^{(3)}_{basket}=(p_1,p_2,\ldots,p_N)=\int_{l_1}\int_{l_2}\int_{l_3} G_D(\alpha_1\cup\alpha_2)\,G_D(\alpha_3\cup\alpha_4)$$

$$\begin{aligned}L^{(3)}_{basket}=(p_1,p_2,\ldots,p_N)=\int_{l_1}\int_{l_2}\int_{l_3}\{&G_D(\alpha_2,\alpha_3,-\alpha_4)G_F(\alpha_1)+G_D(\alpha_1,\alpha_3,-\alpha_4)G_F(\alpha_2)\\&+G_D(-\alpha_1,\alpha_2,\alpha_4)G_F(\alpha_3)+G_D(-\alpha_1,\alpha_2,\alpha_3)G_F(\alpha_2)\\&+G_D(-\alpha_1,\alpha_2,\alpha_3,\alpha_4)+G_D(\alpha_1,\alpha_2,\alpha_3,-\alpha_4)+G_D(-\alpha_1,\alpha_2,\alpha_3,-\alpha_4)\}\end{aligned}$$

Notation

$$G_D(\alpha_1,\ldots,\alpha_N)=\prod_i G_D(\alpha_i)$$

Summary and conclusions

- One Loop Integrals are written as linear combinations of N single cut phase space integrals
- We rederived and extended the one loop Duality Theorem to two- and three-loops and beyond
- Number of cuts equals the number of loops, so that a loop diagram is opened up to a tree diagram
- This is a systematic procedure with the potential to treat numerically higher loop corrections
- Treatment of singularities of the Dual Integrals in one and two loops is in progress
- Treatment of higher order poles, relevant for higher order corrections is under investigation

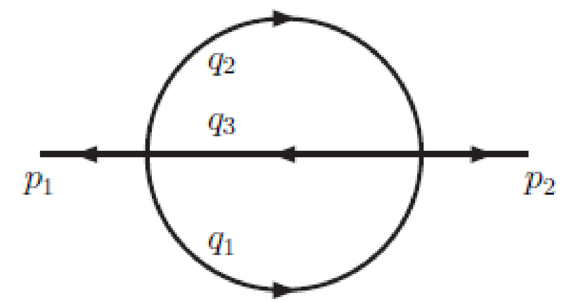
Extra Slides

Example: Massless sunrise 2-Loop 2-point function

The scalar sunrise integral

$$L^{(2)} = \int_{l_1} \int_{l_2} \{ \tilde{\delta}(q_1) \tilde{\delta}(q_2) G_F(q_3) + \tilde{\delta}(-q_1) G_F(q_2) \tilde{\delta}(q_3) + G^*(q_1) \tilde{\delta}(q_2) \tilde{\delta}(q_3) \}$$

We integrate first over l_1



$$\int_{l_1} \tilde{\delta}(l_1) G_F(l_1 + p) = d_\Gamma [p^2 + i0]^{-\epsilon} [1 + \theta(p^2) \theta(-p_0) (e^{2i\pi\epsilon} - 1)]$$

$$\int_{l_1} \tilde{\delta}(l_1) \tilde{\delta}(l_1 + p) = d_\Gamma \; [p^2 + i0]^{-\epsilon} \; \theta(-p^2) (e^{2i\pi\epsilon} - 1)$$

$$d_\Gamma = -\frac{c_\Gamma}{2} \frac{1}{\epsilon(1-2\epsilon)} \frac{1}{\cos(\pi\epsilon)} \qquad c_\Gamma = \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{(4\pi)^{2-\epsilon}\Gamma(1-2\epsilon)}$$

Example: Massless sunrise 2-Loop 2-point function

One final integration over l_2

$$d_\Gamma \int_{l_2} \tilde{\delta}(l_2)[(l_2+p)^{-\epsilon}] = -G_2 \, \frac{\sin(\pi\epsilon)}{\sin(3\pi\epsilon)} \, e^{-2i\pi\epsilon} \, [-p^2 - i0]^{1-2\epsilon} \, [1+\theta(p^2)\theta(-p_0)(e^{2i\pi\epsilon}-1)]$$

$$d_\Gamma \int_{l_2} \tilde{\delta}(l_2)[(l_2+p)^{-\epsilon}]\theta((l_2+p)^2)\theta((l_2+p)_0) = G_2 \, \frac{\sin(\pi\epsilon)}{\sin(3\pi\epsilon)} \, [-p^2 - i0]^{1-2\epsilon} \, [\theta(-p^2) - \theta(p^2)\theta(p_0)e^{-2i\pi\epsilon}]$$

$$G_2 = \frac{\Gamma(-1+2\epsilon)\Gamma^3(1-\epsilon)}{(4\pi)^{4-2\epsilon}\Gamma(3-3\epsilon)}$$

$$L^{(2)}(p) = -G_2(-p^2 - i0)^{1-2\epsilon}$$