Two-Loop Fermionic Integrals in Perturbation Theory on a Lattice

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Where lattice perturbative calculations are either useful or necessary?

- To determine the Λ parameter of QCD in the lattice regularization and its relation to the respective value Λ_{QCD} in the continuum theory.

$$\Lambda = \lim_{a\to 0} a^{-1}\cdot (b_0 g^2(a))^{-\frac{b_1}{2b_0^2}} \cdot e^{-\frac{1}{2b_0 g^2(a)}} \exp\left\{ -\int_0^{g(a)} dt \left(\frac{1}{\beta(t)} + \frac{1}{b_0 t^3} - \frac{b_1}{b_0^2 t} \right) \right\}, \quad (1)$$

where b_i are the coefficients of the expansion

$$\beta(g) = -g^3 \left[b_0 + b_1 g^2 + b_2 g^4 + \ldots \right] \qquad (g \to 0). \quad (2)$$

and $g(a)$ is the solution of the RG equation $a\dfrac{dg}{da} = \beta(g)$.

The relation between the bare lattice and the renormalized coupling (defined as the three-point function at a certain momentum p) is

$$g_R(p) = g(a)\Big[1 + g^2(a)\Big(- b_0 \log ap + C^L + O(a^2p^2 \log ap)\Big) + O(g^4)\Big],$$

whereas for the continuum coupling ($\bar{MS}$ scheme) one has

$$g_R(p) = g_{ms}\Big[1 + g^2_{ms}\Big(- b_0 \log \frac{p}{\mu} + C^{ms}\Big) + O(g^4_{ms})\Big].$$

Combining these two equations, we arrive at

$$g(a) = g_{ms}\Big[1 + g^2(a)\Big(C^{ms} - C^L + b_0 \log a\mu\Big) + O(g^4) + O(a^2)\Big].$$

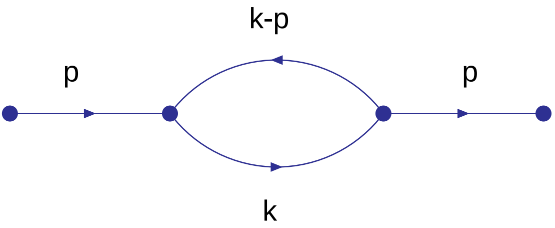
Substituting this relation into the definition of Λ, we arrive at the relation between Λ_{QCD} and Λ_{LAT}. For the pure gauge Wilson action (covariant gauge) one obtains

$$\frac{\Lambda_{ms}}{\Lambda_{lat}} = 28.80934(1). \tag{3}$$

- A determination of the renormalization factors of matrix elements of operators and of the renormalization of the bare parameters of the Lagrangian, like couplings and masses. Perturbation theory is needed to establish the connection of the matrix elements simulated on a lattice with their values in the physical continuum theory. Every lattice action defines a specific regularization scheme, and thus one needs a complete set of renormalization computations in order for the results obtained in Monte Carlo simulations be understood properly.

- Studies of the anomalies on the lattice. Perturbation theory is also important for defining chiral gauge theories on the lattice at all orders in the gauge coupling. Thus the lattice is the only regularization that can preserve both chiral and gauge invariance (without destroying basic features like locality and unitarity).
- study of the general approach to the continuum limit, including the recovery of the continuum symmetries broken by the lattice regularization (like Lorentz or chiral symmetry) in the limit $a \longrightarrow 0$, and the scaling violations, i.e., the corrections to the continuum limit which are of order a^n.

- Perturbative calculations provide the only possibility for an analytical control over the continuum limit. Lattice perturbation theory is tightly connected to the continuum limit of lattice QCD. Because of asymptotic freedom, one has $g_0 \to 0$ when $\mu \to \infty$, which means $a \to 0$. Perturbative calculations play an important role in proving the renormalizability of lattice gauge theories.



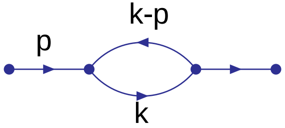
$$= \frac{G_0^2(p)}{(2\pi)^4} \int_{BZ} dk$$

$$\frac{1}{\left(m^2 + \frac{2}{a^2}\sum_{\mu=1}^{4}(1-\cos(k_\mu a))\right)\left(m^2 + \frac{2}{a^2}\sum_{\mu=1}^{4}(1-\cos((p-k)_\mu a))\right)}$$

This integral is extremely complicated and cannot be calculated analytically at finite values of a. In the limit $a \to 0$ it can be evaluated using the

Kawai–Nakayama–Seo method.

The above integrand can be represented in the form



$$= I(k,p,m^2;a) =$$

$$= I(k,0,0;a) + (I(k,p,m^2;a) - I(k,0,0;a)), \quad (4)$$

UV dirergent — UV-finite

IR divergent — IR-divergent

- $I(k,p,m^2;a) - I(k,0,0;a)$ has a smooth continuum limit ($pa \to 0$ and $ma \to 0$).
  - dimensional regularization
  - fictitious mass regularization
  - something similar

- IR divergencies in $I(k,0,0;a)$ and $(I(k,p,m^2;a) - I(k,0,0;a))$ cancel each other
- IR regularization by a fictitious mass is introduced:

$$k^2 \longrightarrow k^2 + \mu_R^2$$

$$I(k,p,m^2;a) = \lim_{\mu_R^2 \to 0} I(k,p,m^2;a,\mu_R^2).$$

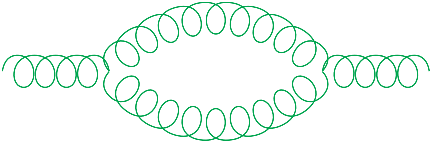
Thus the sought-for integral for the vacuum polarization, is represented as the sum of the integral over the Euclidean momentum space

$$\Pi_C(p) = \lim_{\mu_R^2 \to 0} \int dk \left(\frac{1}{(k^2 + m^2 + \mu_R^2)((k-p)^2 + m^2 + \mu_R^2)} - \frac{1}{(k^2 + \mu_R^2)^2} \right)$$

and the "zero-momentum" integral over the Brillouin zone

$$\Pi_{latt}(p) = \lim_{\mu_R^2 \to 0} \int_{BZ} \frac{dk}{\left(\mu_R^2 + \frac{2}{a^2} \sum_{\mu=1}^{4} (1 - \cos(p_\mu a)) \right)^2}.$$

Gauge Theories



An expression for a typical Feynman diagram involves a plethora of $\hat{k}_\mu$ **to even powers**

$$\hat{k}_\mu = \frac{2}{a}\ \sin\left(\frac{k_\mu a}{2}\right)$$

(odd powers do not give contribution because the domain of integration is symmetric in k_μ, whereas the integrand is antisymmetric).

Since $\hat{k}_\mu^2 = \frac{2}{a^2}(1 - \cos(k_\mu a))$,

a typical Feynman integral has the form

$$F(q, n_1, n_2, n_3, n_4) =$$
$$= \int_{BZ} dk \, \frac{\cos(k_1)^{n_1} \cos(k_2)^{n_2} \cos(k_3)^{n_3} \cos(k_4)^{n_4}}{\Delta_B^q}$$

where

$$\Delta_B = 4 + \mu_B^2 - \cos(k_1) - \cos(k_2) - \cos(k_3) - \cos(k_4),$$

μ_B is the IR regularization mass, δ is an infinitesimal parameter for intermediate regularization.
$F(q; n_1, n_2, n_3, n_4)$ is symmetric in n_1, n_2, n_3, and n_4, $\Longrightarrow$
we consider only the case when $n_1 \geq n_2 \geq n_3 \geq n_4$.

The Burgio-Caracciolo-Pelissetto method

If we introduce an auxiliary regularization parameter δ

$$F(q, n_1, n_2, n_3, n_4) =$$

$$= \lim_{\delta \to 0} \int_{BZ} dk \, \frac{\cos(k_1)^{n_1} \cos(k_2)^{n_2} \cos(k_3)^{n_3} \cos(k_4)^{n_4}}{\Delta_B^{(q+\delta)}}$$

we can employ the method of integration over the Brillouin zone by parts and thus derive the recursion relations **(I)**.

These recursion relations have the form

$$\text{if}\ \ n_\mu \geq 2 \quad \text{then} \quad F(q; ..., n_\mu, ...) =$$

$$= \ F(q; ..., n_\mu - 2, ...) \ -$$

$$-\frac{(n_\mu - 1)F(q-1; ..., n_\mu - 1, ...)}{q - 1 + \delta} \ +$$

$$+\frac{(n_4 - 2)F(q-1; ..., n_\mu - 3, ...)}{q - 1 + \delta},$$

if $n_4 \geq 2$ then $F(q, n_1, n_2, n_3, n_4) = F(q, n_1, n_2, n_3, n_4 - 2)$ −

$$- \frac{(n_4 - 1)F(q-1, n_1, n_2, n_3, n_4 - 1) - (n_4 - 2)F(q-1, n_1, n_2, n_3, n_4 - 3)}{q - 1 + \delta},$$

else if $n_3 \geq 2$ then $F(q, n_1, n_2, n_3, 0) = F(q, n_1, n_2, n_3 - 2, 0)$ −

$$- \frac{(n_3 - 1)F(q-1, n_1, n_2, n_3 - 1, 0) - (n_3 - 2)F(q-1, n_1, n_2, n_3 - 3, 0)}{q - 1 + \delta},$$

else if $n_2 \geq 2$ then $F(q, n_1, n_2, 0, 0) = F(q, n_1, n_2 - 2, 0, 0)$ −

$$- \frac{(n_2 - 1)F(q-1, n_1, n_2 - 1, 0, 0) - (n_2 - 2)F(q-1, n_1, n_2 - 3, 0, 0)}{q - 1 + \delta},$$

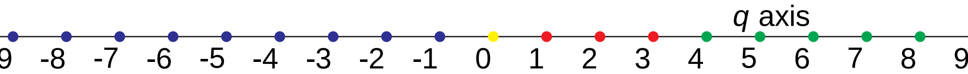
else if $n_1 \geq 2$ then $F(q, n_1, 0, 0, 0) = F(q, n_1 - 2, 0, 0, 0)$ −

$$- \frac{(n_1 - 1)F(q-1, n_1 - 1, 0, 0, 0) - (n_1 - 2)F(q-1, n_1 - 3, 0, 0, 0)}{q - 1 + \delta}.$$

$$
\begin{aligned}
F(q,n_1,n_2,n_3,1) &= (4+\mu_B^2)F(q,n_1,n_2,n_3,0) - F(q-1,n_1,n_2,n_3,0)\\
&- F(q,n_1+1,n_2,n_3,0) - F(q,n_1,n_2+1,n_3,0)\\
&- F(q,n_1,n_2,n_3+1,0),\\
F(q,n_1,n_2,1,0) &= \frac{1}{2}\Big((4+\mu_B^2)F(q,n_1,n_2,0,0) - F(q-1,n_1,n_2,0,0)\\
&- F(q,n_1+1,n_2,0,0) - F(q,n_1,n_2+1,0,0)\Big),\\
F(q,n_1,1,0,0) &= \frac{1}{3}\Big((4+\mu_B^2)F(q,n_1,0,0,0) - F(q-1,n_1,0,0,0)\\
&- F(q,n_1+1,0,0,0)\Big),\\
F(q,1,0,0,0) &= \frac{1}{4}\Big((4+\mu_B^2)F(q,0,0,0,0) - F(q-1,0,0,0,0)\Big).
\end{aligned}
$$

Thus we obtain an expression for each integral $F(q, n_1, n_2, n_3, n_4)$ in terms of the functions

$$G_\delta(q, \mu_B^2) = \int \frac{dk}{(2\pi)^4} \frac{1}{(\Delta_B)^{q+\delta}},$$



- $q \geq 2$ – divergent parts:
  $G_\delta(q, \mu_B^2) = D(q, \mu_B^2) + J(q) + O(\mu_B^2) + O(\delta)$
- $q \leq 0$ – $O(\delta)$ terms: $G_\delta(q, \mu_B^2) = B(q) + \delta J(q) + O(\mu_B^2) + O(\delta^2)$

Isolating the Divergent Part (Fictitious Mass Regularization)

$$
\begin{aligned}
G_\delta(q,\mu_B^2) &= \frac{1}{\Gamma(q+\delta)}\int_0^\infty t^{q-1+\delta}\,dt\;\left[e^{-4t-\mu_B^2 t} I_0^4(t)\right] \\
&= \frac{1}{\Gamma(q+\delta)}\left\{\int_0^1 t^{q-1+\delta}\,dt\;\left[e^{-4t-\mu_B^2 t} I_0^4(t)\right] + \right. \\
&\quad + \int_1^\infty t^{q-1+\delta}\,dt\; e^{-\mu_B^2 t}\left[e^{-4t} I_0^4(t) - \frac{1}{(2\pi t)^2}\sum_{n=0}^{q-2}\frac{b_n}{t^n}\right] \\
&\quad \left. + \int_1^\infty t^{q-1+\delta}\,dt\;\frac{1}{(2\pi t)^2}\, e^{-\mu_B^2 t}\sum_{n=0}^{q-2}\frac{b_n}{t^n}\right\}
\end{aligned}
$$

$$F(q;1,1,1,1) =$$
$$= \int \frac{d^4k}{(2\pi)^4} \frac{\cos(k_1)\cos(k_2)\cos(k_3)\cos(k_4)\{\ ZERO\}}{\Delta_B^{q+\delta}}$$

where

$$ZERO = -\ \Delta_B + 4 + \mu_B^2 - \cos(k_1) - \cos(k_2) - \cos(k_3) - \cos(k_4)$$

Thus we arrive at

$$0 = (4+\mu_B^2) * F(q;1,1,1,1) - 4F(q;2,1,1,1) - F(q-1;1,1,1,1);$$

and the relations are obtained when the above F functions are expressed in terms of $G(q)$.
(Burgio-Caracciolo-Pelissetto, Nucl.Phys.B, 1995)

At $q \geq 4$ we can derive (with FORM and REDUCE packages)

$$J(q) = \frac{1}{384(q-1)(q-2)^2(q-3)} \Big\{ 16(q-2)(q-3)\left[12+25(q-2)(q-3)\right] J(q-1) + 4(q-3)^2\left[-17-35(q-3)^2\right] J(q-2) + 4\left[1+5(q-3)^3(q-4)-5(q-3)(q-4)^2\right] J(q-3) - (q-4)^4 J(q-4)\Big\} + \frac{1}{(q-2)}\left\{D(q) - \frac{25}{24(q-1)}(2q-5)\,D(q-1)\right\} + \frac{1}{96(q-1)(q-2)^2}\Big\{\left[17+105(q-3)^2\right] D(q-2) + \frac{5}{(q-3)}\left[-1-4(q-3)^2(q-4)+2(q-4)^2\right] D(q-3) + \frac{5}{4(q-3)}(q-4)^3\, D(q-4)\Big\};$$

Thus integrals $G(q)$ can be expressed in terms of the quantities

$$\begin{aligned} Z_0 &= 0.154933390231060214084837208 \\ Z_1 &= 0.107781313539874001343391550 \\ F_0 &= 4.369225233874758 \end{aligned} \tag{5}$$

determined from the relations

$$\begin{aligned} F(1,0,0,0,0) &= 2Z_0 + O(\mu_B^2) \\ F(2,0,0,0,0) &= -\frac{l_C}{(2\pi)^2} + \frac{\bar{F}_0}{(2\pi)^2} + O(\mu_B^2) \\ F(3,0,0,0,0) &= \frac{1}{(2\pi)^2}\left(\frac{1}{2\,\mu_B^2} - \frac{l_C}{4} - \frac{13}{48} \right. \\ &\left. + \frac{\bar{F}_0}{4}\right) - \frac{1}{128} + \frac{Z_1}{32} + O(\mu_B^2), \end{aligned} \tag{6}$$

where $\bar{F}_0 = F_0 - \ln 2$, $l_C = \ln \mu_B^2 + C$, $C = 0.577...$ is the Euler constant.

The Wilson-fermion propagator is given by

$$\frac{m + \Delta + i \sum_{\mu=1}^{4} \gamma^{\mu} s_{\mu}(p)}{(m+\Delta)^2 + s^2(p)} \tag{7}$$

where

$$\begin{aligned} \Delta &= \frac{r}{a} \sum_{\mu=1}^{4} (1 - \cos(p_{\mu} a)), \\ s_{\mu}(p) &= \frac{1}{a} \sin(p_{\mu} a). \end{aligned} \tag{8}$$

r is the Wilson parameter.

Fermion Integrals

$$F(p,q;n_1,n_2,n_3,n_4) =$$
$$= \lim_{\delta\to 0} \int \frac{d^4k}{(2\pi)^4} \frac{\cos^{n_1}(k_1)\cos^{n_2}(k_2)\cos^{n_3}(k_3)\cos^{n_4}(k_4)}{\Delta_B^q \Delta_F^{p+\delta}}$$

where

$$\begin{aligned} \Delta_F &= 10 - 4\sum_{\mu=1}^{4}\cos(k_\mu) \\ &\quad + \sum_{1\le\mu<\nu\le 4}\cos(k_\mu)\cos(k_\nu) + \mu_B^2 \end{aligned} \tag{9}$$

$$\Delta_B = 4 + \mu_B^2 - \cos(k_1) - \cos(k_2) - \cos(k_3) - \cos(k_4).$$

Making use of the recursion relations

$$
\begin{aligned}
&F(p,q,...,l,...) = F(p,q,...,l-2,...)\\
&+\mu_B^2\,(F(p,q,...,l-1,...) - F(p,q,...,l-3,...))\\
&-\ (F(p,q-1,...,l-1,...) - F(p,q-1,...,l-3,...))\\
&-q\ \frac{(F(p-1,q+1,...,l-1,...) - F(p-1,q+1,...,l-3,...))}{p-1+\delta}\\
&-\ \frac{((l-2)F(p-1,q,...,l-2,...) - (l-3)F(p-1,q,...,l-4,...))}{p-1+\delta}
\end{aligned}
$$

$$
\begin{aligned}
&F(p,q;n1,n2,n3,2) = \\
&F(p,q-2,n1,n2,n3,0) - 2\,\mu_B^2\,F(p,q-1,n1,n2,n3,0) \\
&-2\,F(p-1,q,n1,n2,n3,0) + (4 + 2\,\mu_B^2 + \mu_B^4)\,F(p,q,n1,n2,n3,0) \\
&-F(p,q,n1+2,n2,n3,0) - F(p,q,n1,n2+2,n3,0) \\
&-F(p,q,n1,n2,n3+2,0), \\
&F(p,q,n1,n2,n3,1) = (\mu_B^2 + 4)\,F(p,q,n1,n2,n3,0) \\
&-F(p,q-1,n1,n2,n3,0) - F(p,q,n1+1,n2,n3,0) \\
&-F(p,q,n1,n2+1,n3,0) - F(p,q,n1,n2,n3+1,0)
\end{aligned}
$$

we can express $F(p,q;n_1,n_2,n_3,n_4)$ in terms of the integrals

$$
G(p,q) = \int \frac{d^4k}{(2\pi)^4}\,\frac{1}{\Delta_B^q\,\Delta_F^{p+\delta}}
$$

$$F(p,q;1,1,1,1) =$$
$$= \int \frac{d^4k}{(2\pi)^4} \frac{\cos(k_1)\cos(k_2)\cos(k_3)\cos(k_4)\{\, ZERO\}}{\Delta_B^q \Delta_F^{p+\delta}}$$

where

$$\begin{aligned} ZERO &= -\Delta_F + 10 - 4\sum_{\mu=1}^{4}\cos(k_\mu) \\ &+ \sum_{1\le\mu<\nu\le4} \cos(k_\mu)\cos(k_\nu) + \mu_B^2 \end{aligned} \qquad (10)$$

or

$$ZERO = -\Delta_B + 4 + \mu_B^2 - \cos(k_1) - \cos(k_2) - \cos(k_3) - \cos(k_4)$$

Identities of the second type

From the above we derive

$$\begin{aligned} 0 \;=\; & (4+2\mu_B^2+\mu_B^4)F(p+1,-q;1,1,1,1) \\ & -2\mu_B^2 F(p+1,-q-1;1,1,1,1) \\ & -4F(p+1,-q;3,1,1,1) \\ & +F(p+1,-q-2;1,1,1,1)-2F(p,-q;1,1,1,1); \end{aligned} \tag{11}$$

and the recursion relations are obtained
when the above F functions
are expressed in terms of $G(p,q)$.

(Burgio-Caracciolo-Pelissetto, Nucl.Phys.B, 1995)

It is convenient to represent

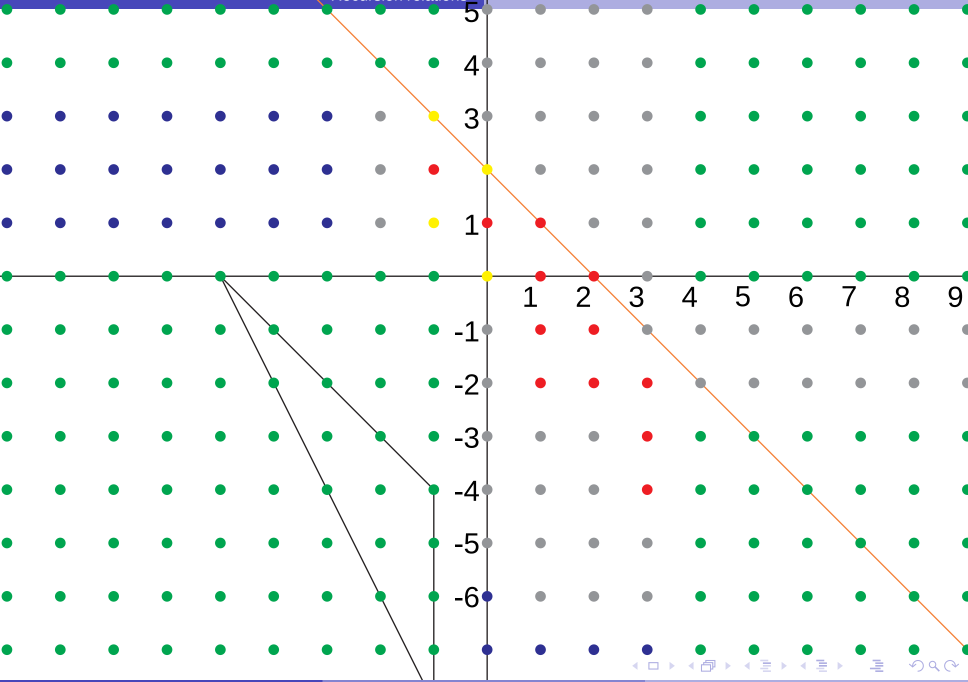
$$G_\delta(p,q) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{\Delta_B^q \Delta_F^{p+\delta}}$$

in the form

$$\begin{aligned} G_\delta(p,q) &= D(p,q;\mu_B^2) + B(p,q) \\ &\quad + \delta\left(L(p,q;\mu_B^2) + J(p,q)\right) + O(\delta^2), \qquad p \le 0; \\ G_\delta(p,q) &= D(p,q;\mu_B^2) + J(p,q) + O(\delta), \qquad p > 0. \end{aligned} \tag{12}$$

$$
\begin{aligned}
&J(-p,-q) = (24J(-p+1,-q-1)(q+2)(q+3)(-5p+3)\\
&+2J(-p+1,-q-2)(6q^3-21(p-3)q^2\\
&+q(-20p^2-91p+207)+12(-5p^2-7p+18))\\
&+(p-1)(24J(-p+2,-q-2)(q+3)(-11p+14)\\
&+4J(-p+2,-q-3)(15q^2+q(-20p+121)+5(-6p^2+4p+35))\\
&+J(-p+2,-q-4)(21q^2+8q(p+18)+2(-5p^2+26p+120)))\\
&+(p-1)(p-2)(24J(-p+3,-q-3)(-15p+31)\\
&+2J(-p+3,-q-4)(66q+73p+85)\\
&+4J(-p+3,-q-5)(10q+10p+23)\\
&+6J(-p+3,-q-6)q\\
&+6(p+3)J(-p+3,-q-6))\\
&+(p-1)(p-2)(p-3)(192J(-p+4,-q-4)\\
&\qquad -20J(-p+4,-q-5)-3J(-p+4,-q-6))\\
&)/24/(q+1)/(q+2)/(q+3)
\end{aligned}
$$

$$
\begin{aligned}
&+(24B(-p+1,-q-1)(q+2)(q+3)(-10p^3+39p^2-36p+3)\\
&+B(-p+1,-q-2)(q+3)((3(p-2)^2-1)(12q^2+6q-128)\\
&\qquad+(p-2)^2(-84q(p-2)-8(5(p-2)^2+54(p-2)+5)))\\
&+(p-1)^2(24B(-p+2,-q-2)(q+3)(-11p^2+28p-4)\\
&+B(-p+2,-q-3)(60q^2(2p-5)+4q(-20p^2+242p-485)\\
&\qquad+20(-26p^2+142p-199))\\
&+B(-p+2,-q-4)(21q^2(2p-5)+8q(p^2+36p-96)\\
&\qquad+2(p^2+300p-756)))\\
&+(p-1)^2(p-2)^2(-336B(-p+3,-q-3)\\
&\quad+4B(-p+3,-q-4)(33q+152)\\
&\quad+4B(-p+3,-q-5)(10q+53)+6B(-p+3,-q-6)(q+6))\\
&+24B(-p,-q)(q+1)(q+2)(q+3)(-3p^2+12p-11)\\
&)/24/(p-1)/(p-2)/(p-3)/(q+1)/(q+2)/(q+3);
\end{aligned}
$$


5
4
3
1
1
2
3
4
5
6
7
8
9
-1
-2
-3
-4
-5
-6

$G(p,q)$ can be expressed in therms of the quantities

$$
\begin{aligned}
& Y_4 = \frac{J(1,0)}{2}, \qquad Y_5 = J(1,-1), \\
& Y_6 = 2J(1,-2), \qquad Y_7 = \frac{J(2,-1)}{2}, \\
& Y_8 = J(2,-2), \qquad Y_9 = \frac{J(3,-2)}{2}, \\
& Y_{10} = J(3,-3), \qquad Y_{11} = 2J(3,-4), \\
& Y_0 = \frac{J(2,0)}{4} - \frac{F_0}{16\pi^2}
\end{aligned}
\tag{13}
$$

and

$$
\begin{aligned}
Y_1 &= \frac{1}{48} - \frac{1}{4}\,Z_0 - \frac{1}{24}\,J(-1,2) + \frac{1}{12}\,J(0,1) + \frac{1}{12}\,J(1,0);\\
Y_2 &= \frac{1}{6} - \frac{1}{\pi^2} - Z_0 - \frac{1}{6}\,J(-1,2) + \\
&\quad \frac{1}{3}\,J(0,1) - \frac{1}{24}\,J(1,-2) - \frac{1}{12}\,J(1,-1) - \\
&\quad -\frac{17}{8}\,J(1,0) + 4\,J(1,1) - \frac{1}{48}\,J(2,-2) \\
&\quad +\frac{25}{6}\,J(2,-1) - 4\,J(2,0);\\
Y_3 &= -\frac{1}{384\pi^2} - F_0\,\frac{1}{128\pi^2} + \frac{1}{96}\,Z_0 - \\
&\quad \frac{1}{48}\,J(-1,3) + \frac{1}{192}\,J(0,1) + \frac{1}{48}\,J(0,2) + \frac{1}{48}\,J(1,1);
\end{aligned}
$$

Y_0	$-$ 0.01849765846791657356
Y_1	0.00376636333661866811
Y_2	0.00265395729487879354
Y_3	0.00022751540615147107
$Y_4 = \mathcal{F}(1,0)$	0.08539036359532067914
$Y_5 = \mathcal{F}(1,-1)$	0.46936331002699614475
$Y_6 = \mathcal{F}(1,-2)$	3.39456907367713000586
$Y_7 = \mathcal{F}(2,-1)$	0.05188019503901136636
$Y_8 = \mathcal{F}(2,-2)$	0.23874773756341478520
$Y_9 = \mathcal{F}(3,-2)$	0.03447644143803223145
$Y_{10} = \mathcal{F}(3,-3)$	0.13202727122781293085
$Y_{11} = \mathcal{F}(3,-4)$	0.75167199030295682254

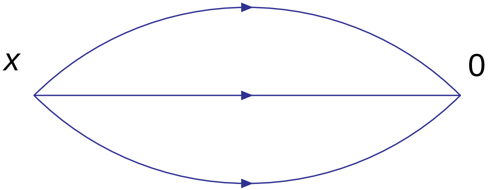
Table: New constants appearing in the general fermionic case.

(S. Capitani, *Phys. Rep.*, **v. 382** (2003))

The respective programs can be found on the web page of the ITEP Lattice group
http://www.lattice.itep.ru/∼pbaivid/lattpt/
The results presented there are as follows:

- The program for a computation of $F(p,q;n_1,n_2,n_3,n_4$ at $0 \leq p,q \leq 9$ and $n_1+n_2+n_3+n_4 \leq 25$ that can be readily used by anyone.
- The values of the functions $J(p,q)$ and $B(p,q)$ at $-26 \leq p \leq 0, \quad -56-2p \leq q \leq 34$ and the values of $J(p,q)$ at $1 \leq p \leq 9, \quad -28 \leq q \leq 33-p$;
- The explicit expressions for $F(p,q;n_1,n_2,n_3,n_4$ at some particular values of p and q and $n_1 \leq 6$.

Two-Loop Integrals



$$\equiv A(k) =$$

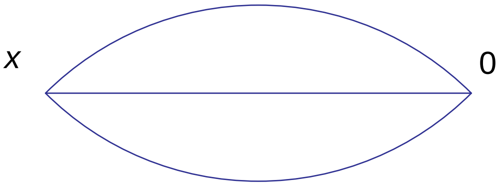
$$= \sum_{x\in\Lambda} e^{-ikx} G^3(x)$$

where

$$G(x) = \frac{1}{(2\pi)^4} \int_{BZ} e^{-ikx} \frac{1}{2\Delta}, \tag{14}$$

Δ is the bosonic or fermionic propagator.

Such diagrams in the bosonic case were computed by
M. Lüscher and P. Weisz (Nucl.Phys. **B445** (1995)):



$$= A_B(x)$$

- The value of $A_B(p)$ at zero momentum
- Asymptotic expansion of $A_B(p)$ when $p \to 0$
- Similar integrals with nontrivial numerators

We outline the **LW** method by considering $A(0) = \sum_{x\in\Lambda} G^3(x)$:

$$
\begin{aligned}
A(0) &= G^3(0) + \sum_{x\in\Lambda'} G^3_{as}(x) \\
&\quad + \sum_{x\in\{\mathcal{F}_N\}} \Big(G^3(x) - G^3_{as}(x) \Big) \\
&\quad + \sum_{x\in\{\Lambda'\setminus\mathcal{F}_N\}} \Big(G^3(x) - G^3_{as}(x) \Big)
\end{aligned}
\qquad (15)
$$

$$
\mathcal{F}_N = \{x \in \Lambda :\ |x_1| + |x_2| + |x_3| + |x_4| \le N\}
$$

$$
\Lambda = \{x :\ x_\mu \in \mathbb{Z}\},\ \ \Lambda' = \Lambda \setminus \{0\}.
$$

The Lüscher–Weisz method is based on

- recursion formulas for the free bosonic propagator in the x-representation,

$$G_B(x+\hat{\mu}) = G_B(x-\hat{\mu}) + \frac{2x_\mu}{\left(\sum_{\nu=1}^{4} x_\nu\right)} \sum_{\lambda=1}^{4} (G_B(x) - G_B(x-\hat{\lambda}));$$

- asymptotic expansion of $G(x)$ at $x \to \infty$.

In the fermionic case,
there are no analogous recursion relations,
and asymptotic formulas are little more complicated.

First we compute

$$G_B(x) = \frac{1}{(2\pi)^4}\int_{BZ} e^{-ikx}\frac{1}{2\Delta_B} \tag{16}$$
$$G_F(x) = \frac{1}{(2\pi)^4}\int_{BZ} e^{-ikx}\frac{1}{2\Delta_F}$$

over the domain $\mathcal{F}_{48}^{+}$:

$$x_1 \geq x_2 \geq x_3 \geq x_4 \geq 0 \tag{17}$$
$$x_1 + x_2 + x_3 + x_4 \leq 48$$

Here

$$BZ = \left\{ k : \frac{-\pi}{a} \leq k_\mu \leq \frac{\pi}{a} \right\}$$

The integrals

$$\bar{G}(x_1,x_2,x_3,x_4) = \int \frac{d^4k}{(2\pi)^4} \frac{\cos(k_1x_1)\cos(k_2x_2)\cos(k_3x_3)\cos(k_4x_4)}{\Delta_F}$$

can be expressed in terms of the quantities

$$F(p,q;n_1,n_2,n_3,n_4) = \int \frac{d^4k}{(2\pi)^4} \frac{\cos^{n_1}(k_1)\cos^{n_2}(k_2)\cos^{n_3}(k_3)\cos^{n_4}(k_4)}{\Delta_B^q \Delta_F^{p+\delta}}$$

by making use of relations

$$\cos(nx) = 2^{n-1}\cos^n x \; + \; \frac{n}{2}\sum_{k=0}^{[n/2]-1} \frac{(-1)^{k+1}}{k+1} C^k_{n-k-2}(2\cos x)^{n-2k-2}$$

Using the quantities $F(p, q; n_1, n_2, n_3, n_4)$ evaluated by the Burgio-Caracciolo-Pelissetto method and FORM package, we compute the values $G(x_1, x_2, x_3, x_4)$ at $|x_1| + |x_2| + |x_3| + |x_4| \leq 48$. This takes about three months on a PC.

The result is a linear combination

$$G(x_1, x_2, x_3, x_4) = \sum_{n=4}^{11} c_n(x) Y_n + c_\pi(x) \frac{1}{\pi^2} + c(x)$$

.

$c_n(x)$ show exponential growth with $|x|$, therefore Y_n should be evaluated to a high precision.

First we compute $G_F(x)$ at the points

$$\begin{aligned} x_1 &= (96,0,0,0); \\ x_2 &= (95,0,0,0); \\ x_3 &= (95,1,0,0); \\ x_4 &= (94,2,0,0); \\ x_5 &= (94,1,1,0); \\ x_6 &= (93,3,0,0); \\ x_7 &= (93,2,1,0); \\ x_8 &= (93,2,0,0). \end{aligned} \tag{18}$$

The result furnishes a linear combination of the constants $Y_4 \div Y_{11}$, 1, and $\frac{1}{(2\pi)^2}$ with rational coefficients.

Thus we arrive at the system of linear equations for $Y_4 \div Y_{11}$,

$$\sum_{j=1}^{8} M_{Nj} Y_{j+3} = C_N - G_F(x_N), \qquad (19)$$

the right-hand side involves the linear combinations C_N of 1 and $\dfrac{1}{(\pi)^2}$ with rational coefficients as well as the eight values of $G_F(x_N)$. The coefficients M_{Nj} in the left-hand side are rational numbers increasing exponentially with $|x_N|$, the same is valid for C_N. Our computation of the values of Y_n is based on the fact that, at large $|x_N|$, $G_F(x_N)$ can be neglected as compared to C_N. In fact, we do not simply neglect $G_F(x_N)$ but use the asymptotic expansion

Thus we arrive at

$$
\begin{aligned}
Y_4 &= 0.085390363595320679135167028885334120581941471274432\,65(1)\\
Y_5 &= 0.469363310026996144753475397057518034820462958875231\,84(1)\\
Y_6 &= 3.394569073677130005860086897023744964536852723137335\,03(1)\\
Y_7 &= 0.051880195039011366364902287664715799409680127572915\,08(1)\\
Y_8 &= 0.238747737563414785202336139303869704452801949834779\,88(1)\\
Y_9 &= 0.034476441438032231453961881442431936001212771247157\,84(1)\\
Y_{10} &= 0.132027271227812930853147310981965969711971447959594\,77(1)\\
Y_{11} &= 0.751671990302956822535431485907781109910112771931448\,03(1)
\end{aligned}
$$

That is, to a precision of 54 digits.

$$A(0) = G^3(0) + \sum_{x\in\Lambda'} G^3_{as}(x) \tag{20}$$

decreases rapidly

$$+ \sum_{x\in\{\mathcal{F}_N\}} \Big(G^3(x) - G^3_{as}(x)\Big)$$

can be neglected

$$+ \sum_{x\in\{\Lambda'\backslash\mathcal{F}_N\}} \Big(G^3(x) - G^3_{as}(x)\Big)$$

$$\mathcal{F}_N = \{x \in \Lambda : \ |x_1| + |x_2| + |x_3| + |x_4| \le N\}$$

$$\Lambda = \{x : \ x_\mu \in \mathbb{Z}\}, \quad \Lambda' = \Lambda\backslash\{0\}.$$

Asymptotic behavior

We compute asymptotic behavior at $x \to \infty$ of the quantities

$$\begin{aligned}
G_{B[F]} &= \int \frac{dp}{(2\pi^4)} \frac{e^{-ipx}}{D_{B[F]}(p)} \, , \\
K_{B[F]} &= \int \frac{dp}{(2\pi^4)} \frac{(e^{-ipx} - 1)}{D^2_{B[F]}(p)} \, , \\
L_{B[F]} &= \int \frac{dp}{(2\pi^4)} \frac{\left(e^{-ipx} - 1 + \frac{x^2}{8} \, (4 - \sum_{\mu=1}^{4} \cos^2 k_\mu) \right)}{D^3_{B[F]}(p)} \, ,
\end{aligned} \tag{21}$$

which are defined to be infrared finite.

The functions $G_B(x)$, $G_F(x)$, $K_B(x)$, $K_F(x)$, $L_B(x)$, and $L_F(x)$ can be expressed in terms of the quantities

$$\begin{aligned} J_n^B(m_R,x) &= \int \frac{dp}{(2\pi^4)} \frac{e^{-ipx}}{(D_B(p)+m_R^2)^n} \,, \\ J_n^F(m_R,x) &= \int \frac{dp}{(2\pi^4)} \frac{e^{-ipx}}{(D_F(p)+m_R^2)^n} \end{aligned} \tag{22}$$

as follows:

$$\begin{aligned} K_B &= \lim_{m_R\to 0}\left(J_2^B(m_R,x) - \frac{1}{4} F_R\left(0,2;0,0,0,0;\frac{m_R^2}{2}\right)\right) \,, \\ K_F &= \lim_{m_R\to 0}\left(J_2^F(m_R,x) - \frac{1}{4} F_R(2,0;0,0,0,0;\frac{m_R^2}{2})\right) \end{aligned} \tag{23}$$

The derivation of asymptotic behavior of the quantity

$$J_n^F(m_R, x) = \int \frac{dp}{(2\pi^4)} \frac{e^{-ipx}}{(D_F(p) + m_R^2)^n}$$

is based on the asymptotic expansion of the denominator of the integrand

$$\frac{1}{(D_F(p) + m_R)^n} \simeq \sum_{k=0}^{\infty} \sum_{\alpha=4k}^{\infty} \frac{Q_F^\alpha(n,k;p)}{(p^2 + m_R^2)^{k+n}}$$

where $Q_B^\alpha(n,k;p)$ is a polynomial in p of degree α.
Then one can employ the relation

$$\int \frac{dp}{(2\pi^4)} \frac{e^{-ipx}}{(p^2 + m_R^2)^n} \simeq \frac{2\pi^2}{\Gamma(n)} \left(\frac{x}{2m_R}\right)^{n-2} K_{n-2}(m_R|x|),$$

where $K_\nu(z)$ is the McDonald function.
First we tend $m_R \to 0$ and only then $|x| \to \infty$.

Using the designations

$$[x^2] = x_1^2 + ... + x_4^2, \quad ... \qquad [x^8] = x_1^8 + ... + x_4^8,$$

we can represent the rusults as follows:

$$\begin{aligned} G_B(x) &\simeq \frac{1}{[x^2]} + \left(-\frac{1}{[x^2]^2} + 2\,\frac{[x^4]}{[x^2]^4}\right) \\ &+ \left(-4\,\frac{1}{[x^2]^3} - 48\,\frac{[x^6]}{[x^2]^6} + 16\,\frac{[x^4]}{[x^2]^5} + 40\,\frac{[x^4]^2}{[x^2]^7}\right) \end{aligned} \tag{24}$$

$$\begin{aligned} G_F(x) &\simeq \frac{1}{[x^2]} + \left(-\,4\frac{1}{[x^2]^2} + 8\,\frac{[x^4]}{[x^2]^4}\right) \\ &+ \left(-40\frac{1}{[x^2]^3} - 768\,\frac{[x^6]}{[x^2]^6} + 208\,\frac{[x^4]}{[x^2]^5} + 640\,\frac{[x^4]^2}{[x^2]^7}\right) \end{aligned} \tag{25}$$

($G_B^{as}(x)$ was found by Lüscher and Weisz (1995).)

Summation of an asymptotic expansion, an example

$$Z(s) = \sum_{\Lambda'} \frac{1}{[x^2]^s} =$$

$$= \frac{\pi^s}{\Gamma(s)} \sum x \in \Lambda' \Big(\alpha_{s-1}(\pi x^2) + E_{s-1}(\pi x^2) \Big)$$

where ($n > 0$)

$$E_n(z) = \int_1^\infty \frac{e^{-tz}}{t^n}\, dt$$

$$\alpha_n(z) = \int_1^\infty e^{-tz} t^n\, dt$$

$$Q_1^{BBB} = \int_{BZ} \frac{d^4k}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \sum_{\mu=1}^{4} \frac{\hat{k}_\mu^2 \hat{q}_\mu^2}{D_B(k)D_B(q)D_B(r)} \qquad (26)$$

$$Q_1^{BBF} = \int_{BZ} \frac{d^4k}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \sum_{\mu=1}^{4} \frac{\hat{k}_\mu^2 \hat{q}_\mu^2}{D_B(k)D_B(q)D_F(r)}$$

and so on.
In a similar way, we can consider

$$Q_2^{BBF} = \int_{BZ} \frac{d^4k}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \sum_{\mu=1}^{4} \frac{\hat{k}_\mu^2 \hat{q}_\mu^2 \hat{r}_\mu^2}{D_B(k)D_B(q)D_F(r)} \qquad (27)$$

etc.

The results of my computations are as follows:

$$
\begin{aligned}
Q_1^{BBB} &= 0.042306368(1) \\
Q_1^{FBB} &= 0.020079702(3) \\
Q_1^{BBF} &= 0.024555253(3) \\
Q_1^{FFB} &= 0.00969896(1) \\
Q_1^{BFF} &= 0.01173224(1) \\
Q_1^{FFF} &= 0.00576013(3) \\
Q_2^{BBB} &= 0.05462397818(1) \\
Q_2^{BBF} &= 0.02659175158(3) \\
Q_2^{BFF} &= 0.0130373237(1) \\
Q_2^{FFF} &= 0.0064945681(3)
\end{aligned} \tag{28}
$$

Summary and Outlook

- The Burgio-Caracciolo-Pelissetto algorithm is realized on a computer
- The Lüscher-Weisz method is extended to the case of fermions
- $G(p,q)$ are found over a large domain of values of p and q, making it possible to compute $G_F(x)$ at $|x| \leq 96$
- $G_F(x)$ is evaluated at $|x| \leq 48$
- Values of some Y_n are computed to a better precision (54 significant digits) (Z_0 and Z_1 are known to 396 significant digits :-))
- Some two-loop fermionic integrals are evaluated to a precision of 10 significant digits

The work is in progress!