The $O(\alpha_s^3\, T_F^2)$ contributions to the Heavy Flavor Wilson Coefficients of $F_2(x, Q^2)$ for $Q^2 \gg m^2$

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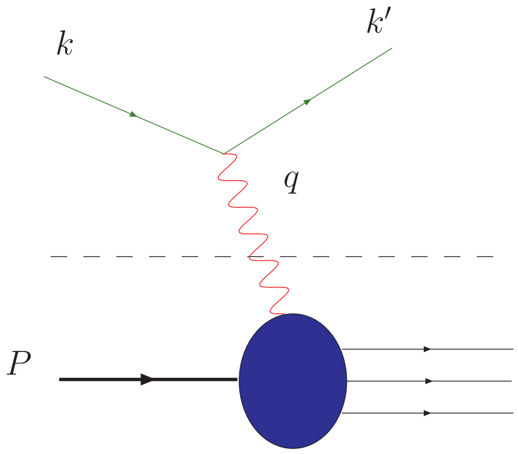
- Introduction
- The Method
- Results in $O\left(N_F T_F^2 C_{A,F}\right)$
- First Contributions $\propto T_F^2$ and $O(m_c^2/m_b^2)$
- Conclusions

[Based on: "The $O(\alpha_s^3)$ Massive Operator Matrix Elements of $O(n_f)$ for the Structure Function $F_2(x, Q^2)$ and Transversity", Nuclear Physics B 844 (2011), pp. 26-54, arXiv:1008.3347]

1. Introduction

Deep–Inelastic Scattering (DIS):



$\longrightarrow \quad L_{\mu\nu}$

$\longrightarrow \quad W_{\mu\nu}$

$$Q^2 := -q^2, \quad x := \frac{Q^2}{2pq} \quad \text{Bjorken–x}$$

$$\nu := \frac{Pq}{M} \ ,$$

$$\frac{d\sigma}{dQ^2\, dx} \sim W_{\mu\nu} L^{\mu\nu}$$

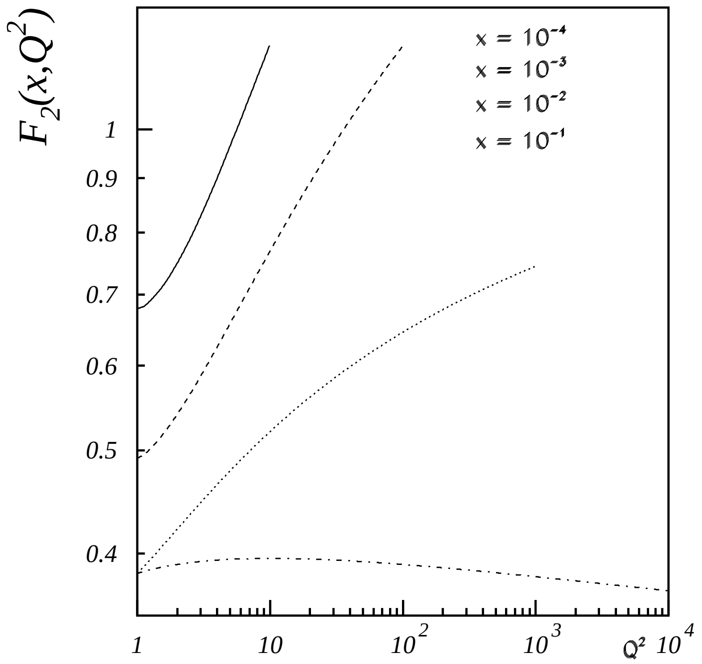
$$W_{\mu\nu}(q,P,s) = \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P,s \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P,s \rangle$$

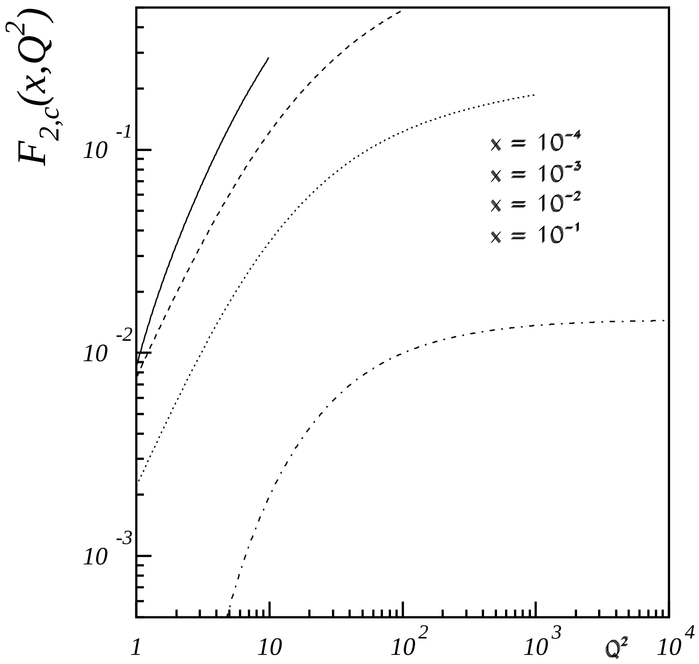
$$\text{unpol.} \left\{ \ = \frac{1}{2x} \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) F_L(x,Q^2) + \frac{2x}{Q^2} \left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu} \right) F_2(x,Q^2) \right.$$

$$\text{pol.} \left\{ \ -\frac{M}{2Pq} \varepsilon_{\mu\nu\alpha\beta} q^\alpha \left[s^\beta g_1(x,Q^2) + \left(s^\beta - \frac{sq}{Pq} p^\beta \right) g_2(x,Q^2) \right] \ . \right.$$

Structure Functions: $F_{2,L}$

contain light and heavy quark contributions





LO charm contributions: PDFs from [Alekhin, Melnikov, Petriello, 2006.]

→ different scaling violations,

→ massive contributions at lower values of x are of order 20%-35%.

Hence for the prediction of cross sections at the LHC the precise knowledge of all PDFs is needed.

2. Status of Heavy Flavor Contributions to DIS Structure Functions

Leading Order: [Witten, 1976; Babcock, Sivers, 1978;Shifman, Vainshtein, Zakharov, 1978; Leveille, Weiler, 1979; Glück, Reya, 1979; Glück, Hoffmann, Reya, 1982.]

Next-to-Leading Order : [Laenen, Riemersma, Smith, van Neerven, 1993, 1995]

asymptotic: [Buza, Matiounine, Smith, Migneron, van Neerven, 1996] via IBP

$(Q^2 \gg m^2)$ [Bierenbaum, Blümlein, Klein, 2007] via ${}_pF_q$'s, more compact results

NLO fast Mellin space implementation: [Alekhin, Blümlein 2003]

NNLO, $Q^2 \gg m^2$: contribs. to F_L for all N: [Blümlein, De Freitas, van Neerven, Klein 2006]

contributions to F_2 $(N = 2...10(14))$: [Bierenbaum, Blümlein, Klein 2009]

contributions to transversity $(N = 1...13)$: [Blümlein, Klein, Tödtli 2009]

$\Longrightarrow$ Moment-reference

all $O(\alpha_s^3) \times \ln^k \left(\frac{Q^2}{m^2}\right)$ terms for massive OMEs for general N:

[Bierenbaum, Blümlein, Klein 2010]

Computation of all 3-loop ladder graphs: [Blümlein, Hasselhuhn, Klein, Schneider]

Goal: Calculate the 3-loop massive Wilson coefficients for $F_2(x, Q^2)$ in the region $Q^2 \gtrsim 10m^2$ for general values of N.

- in the asymptotic region F_L is known for general values of N to NNLO [Blümlein, De Freitas, van Neerven, Klein, 2006.]
- F_2 for n_f massless and one heavy quark flavor:

$$
\begin{aligned}
F_{(2,L)}^{Q\overline{Q}}(x, n_f+1, Q^2, m^2) \quad = \quad & \sum_{k=1}^{n_f} e_k^2 \Biggl\{ L_{q,(2,L)}^{\mathsf{NS}}\left(x, n_f+1, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \otimes \left[f_k(x,\mu^2,n_f) + f_{\overline{k}}(x,\mu^2,n_f)\right] \\
+ \frac{1}{n_f}\Biggl[L_{q,(2,L)}^{\mathsf{PS}}\left(x, n_f+1, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) & \otimes \Sigma(x,\mu^2,n_f) + L_{g,(2,L)}^{\mathsf{S}}\left(x, n_f+1, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \otimes G(x,\mu^2,n_f) \Biggr] \Biggr\} \\
+ e_Q^2 \Biggl[H_{q,(2,L)}^{\mathsf{PS}}\left(x, n_f+1, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) & \otimes \Sigma(x,\mu^2,n_f) + H_{g,(2,L)}^{\mathsf{S}}\left(x, n_f+1, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \otimes G(x,\mu^2,n_f) \Biggr]
\end{aligned}
$$

- $\otimes$ denotes the Mellin convolution $[A \otimes B](x) = \int_0^1 \int_0^1 dx_1 dx_2 \; \delta(x - x_1 x_2) A(x_1) B(x_2)$,
- The asymptotic representation for $F_2(x, Q^2)$ becomes effective at $Q^2 \geq 10 \cdot m^2$

- In this limit the massive Wilson coefficients up to $O(a_s^3)$ read

$$
\begin{aligned}
L_{q,(2,L)}^{\mathsf{NS}}(n_f+1) &= a_s^2\Big[A_{qq,Q}^{(2),\mathsf{NS}}(n_f+1)\delta_2 + \hat{C}_{q,(2,L)}^{(2),\mathsf{NS}}(n_f)\Big] \\
&+ a_s^3\Big[A_{qq,Q}^{(3),\mathsf{NS}}(n_f+1)\delta_2 + A_{qq,Q}^{(2),\mathsf{NS}}(n_f+1)C_{q,(2,L)}^{(1),\mathsf{NS}}(n_f+1) + \hat{C}_{q,(2,L)}^{(3),\mathsf{NS}}(n_f)\Big] \\
L_{q,(2,L)}^{\mathsf{PS}}(n_f+1) &= a_s^3\Big[\ A_{qq,Q}^{(3),\mathsf{PS}}(n_f+1)\ \delta_2 + A_{gq,Q}^{(2)}(n_f)\ \ n_f\tilde{C}_{g,(2,L)}^{(1)}(n_f+1) + n_f\hat{\tilde{C}}_{q,(2,L)}^{(3),\mathsf{PS}}(n_f)\Big] \\
L_{g,(2,L)}^{\mathsf{S}}(n_f+1) &= a_s^2 A_{gg,Q}^{(1)}(n_f+1)n_f\tilde{C}_{g,(2,L)}^{(1)}(n_f+1) + a_s^3\Big[\ A_{qg,Q}^{(3)}(n_f+1)\ \delta_2 \\
&+A_{gg,Q}^{(1)}(n_f+1)\ \ n_f\tilde{C}_{g,(2,L)}^{(2)}(n_f+1) + A_{gg,Q}^{(2)}(n_f+1)\ \ n_f\tilde{C}_{g,(2,L)}^{(1)}(n_f+1) \\
&+\ A_{Qg}^{(1)}(n_f+1)\ \ n_f\tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(n_f+1) + n_f\hat{\tilde{C}}_{g,(2,L)}^{(3)}(n_f)\Big]\ , \\
H_{q,(2,L)}^{\mathsf{PS}}(n_f+1) &= a_s^2\Big[\ A_{Qq}^{(2),\mathsf{PS}}(n_f+1)\ \delta_2 +\ \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(n_f+1)\Big] + a_s^3\Big[\ A_{Qq}^{(3),\mathsf{PS}}(n_f+1)\ \delta_2 \\
&+\ \tilde{C}_{q,(2,L)}^{(3),\mathsf{PS}}(n_f+1) + A_{gq,Q}^{(2)}(n_f+1)\ \tilde{C}_{g,(2,L)}^{(1)}(n_f+1) \\
&+A_{Qq}^{(2),\mathsf{PS}}(n_f+1)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(n_f+1)\Big]\ , \\
H_{g,(2,L)}^{\mathsf{S}}(n_f+1) &= a_s\Big[\ A_{Qg}^{(1)}(n_f+1)\ \delta_2 +\ \tilde{C}_{g,(2,L)}^{(1)}(n_f+1)\Big] + a_s^2\Big[\ A_{Qg}^{(2)}(n_f+1)\ \delta_2 \\
&+\ A_{Qg}^{(1)}(n_f+1)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(n_f+1) +\ A_{gg,Q}^{(1)}(n_f+1)\ \tilde{C}_{g,(2,L)}^{(1)}(n_f+1) \\
&+\ \tilde{C}_{g,(2,L)}^{(2)}(n_f+1)\Big] +\ a_s^3\Big[\ A_{Qg}^{(3)}(n_f+1)\ \delta_2 +\ A_{Qg}^{(2)}(n_f+1)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(n_f+1) \\
&+\ A_{gg,Q}^{(2)}(n_f+1)\ \tilde{C}_{g,(2,L)}^{(1)}(n_f+1) +\ A_{Qg}^{(1)}(n_f+1)\Big\{C_{q,(2,L)}^{(2),\mathsf{NS}}(n_f+1) \\
&+\ \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(n_f+1)\Big\} +\ A_{gg,Q}^{(1)}(n_f+1)\ \tilde{C}_{g,(2,L)}^{(2)}(n_f+1) +\ \tilde{C}_{g,(2,L)}^{(3)}(n_f+1)\Big]
\end{aligned}
$$

Renormalization

[Bierenbaum, Blümlein, Klein 2009]

$$\hat{\hat{A}}_{ij} = \delta_{ij} + \sum_{k=1}^{\infty} \hat{a}_s^k \sum_{l=-k}^{0} \frac{\hat{\hat{A}}_{ij}^{(k,l)}}{\varepsilon^l}$$

- Mass renormalization (on–mass shell scheme)
- Charge renormalization
- Renormalization of ultraviolet singularities
  $\Longrightarrow$ are absorbed into the Z–factors given in terms of anomalous dimensions γ_{ij}.
- Factorization of collinear singularities
  $\Longrightarrow$ are factored into the Γ–factors Γ_{NS}, $\Gamma_{ij,S}$.
  Only for massless quarks $\Gamma = Z^{-1}$ holds.
  Here: Γ–matrices apply to parts of the diagrams with massless lines only .

Generic formula for operator renormalization and mass factorization:

$$\boxed{A_{ij} = Z_{il}^{-1} \hat{A}_{lm} \Gamma_{mj}^{-1}}$$

$\Longrightarrow$ $O(\varepsilon^2)$–terms of the 1–loop OMEs and $O(\varepsilon)$–terms of the 2–loop OMEs are needed for renormalization at 3–loops.

- Renormalization allows to express pole terms through lower order contributions to the OMEs, anomalous dimensions, etc.
- Through this the $N_F T_F^2$ contributions to the 3-loop anomalous dimension can be determined for general values of N
- e.g. A_{Qg}^3:

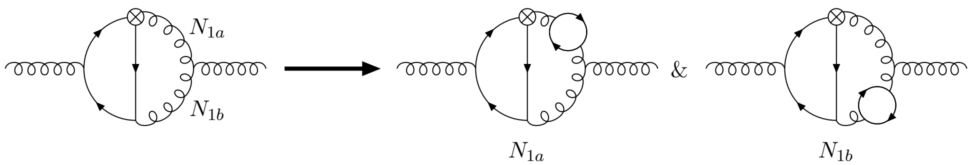
$$
\begin{aligned}
\hat{\hat{A}}_{Qg}^{(3)} = & \left(\frac{\hat{m}^2}{\mu^2}\right)^{3\varepsilon/2}\Biggl[\frac{\hat{\gamma}_{qg}^{(0)}}{6\varepsilon^3}\Biggl((n_f+1)\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}+\gamma_{qq}^{(0)}\Bigl[\gamma_{qq}^{(0)}-2\gamma_{gg}^{(0)}-6\beta_0-8\beta_{0,Q}\Bigr]+8\beta_0^2 \\
& +28\beta_{0,Q}\beta_0+24\beta_{0,Q}^2+\gamma_{gg}^{(0)}\Bigl[\gamma_{gg}^{(0)}+6\beta_0+14\beta_{0,Q}\Bigr]\Biggr)+\frac{1}{6\varepsilon^2}\Biggl(\hat{\gamma}_{qg}^{(1)}\Bigl[2\gamma_{qq}^{(0)}-2\gamma_{gg}^{(0)} \\
& -8\beta_0-10\beta_{0,Q}\Bigr]+\hat{\gamma}_{qg}^{(0)}\Bigl[\hat{\gamma}_{qq}^{(1),\mathsf{PS}}\{1-2n_f\}+\gamma_{qq}^{(1),\mathsf{NS}}+\hat{\gamma}_{qq}^{(1),\mathsf{NS}}+2\hat{\gamma}_{gg}^{(1)}-\gamma_{gg}^{(1)}-2\beta_1 \\
& -2\beta_{1,Q}\Bigr]+6\delta m_1^{(-1)}\hat{\gamma}_{qg}^{(0)}\Bigl[\gamma_{gg}^{(0)}-\gamma_{qq}^{(0)}+3\beta_0+5\beta_{0,Q}\Bigr]\Biggr)+\frac{1}{\varepsilon}\Biggl(\frac{\hat{\gamma}_{qg}^{(2)}}{3}-n_f\frac{\hat{\tilde{\gamma}}_{qg}^{(2)}}{3} \\
& +\hat{\gamma}_{qg}^{(0)}\Bigl[a_{gg,Q}^{(2)}-n_f a_{Qq}^{(2),\mathsf{PS}}\Bigr]+a_{Qg}^{(2)}\Bigl[\gamma_{qq}^{(0)}-\gamma_{gg}^{(0)}-4\beta_0-4\beta_{0,Q}\Bigr]+\frac{\hat{\gamma}_{qg}^{(0)}\zeta_2}{16}\Bigl[\gamma_{gg}^{(0)}\Bigl\{2\gamma_{qq}^{(0)} \\
& -\gamma_{gg}^{(0)}-6\beta_0+2\beta_{0,Q}\Bigr\}-(n_f+1)\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}+\gamma_{qq}^{(0)}\Bigl\{-\gamma_{qq}^{(0)}+6\beta_0\Bigr\}-8\beta_0^2 \\
& +4\beta_{0,Q}\beta_0+24\beta_{0,Q}^2\Bigr]+\frac{\delta m_1^{(-1)}}{2}\Bigl[-2\hat{\gamma}_{qg}^{(1)}+3\delta m_1^{(-1)}\hat{\gamma}_{qg}^{(0)}+2\delta m_1^{(0)}\hat{\gamma}_{qg}^{(0)}\Bigr] \\
& +\delta m_1^{(0)}\hat{\gamma}_{qg}^{(0)}\Bigl[\gamma_{gg}^{(0)}-\gamma_{qq}^{(0)}+2\beta_0+4\beta_{0,Q}\Bigr]-\delta m_2^{(-1)}\hat{\gamma}_{qg}^{(0)}\Biggr)+a_{Qg}^{(3)}\Biggr]
\end{aligned}
$$

- Renormalized expression for $A_{Qg}^{(3)}$:

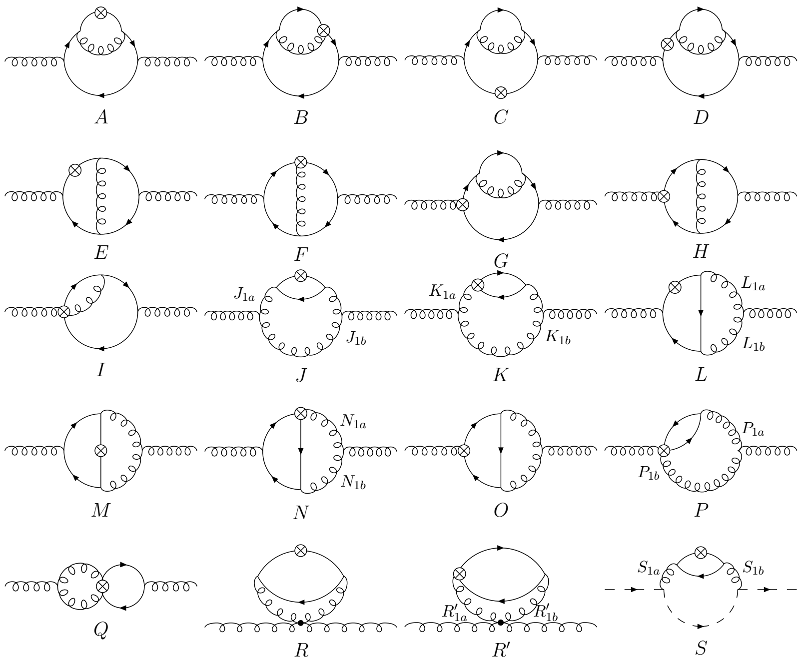
$$
\begin{aligned}
A_{Qg}^{(3),\overline{\mathsf{MS}}} \quad = \quad & \frac{\hat{\gamma}_{qg}^{(0)}}{48}\Biggl\{(n_f+1)\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}+\gamma_{gg}^{(0)}\Bigl(\gamma_{gg}^{(0)}-2\gamma_{qq}^{(0)}+6\beta_0+14\beta_{0,Q}\Bigr)+\gamma_{qq}^{(0)}\Bigl(\gamma_{qq}^{(0)} \\
& -6\beta_0-8\beta_{0,Q}\Bigr)+8\beta_0^2+28\beta_{0,Q}\beta_0+24\beta_{0,Q}^2\Biggr\}\ln^3\Bigl(\frac{m^2}{\mu^2}\Bigr)+\frac{1}{8}\Biggl\{\hat{\gamma}_{qg}^{(1)}\Bigl(\gamma_{qq}^{(0)}-\gamma_{gg}^{(0)} \\
& -4\beta_0-6\beta_{0,Q}\Bigr)+\hat{\gamma}_{qg}^{(0)}\Bigl(\hat{\gamma}_{gg}^{(1)}-\gamma_{gg}^{(1)}+(1-n_f)\hat{\gamma}_{qq}^{(1),\mathsf{PS}}+\gamma_{qq}^{(1),\mathsf{NS}}+\hat{\gamma}_{qq}^{(1),\mathsf{NS}}-2\beta_1 \\
& -2\beta_{1,Q}\Bigr)\Biggr\}\ln^2\Bigl(\frac{m^2}{\mu^2}\Bigr)+\Biggl\{\frac{\hat{\gamma}_{qg}^{(2)}}{2}-n_f\frac{\hat{\hat{\gamma}}_{qg}^{(2)}}{2}+\frac{a_{Qg}^{(2)}}{2}\Bigl(\gamma_{qq}^{(0)}-\gamma_{gg}^{(0)}-4\beta_0-4\beta_{0,Q}\Bigr) \\
& +\frac{\hat{\gamma}_{qg}^{(0)}}{2}\Bigl(a_{gg,Q}^{(2)}-n_f a_{Qq}^{(2),\mathsf{PS}}\Bigr)+\frac{\hat{\gamma}_{qg}^{(0)}\zeta_2}{16}\Bigl(-(n_f+1)\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}+\gamma_{gg}^{(0)}\Bigl[2\gamma_{qq}^{(0)}-\gamma_{gg}^{(0)}-6\beta_0 \\
& -6\beta_{0,Q}\Bigr]-4\beta_0[2\beta_0+3\beta_{0,Q}]+\gamma_{qq}^{(0)}\Bigl[-\gamma_{qq}^{(0)}+6\beta_0+4\beta_{0,Q}\Bigr]\Bigr)\Biggr\}\ln\Bigl(\frac{m^2}{\mu^2}\Bigr)+\overline{a}_{Qg}^{(2)}\Bigl(\gamma_{gg}^{(0)} \\
& -\gamma_{qq}^{(0)}+4\beta_0+4\beta_{0,Q}\Bigr)+\hat{\gamma}_{qg}^{(0)}\Bigl(n_f\overline{a}_{Qq}^{(2),\mathsf{PS}}-\overline{a}_{gg,Q}^{(2)}\Bigr)+\frac{\hat{\gamma}_{qg}^{(0)}\zeta_3}{48}\Bigl((n_f+1)\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)} \\
& +\gamma_{gg}^{(0)}\Bigl[\gamma_{gg}^{(0)}-2\gamma_{qq}^{(0)}+6\beta_0-2\beta_{0,Q}\Bigr]+\gamma_{qq}^{(0)}\Bigl[\gamma_{qq}^{(0)}-6\beta_0\Bigr]+8\beta_0^2-4\beta_0\beta_{0,Q} \\
& -24\beta_{0,Q}^2\Bigr)+\frac{\hat{\gamma}_{qg}^{(1)}\beta_{0,Q}\zeta_2}{8}+\frac{\hat{\gamma}_{qg}^{(0)}\zeta_2}{16}\Bigl(\gamma_{gg}^{(1)}-\hat{\gamma}_{qq}^{(1),\mathsf{NS}}-\gamma_{qq}^{(1),\mathsf{NS}}-\hat{\gamma}_{qq}^{(1),\mathsf{PS}}+2\beta_1 \\
& +2\beta_{1,Q}\Bigr)+\frac{\delta m_1^{(-1)}}{8}\Bigl(16a_{Qg}^{(2)}+\hat{\gamma}_{qg}^{(0)}\Bigl[-24\delta m_1^{(0)}-8\delta m_1^{(1)}-\zeta_2\beta_0-9\zeta_2\beta_{0,Q}\Bigr]\Bigr) \\
& +\frac{\delta m_1^{(0)}}{2}\Bigl(2\hat{\gamma}_{qg}^{(1)}-\delta m_1^{(0)}\hat{\gamma}_{qg}^{(0)}\Bigr)+\delta m_1^{(1)}\hat{\gamma}_{qg}^{(0)}\Bigl(\gamma_{qq}^{(0)}-\gamma_{gg}^{(0)}-2\beta_0-4\beta_{0,Q}\Bigr) \\
& +\delta m_2^{(0)}\hat{\gamma}_{qg}^{(0)}+a_{Qg}^{(3)}\,.
\end{aligned}
$$

Contributing Diagrams

- 289 diagrams contribute. The are generated using QGRAF with operator insertions [Nogueira, 1991; Bierenbaum, Blüemlein, Klein, 2009]
- Due to symmetry reasons, many of them are identical.
- Many diagrams can be generated from 2-loop diagrams by bubble insertions.
- e.g. from diagram N:

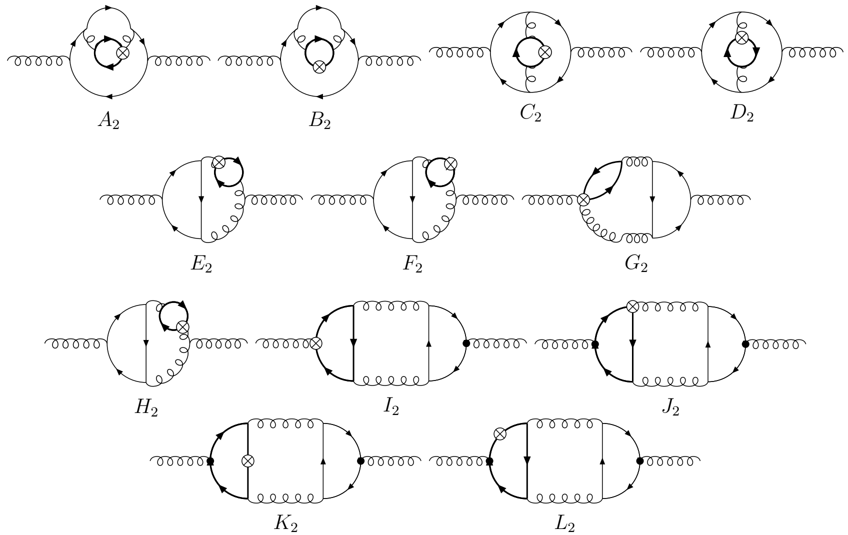


- generating 2-loop diagrams



⇒ contributing diagrams obtained by bubble insertions

- Further diagrams:



- Further diagrams contribute to the other OMEs $A_{qg,Q}$,$A_{qq,Q}^{\rm PS}$,$A_{Qq}^{\rm PS}$,$A_{qq,Q}^{\rm NS}$,$A_{qq,Q}^{\rm NS,TR}$

- Typical Feynman parameter integral after momentum integration

$$I_1 \;=\; \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_4 \int_0^1 dx_5 \; x_1^{2+\varepsilon} x_2^{1-\varepsilon/2} x_5^{1-\varepsilon} (1-x_1)^{\varepsilon/2} (1-x_5)^2 (x_4 - x_5\; x_4 + x_2\; x_5)^N \left(1 - x_5 \;\left(1 - \frac{1}{1-x_1}\right)\right)^{3/2\varepsilon}$$

- Performing the integral gives a linear combination of sums over B-functions and Hypergeometric ${}_PF_Q$

$$I_1 = \frac{\Gamma(1-\varepsilon)\;\Gamma(3+\varepsilon)}{6(N+1)} \quad \Bigg\{ \sum_{j=1}^{N+1} \binom{1+N}{j} \;(-1)^j \; B(2-\varepsilon+j,2)\; B(1+j,2-\varepsilon/2)\; {}_3F_2\left[{-3/2\;\varepsilon,2,3+\varepsilon \atop 4+j-\varepsilon,4}, 1\right]$$
$$+B(3+N-\varepsilon,2)\; B(1,3+N-\varepsilon/2)\; {}_3F_2\left[{-3/2\varepsilon,2,3+\varepsilon \atop 5-\varepsilon,4}; 1\right] \Bigg\}$$

- The generalized hypergeometric function ${}_PF_Q$ is defined by

$${}_PF_Q\left[{a_1, ..., a_P \atop b_1, ..., b_Q}; z\right] = \sum_{i=0}^{\infty} \frac{(a_1)_i ... (a_P)_i}{(b_1)_i ... (b_Q)_i} \frac{z^i}{\Gamma(i+1)} \;.$$

- Now: perform a series expansion in ε and evaluate the remaining sums
- Up to 4 (in)finite sums occur, which are computed using modern summation methods encoded in SIGMA [C. Schneider, 2007]

Mathematical Structure: Harmonic Sums

- only ζ_2, ζ_3, harmonic sums $S_{\vec{a}}(N)$ and rational terms appear

$$
\begin{aligned}
S_{a_1,\ldots,a_m}(N) &= \sum_{n_1=1}^{N} \sum_{n_2=1}^{n_1} \cdots \sum_{n_m=1}^{n_{m-1}} \frac{(\mathrm{sign}(a_1))^{n_1}}{n_1^{|a_1|}} \frac{(\mathrm{sign}(a_2))^{n_2}}{n_2^{|a_2|}} \cdots \frac{(\mathrm{sign}(a_m))^{n_m}}{n_m^{|a_m|}} \,, \\
\text{weight } w &= \sum |a_i|, \; m - \text{depth}
\end{aligned}
$$

[Blümlein, Kurth 1998; Vermaseren 1998]

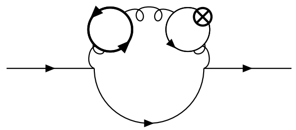
- complete set of algebraic and structural relations is known [Blümlein, 2003, 2009]
- in intermediary steps: generalized harmonic sums occur

$$
\begin{aligned}
\widetilde{S}_{m_1,\ldots}(x_1,\ldots;N) &= \sum_{i_1}^{N} \frac{x_1^{i_1}}{i_1^{m_1}} \sum_{i_2=1}^{i_1-1} \frac{x_2^{i_2}}{i_2^{m_2}} \widetilde{S}_{m_3,\ldots}(x_3,\ldots;i_2) \\
&\quad + \widetilde{S}_{m_1+m_2,m_3,\ldots}(x_1 \cdot x_2, x_3,\ldots;N) \,.
\end{aligned}
$$

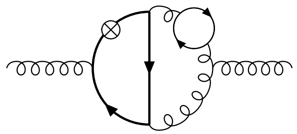
[Moch, Uwer, Weinzierl, 2002]

- can be reduced to nested harmonic sums for $x_i \in \{-1, 1\}$
- in our case: $x_i \in \{-1/2, 1/2, -2, 2\}$
- algebraic and structural relations have been worked out [Ablinger, Blümlein, Schneider, 2010]

3. The Results for the N_F-contributions



$$
\begin{aligned}
A_{qq,Q}^{(3),PS,B} \quad = \quad & C_F T_F^2 N_F \Biggl\{ \Biggl[-\frac{128}{9} \frac{(2+N)(-1+N)}{N^2(1+N)^2} \Biggr] \frac{1}{\varepsilon^3} \\
& + \Biggl[\frac{128}{9} \frac{(2+N)(-1+N)}{N^2(1+N)^2} S_1 - \frac{64}{27} \frac{16N^4+26N^3-25N^2-11N+6}{N^3(1+N)^3} \Biggr] \frac{1}{\varepsilon^2} \\
& + \Biggl[-\frac{64}{9} \frac{(2+N)(-1+N)}{N^2(1+N)^2} S_2 - \frac{64}{9} \frac{(2+N)(-1+N)}{N^2(1+N)^2} S_1^2 - 16/3 \frac{(2+N)(-1+N)}{N^2(1+N)^2} \zeta_2 \\
& + \frac{64}{27} \frac{\left(16N^4+26N^3-25N^2-11N+6\right)}{N^3(1+N)^3} S_1 - \frac{32}{81} \frac{181N^6+447N^5-32N^4-297N^3-92N^2+15N-18}{N^4(1+N)^4} \Biggr] \frac{1}{\varepsilon} \\
& + \frac{128}{27} \frac{(2+N)(-1+N)}{N^2(1+N)^2} S_3 + \frac{64}{9} \frac{(2+N)(-1+N)}{N^2(1+N)^2} S_2 S_1 + \frac{64}{27} \frac{(2+N)(-1+N)}{N^2(1+N)^2} S_1^3 + 16/3 \frac{(2+N)(-1+N)}{N^2(1+N)^2} S_1 \zeta_2 \\
& + \frac{112}{9} \frac{(2+N)(-1+N)}{N^2(1+N)^2} \zeta_3 - \frac{32}{27} \frac{\left(16N^4+26N^3-25N^2-11N+6\right)}{N^3(1+N)^3} S_2 - \frac{32}{27} \frac{\left(16N^4+26N^3-25N^2-11N+6\right)}{N^3(1+N)^3} S_1^2 \\
& - \frac{8}{9} \frac{\left(16N^4+26N^3-25N^2-11N+6\right)}{N^3(1+N)^3} \zeta_2 + \frac{32}{81} \frac{\left(181N^6+447N^5-32N^4-297N^3-92N^2+15N-18\right)}{N^4(1+N)^4} S_1 \\
& - \frac{16}{243} \frac{-3503N^5+4927N^6-5309N^4-929N^3+7210N^7+54+2074N^8+231N^2+9N}{N^5(1+N)^5} \Biggr\}
\end{aligned}
$$



$$
\begin{aligned}
A_{Qg}^{(3),l1} \quad = \quad & N_F T_F^2 C_A \Big\{ \left[\frac{128}{9\,N}\, S_1 + \frac{32}{9}\, \frac{2\,N^3+7\,N^2+6\,N+3}{N^2\,(1+N)^2} \right] \frac{1}{\varepsilon^3} \\
& + \left[-\frac{64}{27}\, \frac{(5\,N+14)}{N\,(1+N)}\, S_1 - \frac{16}{27}\, \frac{P_1(N)}{N^3\,(2+N)\,(1+N)^3} + \frac{64}{9\,N}\, S_1^2 + \frac{128}{9\,N}\, S_2 \right] \frac{1}{\varepsilon^2} \\
& + \left[\frac{128}{9\,N}\, S_1\, S_2 + \frac{416}{27\,N}\, S_3 - \frac{32}{27}\, \frac{(5\,N+14)}{N\,(1+N)}\, S_1^2 + \frac{4}{81}\, \frac{P_2(N)}{N^4\,(2+N)^2\,(1+N)^4} \right. \\
& + \frac{128}{9\,N}\, S_{21} + \frac{16}{3\,N}\, \zeta_2\, S_1 - \frac{16}{27}\, \frac{\left(74\,N^3+121\,N^2+38\,N-27\right)}{N^2\,(1+N)^2}\, S_2 + \frac{64}{27\,N}\, S_1^3 \\
& \left. + \frac{16}{81}\, \frac{\left(47\,N^3+13\,N^2-196\,N-108\right)}{N^2\,(1+N)^2}\, S_1 + \frac{4}{3}\, \frac{\left(2\,N^3+7\,N^2+6\,N+3\right)}{N^2\,(1+N)^2}\, \zeta_2 \right] \frac{1}{\varepsilon} \\
& + \frac{8}{81}\, \frac{P_3(N)}{N^3\,(2+N)\,(1+N)^3}\, S_2 - \frac{2}{9}\, \frac{P_1(N)}{N^3\,(2+N)\,(1+N)^3}\, \zeta_2 - \frac{8}{9}\, \frac{(5\,N+14)}{N\,(1+N)}\, S_1 \zeta_2 \\
& - \frac{64}{27}\, \frac{(5\,N+14)}{N\,(1+N)}\, S_2\, S_1 - \frac{8}{81}\, \frac{\left(616\,N^3+899\,N^2+202\,N-243\right)}{N^2\,(1+N)^2}\, S_3 \\
& - \frac{1}{243}\, \frac{P_4(N)}{(2+N)^3\,N^5\,(1+N)^5} - \frac{64}{27}\, \frac{(5\,N+14)\,(N)}{N\,(1+N)}\, S_{21} - \frac{112}{9\,N}\, \zeta_3\, S_1 + \frac{8}{3\,N}\, \zeta_2\, S_1^2 \\
& + \frac{128}{9\,N}\, S_1\, S_{21} + \frac{16}{3\,N}\, \zeta_2\, S_2 + \frac{64}{9\,N}\, S_1^2\, S_2 \\
& + \frac{8}{81}\, \frac{\left(47\,N^3+13\,N^2-196\,N-108\right)}{N^2\,(1+N)^2}\, S_1^2 - \frac{128}{9\,N}\, S_{2,1,1} + \frac{160}{9\,N}\, S_2 \\
& + \frac{16}{27\,N}\, S_1^4 + \frac{256}{9\,N}\, S_4 - \frac{28}{9}\, \frac{\left(2\,N^3+7\,N^2+6\,N+3\right)}{N^2\,(1+N)^2}\, \zeta_3 - \frac{32}{81}\, \frac{(5\,N+14)}{N\,(1+N)}\, S_1^3 \\
& + \frac{416}{27\,N}\, S_1\, S_3 + \frac{64}{3\,N}\, S_{3,1} - \frac{4}{243}\, \frac{\left(323\,N^5-3972\,N^4-9291\,N^3-4456\,N^2-1080\,N+648\right)}{N^3\,(1+N)^3}\, S_1 \Big\}
\end{aligned}
$$

Gluonic contributions

$$
\begin{aligned}
\hat{a}_{Qg}^{(3),0} \quad = \quad & n_f T_F^2 C_A \Biggl\{ \frac{16(N^2+N+2)}{27N(N+1)(N+2)} \Bigl[108 S_{-2,1,1} - 78 S_{2,1,1} - 90 S_{-3,1} + 72 S_{2,-2} - 6 S_{3,1} \\
& -108 S_{-2,1} S_1 + 42 S_{2,1} S_1 - 6 S_{-4} + 90 S_{-3} S_1 + 118 S_3 S_1 + 120 S_4 + 18 S_{-2} S_2 + 54 S_{-2} S_1^2 \\
& +33 S_2 S_1^2 + 15 S_2^2 + 2 S_1^4 + 18 S_{-2}\zeta_2 + 9 S_2 \zeta_2 + 9 S_1^2 \zeta_2 - 42 S_1 \zeta_3 \Bigr] \\
& +32 \frac{5N^4+14N^3+53N^2+82N+20}{27N(N+1)^2(N+2)^2} \Bigl[6 S_{-2,1} - 5 S_{-3} - 6 S_{-2} S_1 \Bigr] \\
& - \frac{64(5N^4+11N^3+50N^2+85N+20)}{27N(N+1)^2(N+2)^2} S_{2,1} - \frac{16(40N^4+151N^3+544N^2+779N+214)}{27N(N+1)^2(N+2)^2} S_2 S_1 \\
& - \frac{32(65N^6+429N^5+1155N^4+725N^3+370N^2+496N+648)}{81(N-1)N^2(N+1)^2(N+2)^2} S_3 \\
& - \frac{16(20N^4+107N^3+344N^2+439N+134)}{81N(N+1)^2(N+2)^2} S_1^3 + \frac{Q_1(N)}{81(N-1)N^3(N+1)^3(N+2)^3} S_2 \\
& + \frac{32(47N^6+278N^5+1257N^4+2552N^3+1794N^2+284N+448)}{81N(N+1)^3(N+2)^3} S_{-2} \\
& + \frac{8(22N^6+271N^5+2355N^4+6430N^3+6816N^2+3172N+1256)}{81N(N+1)^3(N+2)^3} S_1^2 \\
& + \frac{Q_2(N)}{243(N-1)N^2(N+1)^4(N+2)^4} S_1 + \frac{448(N^2+N+1)(N^2+N+2)}{9(N-1)N^2(N+1)^2(N+2)^2} \zeta_3 \\
& - \frac{16(5N^4+20N^3+59N^2+76N+20)}{9N(N+1)^2(N+2)^2} S_1 \zeta_2 - \frac{Q_3(N)}{9(N-1)N^3(N+1)^3(N+2)^3} \zeta_2 \\
& - \frac{Q_4(N)}{243(N-1)N^5(N+1)^5(N+2)^5} \Biggr\}
\end{aligned}
$$

$$
\begin{aligned}
&+n_f T_F^2 C_F \Biggl\{ \frac{16(N^2+N+2)}{27N(N+1)(N+2)} \Bigl[144 S_{2,1,1} - 72 S_{3,1} - 72 S_{2,1} S_1 + 48 S_4 - 16 S_3 S_1 \\
&-24 S_2^2 - 12 S_2 S_1^2 - 2 S_1^4 - 9 S_1^2 \zeta_2 + 42 S_1 \zeta_3\Bigr] + 32 \frac{10N^3+49N^2+83N+24}{81N^2(N+1)(N+2)} \Bigl[3 S_2 S_1 + S_1^3\Bigr] \\
&-\frac{128(N^2-3N-2)}{3N^2(N+1)(N+2)} S_{2,1} - \frac{Q_5(N)}{81(N-1)N^3(N+1)^3(N+2)^2} S_3 \\
&+\frac{Q_6(N)}{27(N-1)N^4(N+1)^4(N+2)^3} S_2 - \frac{32(10N^4+185N^3+789N^2+521N+141)}{81N^2(N+1)^2(N+2)} S_1^2 \\
&-\frac{16(230N^5-924N^4-5165N^3-7454N^2-10217N-2670)}{243N^2(N+1)^3(N+2)} S_1 \\
&+\frac{16(5N^3+11N^2+28N+12)}{9N^2(N+1)(N+2)} S_1 \zeta_2 - \frac{Q_7(N)}{9(N-1)N^3(N+1)^3(N+2)^2} \zeta_3 \\
&+\frac{Q_8(N)}{9(N-1)N^4(N+1)^4(N+2)^3} \zeta_2 + \frac{Q_9(N)}{243(N-1)N^6(N+1)^6(N+2)^5} \Biggr\}
\end{aligned}
$$

$$
\begin{aligned}
a_{qg,Q}^{(3),0} \quad = \quad & n_f T_F^2 \Biggl\{ C_F \Biggl[\frac{N^2+N+2}{N(N+1)(N+2)} \Bigl[-\frac{56}{9} S_4 + \frac{32}{27} S_3 S_1 + \frac{8}{9} S_2 S_1^2 + \frac{4}{9} S_2^2 + \frac{4}{27} S_1^4 + \frac{256}{9} S_1 \zeta_3 \Bigr] \\
& - \frac{16(10N^3+13N^2+29N+6)}{81N^2(1+N)(2+N)} \left[S_1^3 + 3S_2 S_1 \right] + \frac{32(5N^3-16N^2+N-6)}{81N^2(1+N)(2+N)} S_3 \\
& + \frac{8(109N^4+291N^3+478N^2+324N+40)}{27N^2(1+N)^2(2+N)} S_2 \\
& + \frac{8(215N^4+481N^3+930N^2+748N+120)}{81N^2(1+N)^2(2+N)} S_1^2 - \frac{R_4(N)}{243N^2(1+N)^3(2+N)} S_1 \\
& - \frac{64(N^2+N+2)R_5(N)}{9(N-1)N^3(1+N)^3(2+N)^2} \zeta_3 + \frac{R_6(N)}{243(N-1)N^6(1+N)^6(2+N)^5} \Biggr] \\
& + C_A \Biggl[\frac{N^2+N+2}{N(N+1)(N+2)} \Bigl[-\frac{56}{9} S_4 - \frac{128}{9} S_{-4} + \frac{160}{27} S_3 S_1 - \frac{4}{9} S_2^2 + \frac{8}{9} S_2 S_1^2 \\
& - \frac{4}{27} S_1^4 - \frac{64}{9} S_{2,1} S_1 - \frac{128}{9} S_{3,1} + \frac{64}{9} S_{2,1,1} - \frac{256}{9} \zeta_3 S_1 \Bigr] \\
& + \frac{32(5N^4+20N^3+41N^2+49N+20)}{81N(1+N)^2(2+N)^2} \left[S_1^3 + 12 S_{2,1} - 3 S_2 S_1 \right] \\
& + \frac{64}{81} \frac{(5N^4+38N^3+59N^2+31N+20)}{N(1+N)^2(2+N)^2} S_3 + \frac{128}{27} \frac{(5N^2+8N+10)}{N(1+N)(2+N)} S_{-3} \\
& + \frac{512}{9} \frac{(N^2+N+1)(N^2+N+2)}{(N-1)N^2(1+N)^2(2+N)^2} \zeta_3 - \frac{16R_7(N)}{81N(1+N)^3(2+N)^3} S_2 \\
& - \frac{32(121N^3+293N^2+414N+224)}{81N(1+N)^2(2+N)} S_{-2} - \frac{R_8(N)}{81N(1+N)^3(2+N)^3} S_1^2 \\
& + \frac{16R_9(N)}{243(N-1)N^2(1+N)^4(2+N)^4} S_1 + \frac{8R_{10}(N)}{243(N-1)N^5(1+N)^5(2+N)^5} \Biggr] \Biggr\}
\end{aligned}
$$

(complete OME)

$$
\begin{aligned}
\gamma_{qg}^{(2)} \quad = \quad & \frac{n_f^2 T_F^2}{(N+1)(N+2)} \Biggl\{ C_A \Biggl[(N^2+N+2) \Biggl(\frac{128}{3N} S_{2,1} + \frac{128}{3N} S_{-3} + \frac{64}{9N} S_3 + \frac{32}{9N} S_1^3 \\
& - \frac{32}{3N} S_2 S_1 \Biggr) - \frac{128(5N^2+8N+10)}{9N} S_{-2} - \frac{64(5N^4+26N^3+47N^2+43N+20)}{9N(N+1)(N+2)} S_2 \\
& - \frac{64(5N^4+20N^3+41N^2+49N+20)}{9N(N+1)(N+2)} S_1^2 + \frac{64 P_1(N)}{27N(N+1)^2(N+2)^2} S_1 \\
& + \frac{16 P_2(N)}{27(N-1)N^4(N+1)^3(N+2)^3} \Biggr] \\
& + C_F \Biggl[\frac{32}{9} \frac{N^2+N+2}{N} \left\{ 10 S_3 - S_1^3 - 3 S_1 S_2 \right\} \\
& + \frac{32(5N^2+3N+2)}{3N^2} S_2 + \frac{32(10N^3+13N^2+29N+6)}{9N^2} S_1^2 \\
& - \frac{32(47N^4+145N^3+426N^2+412N+120)}{27N^2(N+1)} S_1 + \frac{4 P_3(N)}{27(N-1)N^5(N+1)^4(N+2)^3} \Biggr] \Biggr\}
\end{aligned}
$$

agreement with [Moch, Vermaseren, Vogt 2004]

- Flavor non-singlet contributions:

Vector current

$$\hat{\gamma}_{qq}^{(2),NS} = C_F T_F^2 N_F \left\{ -\frac{256}{27} S_1 - \frac{1280}{27} S_2 + \frac{256}{9} S_3 + \frac{16}{27} \frac{(51 N^6 + 153 N^5 + 57 N^4 + 35 N^3 + 96 N^2 + 16 N - 24)}{N^3 (1+N)^3} \right\}$$

agreement with [Gracey 1993; Moch, Vermaseren, Vogt 2004]

$$a_{qq,Q}^{(3),NS} = C_F T_F^2 N_F \left\{ -\frac{55552}{729} S_1 + \frac{448}{27} \zeta_3 S_1 - \frac{160}{27} \zeta_2 S_1 + \frac{640}{27} S_2 + \frac{32}{9} \zeta_2 S_2 - \frac{320}{81} S_3 + \frac{64}{27} S_4 \right.$$
$$\left. + \frac{2}{729} \frac{P_1(N)}{N^4 (1+N)^4} - \frac{112}{27} \frac{(3 N^2 + 3 N + 2)}{N (1+N)} \zeta_3 + \frac{4}{27} \frac{(3 N^4 + 6 N^3 + 47 N^2 + 20 N - 12)}{N^2 (1+N)^2} \zeta_2 \right\}$$

Transversity

$$\hat{\gamma}_{qq}^{(2),TR} = C_F T_F^2 N_F \left\{ -\frac{256}{27} S_1 - \frac{1280}{27} S_2 + \frac{256}{9} S_3 + \frac{16}{9} \frac{(17 N^2 + 17 N - 8)}{N (1+N)} \right\}$$

agreement with [Gracey 2003]

$$a_{Qq}^{(3),TR} = C_F T_F^2 N_F \left\{ -\frac{55552}{729} S_1 + \frac{448}{27} \zeta_3 S_1 - \frac{160}{27} \zeta_2 S_1 + \frac{640}{27} S_2 + \frac{32}{9} \zeta_2 S_2 - \frac{320}{81} S_3 + \frac{64}{27} S_4 \right.$$
$$\left. + \frac{2}{243} \frac{(3917 N^4 + 7834 N^3 + 4157 N^2 - 48 N - 144)}{N^2 (1+N)^2} - \frac{112}{9} \zeta_3 + \frac{4}{9} \zeta_2 \right\}$$

- Flavor pure singlet contributions:

$$
\begin{aligned}
\hat{\gamma}_{qq}^{(2),PS} &= C_F T_F^2 N_F \Biggl\{ -\frac{64}{3} \frac{\left(N^2+N+2\right)^2}{(-1+N)\,N^2\,(1+N)^2\,(2+N)} \left(S_1^2+S_2\right) + \frac{128}{9} \frac{\left(68\,N^5+37\,N^6+8\,N^7-11\,N^4-86\,N^3-56\,N^2-104\,N-48\right)\,S_1}{N^3\,(1+N)^3\,(2+N)^2\,(-1+N)} \\
&\quad -\frac{128}{27} \frac{P_1(N)}{(-1+N)\,N^4\,(1+N)^4\,(2+N)^3} \Biggr\} \\
P_1(N) &= 1353\,N^7+1200\,N^8-317\,N^6-1689\,N^5-2103\,N^4-2672\,N^3+144-48\,N-1496\,N^2+392\,N^9+52\,N^{10}
\end{aligned}
$$

agreement with [Moch, Vermaseren, Vogt 2004; (Blümlein, Kauers, Klein, Schneider, 2009)]

$$
\begin{aligned}
a_{qq,Q}^{(3),PS} &= C_F T_F^2 N_F \Biggl\{ \frac{128}{27} \frac{\left(N^2+N+2\right)^2}{(-1+N)\,N^2\,(1+N)^2\,(2+N)} S_1^3 - \frac{64}{27} \frac{\left(266\,N^4+181\,N^5+269\,N^3+230\,N^2+74\,N^6+16\,N^7+44\,N-24\right)}{N^3\,(-1+N)\,(2+N)^2\,(1+N)^3} S_1^2 \\
&\quad +\frac{128}{9} \frac{\left(N^2+N+2\right)^2}{(-1+N)\,N^2\,(1+N)^2\,(2+N)} S_1 S_2 + \frac{64}{81} \frac{P_3(N)}{(-1+N)\,N^4\,(1+N)^4\,(2+N)^3} + \frac{32}{3} \frac{\left(N^2+N+2\right)^2}{(-1+N)\,N^2\,(1+N)^2\,(2+N)} \zeta_2 S_1 \\
&\quad -\frac{64}{27} \frac{\left(266\,N^4+181\,N^5+269\,N^3+230\,N^2+74\,N^6+16\,N^7+44\,N-24\right)}{N^3\,(-1+N)\,(2+N)^2\,(1+N)^3} S_2 + \frac{256}{27} \frac{\left(N^2+N+2\right)^2}{(-1+N)\,N^2\,(1+N)^2\,(2+N)} S_3 \\
&\quad -\frac{32}{243} \frac{P_4(N)}{N^5\,(-1+N)\,(2+N)^4\,(1+N)^5} - \frac{16}{9} \frac{P_5(N)}{N^3\,(-1+N)\,(2+N)^2\,(1+N)^3} \zeta_2 + \frac{224}{9} \frac{\left(N^2+N+2\right)^2}{(-1+N)\,N^2\,(1+N)^2\,(2+N)} \zeta_3 \Biggr\}
\end{aligned}
$$

(complete OME)

$$
\begin{aligned}
a_{Qq}^{(3),PS} &= C_F T_F^2 N_F \Biggl\{ -\frac{16}{27} \frac{\left(N^2+N+2\right)^2}{(-1+N)\,N^2\,(1+N)^2\,(2+N)} S_1^3 + \frac{16}{27} \frac{\left(68\,N^5+37\,N^6+8\,N^7-11\,N^4-86\,N^3-56\,N^2-104\,N-48\right)}{N^3\,(1+N)^3\,(2+N)^2\,(-1+N)} S_1^2 \\
&\quad -\frac{208}{9} \frac{\left(N^2+N+2\right)^2}{(-1+N)\,N^2\,(1+N)^2\,(2+N)} S_1 S_2 - \frac{32}{81} \frac{P_6(N)}{(-1+N)\,N^4\,(1+N)^4\,(2+N)^3} S_1 - \frac{16}{3} \frac{\left(N^2+N+2\right)^2}{(-1+N)\,N^2\,(1+N)^2\,(2+N)} \zeta_2 S_1 \\
&\quad +\frac{208}{27} \frac{68\,N^5+37\,N^6+8\,N^7-11\,N^4-86\,N^3-56\,N^2-104\,N-48}{N^3\,(1+N)^3\,(2+N)^2\,(-1+N)} S_2 - \frac{1760}{27} \frac{\left(N^2+N+2\right)^2}{(-1+N)\,N^2\,(1+N)^2\,(2+N)} S_3 \\
&\quad +\frac{32}{243} \frac{P_7(N)}{N^5\,(1+N)^5\,(2+N)^4\,(-1+N)} + \frac{224}{9} \frac{\left(N^2+N+2\right)^2}{(-1+N)\,N^2\,(1+N)^2\,(2+N)} \zeta_3 \\
&\quad +\frac{16}{9} \frac{\left(68\,N^5+37\,N^6+8\,N^7-11\,N^4-86\,N^3-56\,N^2-104\,N-48\right)}{N^3\,(1+N)^3\,(2+N)^2\,(-1+N)} \zeta_2 \Biggr\}
\end{aligned}
$$

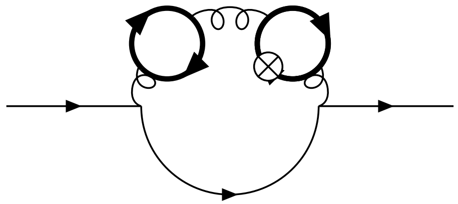
Mellin moments for the PS-, NS- and Transversity-contributions to the anomalous dimensions and constant terms a_{ij} (prefactor $T_F^2 N_F$ taken out)

agreement with [Bierenbaum, Blümlein, Klein, 2009; Blümlein, Klein, Tödtli, 2009]

N	2	8
$\hat{\gamma}_{qg}^{(2)}$	$\frac{16928}{243}C_A - \frac{2768}{243}C_F$	$-\frac{20758849082}{2755620000}C_A + \frac{15806595692962}{1620304560000}C_F$
$a_{Qg}^{(3)}$	$\left(\frac{-6706}{2187} - \frac{616}{81}\zeta_3 - \frac{250}{81}\zeta_2\right)C_A$ $+\left(\frac{158}{243} + \frac{896}{81}\zeta_3 + \frac{40}{9}\zeta_2\right)C_F$	$\left(\frac{24718362393463}{1322697600000} - \frac{125356}{18225}\zeta_3 + \frac{2118187}{2916000}\zeta_2\right)C_A$ $+\left(-\frac{291376419801571603}{32665339929600000} + \frac{887741}{174960}\zeta_3 - \frac{139731073}{1143072000}\zeta_2\right)C_F$
$a_{qg,Q}^{(3)}$	$\left(\frac{83204}{2187} - \frac{616}{81}\zeta_3 + \frac{290}{81}\zeta_2\right)C_A$ $+\left(-\frac{5000}{243} + \frac{896}{81}\zeta_3 - \frac{4}{3}\zeta_2\right)C_F$	$\left(\frac{157327027056457}{3968092800000} - \frac{125356}{18225}\zeta_3 + \frac{7917377}{2268000}\zeta_2\right)C_A$ $+\left(-\frac{201046808090490443}{10888446643200000} + \frac{887741}{174960}\zeta_3 - \frac{3712611349}{3429216000}\zeta_2\right)C_F$
$\hat{\gamma}_{qq}^{(2),PS}$	$-\frac{10048}{243}C_F$	$-\frac{13131081443}{6751269000}C_F$
$a_{Qq}^{(3),PS}$	$\left(-\frac{76408}{2187} - \frac{112}{81}\zeta_2 + \frac{896}{81}\zeta_3\right)C_F$	$\left(-\frac{16194572439593}{15122842560000} - \frac{343781}{14288400}\zeta_2 + \frac{1369}{3645}\zeta_3\right)C_F$
$a_{qq,Q}^{(3),PS}$	$\left(-\frac{100096}{2187} - \frac{256}{81}\zeta_2 + \frac{896}{81}\zeta_3\right)C_F$	$\left(-\frac{20110404913057}{27221116608000} + \frac{135077}{4762800}\zeta_2 + \frac{1369}{3645}\zeta_3\right)C_F$
$\gamma_{qq}^{(2),NS}$	$-\frac{3584}{243}C_F$	$-\frac{38920977797}{1125211500}C_F$
$a_{qq,Q}^{(3),NS}$	$\left(-\frac{100096}{2187} - \frac{256}{81}\zeta_2 + \frac{896}{81}\zeta_3\right)C_F$	$\left(-\frac{4763338626853463}{340263957600000} - \frac{36241943}{3572100}\zeta_2 + \frac{39532}{1215}\zeta_3\right)C_F$
$\gamma_{qq}^{(2),TR}$	$-\frac{368}{27}C_F$	$-\frac{711801943}{20837250}C_F$
$a_{qq,Q}^{(3),TR}$	$\left(-\frac{4390}{81} - 4\zeta_2 + \frac{112}{9}\zeta_3\right)C_F$	$\left(-\frac{29573247248999}{210039480000} - \frac{2030251}{198450}\zeta_2 + \frac{4408}{135}\zeta_3\right)C_F$

4. The T_F^2 contributions

Two massive lines with $m_1^2 = m_2^2$



- Problem: Feynman-Parameter integrals can not be mapped directly onto hypergeometric functions:

$$I = \int_0^1 dx \int_0^1 dy \int_0^1 dz\; x^{\alpha_1}(1-x)^{\beta_1} y^{\alpha_2}(1-y)^{\beta_2} z^{\alpha_3}(1-z)^{\beta_3} \left(\frac{z}{x(1-x)} + \frac{1-z}{y(1-y)}\right)^{\gamma}$$

- $\rightarrow$ use Mellin-Barnes representation at the momentum level
- we obtain integrals like

$$I_1 = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} ds \frac{\Gamma(-s)\,\Gamma\left(s-\frac{\varepsilon}{2}\right)\Gamma\left(s-\frac{3}{2}\varepsilon\right)\Gamma(\varepsilon-1-s)\,\Gamma^2\left(2-s+\frac{\varepsilon}{2}\right)\Gamma(3+s-\varepsilon)\,\Gamma(s+N-\varepsilon)}{\Gamma(4-2s+\varepsilon)\,\Gamma(3+2s+N-2\varepsilon)}$$

- these contour integrals can me mapped on a linear combination of hypergeometric ${}_pF_q$'s containing also half-integer values, e.g.:

$$I_2 = {}_5F_4\left[\begin{matrix} -\frac{1}{2}\varepsilon, -\frac{3}{2}\varepsilon, 3-\varepsilon, N-\varepsilon, -\frac{3}{2}-\frac{1}{2}\varepsilon \\ 2+\frac{N}{2}-\varepsilon, \frac{3}{2}+\frac{N}{2}-\varepsilon, 2-\varepsilon, -1-\frac{1}{2}\varepsilon \end{matrix}; 1\right]$$

- $\rightarrow$ expand in ε
- perform infinite sums, new classes of sums contribute, e.g.:

$$\sum_{n=2}^{\infty}\frac{1}{n}S_1(N+2n),\ \sum_{n=2}^{\infty}\frac{1}{2n-3}S_1(N+2n),\ \sum_{n=2}^{\infty}\frac{1}{n}\frac{\Gamma(2n)\Gamma(n+N)}{\Gamma(n)\Gamma(2n+N)}S_1(n)$$

- $\rightarrow$ potential for new mathematical structures in the results

First Results

- Flavor non-singlet contributions:

Vector current

$$\hat{\gamma}_{qq}^{(2),NS} = C_F T_F^2 \left(\frac{128 S_3}{9} - \frac{640 S_2}{27} - \frac{128 S_1}{27} + \frac{8\left(51N^6+153N^5+57N^4+35N^3+96N^2+16N-24\right)}{27N^3(N+1)^3} \right)$$

agreement with [Gracey 1993; Moch, Vermaseren, Vogt 2004]

$$\begin{aligned}\hat{a}_{qq,Q}^{(3),NS} = T_F^2 C_F \Bigg\{ & \frac{128}{27} S_4 - \frac{1024}{27}\zeta_3 S_1 + \frac{64}{9}\zeta_2 S_2 + \frac{256\left(3N^2+3N+2\right)}{27N(N+1)}\zeta_3 - \frac{320}{27}\zeta_2 S_1 - \frac{640}{81} S_3 \\ & + \frac{8\left(3N^4+6N^3+47N^2+20N-12\right)}{27N^2(N+1)^2}\zeta_2 + \frac{1856}{81} S_2 - \frac{19424}{729} S_1 \\ & - \frac{4\left(417N^8+1668N^7-4822N^6-12384N^5-6507N^4+740N^3+216N^2+144N+432\right)}{729N^4(N+1)^4} \Bigg\}\end{aligned}$$

Transversity

$$\hat{\gamma}_{qq}^{(2),TR} = C_F T_F^2 \left\{ \frac{128 S_3}{9} - \frac{640 S_2}{27} - \frac{128 S_1}{27} + \frac{8\left(17N^2+17N-8\right)}{9N(N+1)} \right\}$$

agreement with [Gracey 2003]

$$\begin{aligned}a_{qq,Q}^{(3),TR} = C_F T_F^2 \Bigg\{ & \frac{128}{27} S_4 - \frac{1024}{27}\zeta_3 S_1 + \frac{64}{9} S_2 \zeta_2 + \frac{256}{9}\zeta_3 - \frac{320}{27} S_1 \zeta_2 - \frac{640}{81} S_3 \\ & + \frac{8}{9}\zeta_2 + \frac{1856}{81} S_2 - \frac{19424}{729} S_1 - \frac{4\left(139N^4+278N^3-101N^2+48N+144\right)}{243N^2(N+1)^2} \Bigg\}\end{aligned}$$

- Flavor pure singlet contributions:

$$\hat{a}_{Qq}^{(3),PS} = \frac{T_F^2 C_F}{(N-1)N^2(N+1)^2(2+N)}\Bigg\{\left(N^2+N+2\right)^2\left(\frac{32}{27}S_1^3-\frac{512}{27}S_3+\frac{128}{3}S_{2,1}-\frac{1024}{9}\zeta_3-\frac{160}{9}S_2S_1\right.$$
$$\left.+\frac{32}{3}\zeta_2S_1\right)-\frac{32P_1(N)}{9N(N+2)}\zeta_2+\frac{32P_2(N)}{27N(N+2)(N+3)(N+4)(N+5)}S_2-\frac{32P_3(N)}{27N(N+1)(N+2)(N+3)(N+4)(N+5)}S_1^2$$
$$+\frac{64P_4(N)}{81N^2(N+1)^2(N+2)^2(N+3)(N+4)(N+5)}S_1-\frac{64P_5(N)}{243N^3(N+1)^2(N+2)^3(N+3)(N+4)(N+5)}\Bigg\}.$$

$$\hat{\gamma}_{qq}^{(3),PS} = \frac{T_F^2 C_F}{(N-1)N^2(N+1)^2(2+N)}\Bigg\{-\frac{32}{3}\left(N^2+N+2\right)^2\left(S_1^2+S_2\right)$$
$$+\frac{64P_6(N)}{9N(N+1)(N+2)}S_1-\frac{64P_7(N)}{27N^2(N+1)^2(N+2)^2}\Bigg\},$$

with

$$P_6(N) = 8N^7+37N^6+68N^5-11N^4-86N^3-56N^2-104N-48\,,$$
$$P_7(N) = 52N^{10}+392N^9+1200N^8+1353N^7-317N^6-1689N^5-2103N^4$$
$$-2672N^3-1496N^2-48N+144\,.$$

agreement with [Moch, Vermaseren, Vogt 2004]

Two massive lines with $m_1^2 \neq m_2^2$

$$
\begin{aligned}
a_{qq,Q}^{(3),\mathrm{NS}} = \; & C_F T_F^2 \Biggl\{ \\
& -\frac{39N^4+78N^3+9N^2+2N+16}{18N^2(N+1)^2}\log^2(x) + \frac{3N^2+3N+2}{72N(N+1)}\Biggl[-\frac{32}{3}\log^3(x) + \Bigl\{-10x+64\log(1-x)-10\frac{1}{x}+\Bigl\{5x^{3/2}+27\sqrt{x}+27\frac{1}{\sqrt{x}}+5x^{-3/2}\Bigr\} \\
& \times\log\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)\Bigr\}\log^2(x) - 40x\log(x) + 40\frac{1}{x}\log(x) + \Bigl\{-20x^{3/2}-\frac{8}{3}-20x^{-3/2}\left(27\sqrt{x}+64\right)-108\frac{1}{\sqrt{x}}\Bigr\}\mathrm{Li}_2\left(\sqrt{x}\right)\log(x) - 80 + \Bigl\{40x^{3/2} \\
& +\frac{8}{3}\left(27\sqrt{x}+64\right)+216\frac{1}{\sqrt{x}}+40x^{-3/2}\Bigr\}\mathrm{Li}_3\left(\sqrt{x}\right)\Biggr] + S_1\Biggl[\frac{16}{27}\log^3(x) + \Bigl\{\frac{2}{9}+\frac{5}{9}x-\frac{32}{9}\log(1-x)+\frac{5}{9}\frac{1}{x}+\Bigl(-\frac{5}{18}x^{3/2}-\frac{3}{2}\sqrt{x}-\frac{3}{2}\frac{1}{\sqrt{x}} \\
& -\frac{5}{18}x^{-3/2}\Bigr)\log\left(\frac{\sqrt{x}+1}{1-\sqrt{x}}\right)\Bigr\}\log^2(x) + \Bigl\{\frac{20}{9}x-\frac{20}{9}\frac{1}{x}+\Bigl(\frac{10}{9}x^{3/2}+\frac{2}{9}\left(27\sqrt{x}+64\right)+6\frac{1}{\sqrt{x}}+\frac{10}{9}x^{-3/2}\Bigr)\mathrm{Li}_2\left(\sqrt{x}\right)\Bigr\}\log(x) - \frac{25904}{729} \\
& +\frac{40}{9}x + \left(-\frac{20}{9}x^{3/2}-\frac{4}{9}\left(27\sqrt{x}+64\right)-\frac{12}{\sqrt{x}}-\frac{20}{9x^{3/2}}\right)\mathrm{Li}_3\left(\sqrt{x}\right)+\frac{40}{9x}\Biggr] + S_2\Bigl\{\frac{16}{9}\log^2(x)+\frac{1856}{81}\Bigr\} - \frac{10\left(3N^2+3N+2\right)}{9N(N+1)x} + \frac{8P_1(N)}{729N^4(N+1)^4} \\
& -\frac{640}{81}S_3+\frac{128}{27}S_4+\zeta_3\left[\frac{128}{27}S_1-\frac{32\left(3N^2+3N+2\right)}{27N(N+1)}\right]+\zeta_2\left[\frac{8\left(3N^4+6N^3+47N^2+20N-12\right)}{27N^2(N+1)^2}-\frac{320}{27}S_1+\frac{64}{9}S_2\right]\Biggr\}\,,
\end{aligned}
$$

with $x = (m_c/m_b)^2$

5. Conclusion

- We computed the $O(\alpha_s^3 N_F T_F^2)$ contributions to all the OMEs A_{ij} which contribute to the nucleonic structure function $F_2(x, Q^2)$ and transversity for general values of the Mellin variable N.
- These calculations constitute first complete expressions for one color factor to the heavy flavor Wilson Coefficients for $F_2(x, Q^2)$ at $O(a_s^3)$. The Wilson Coefficients $L_{qq,Q}^{\rm PS}$ and $L_{qg,Q}^{\rm S}$ are now completly known.
- Along with the computation of the massive OMEs we obtained the corresponding parts of the 3-loop anomalous dimensions and confirmed analytically results given in the literature, partly for the first time.
- The method used provides most compact results underlining the strength of the approach relying on the use of generalized hypergeometric and related functions combined with modern summation methods.
- First results have been obtained for the $O(\alpha_s^3 T_F^2)$ terms resulting from the graphs with two massive lines with equal and non-equal masses.