Automorphic Black Holes as Probes of Geometry

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Outline:

1. Unifying different automorphic themes:
   automorphic entropy and automorphic motives
2. Chaudhuri-Hockney-Lykken models
3. Automorphic black hole entropy in CHL_N models via Siegel modular forms
4. Siegel entropy via the Maaß-Skoruppa lift
5. Motivic interpretation of CHL_N entropy

Extended Agenda

The general background of this talk is provided by a conceptual framework obtained by combining

$$\text{Langlands} \ + \ \text{Grothendieck}$$

Here

$$\text{Langlands} \ \text{“}=\text{”} \ \text{theory of automorphic forms}$$

is quite natural in physics, in particular string theory.

Grothendieck on the other hand has not been a mainstream in physics, but since

$$\text{Grothendieck} \ \text{“}=\text{”} \ \text{geometry},$$

it is not too alien.

The combination of these two provides a new way to think about physical problems.

Basic Question about Black Holes

- *Vague question:* How much information about the structure of spacetime can be extracted from black holes?

- *More concrete question:* Given a black hole what geometric information can be extracted from its entropy?

  This question leads to

- *Two leitmotifs:*
  automorphic black hole entropy and automorphic spacetime.

LM1: Automorphic Black Holes

- First leitmotif: the statistical entropy is determined by degeneracies that appear as Fourier coefficients of automorphic forms.
- In recent years explicit examples of such automorphic black holes have been discovered:
  - Early start by Dijkgraaf-Verlinde-Verlinde in 1996;
  - Jatkar-Sen05 extended this work to prime CHL_p models.
  - Extension of Jatkar-Sen05 by Govindaraja-Krishna09 to non-prime CHL_N models.
- This is good for several reasons:
  Automorphic forms are highly restricted, forming finite dimensional function spaces.

  Hence a finite amount of computation determines the whole function.

LM2: Automorphic Varieties

Second leitmotif: automorphic geometry.

- The problem of whether geometry is automorphic can be traced back to Felix Klein and his student Adolf Hurwitz in the late 19th century.
- In the 1920s and 1930s Hecke did some isolated work on this problem (for $\Gamma \subset \mathrm{SL}(2, \mathbb{Z})$ forms).
- It was made popular by Taniyama and Shimura and Weil for elliptic curves. In particular Weil's observation was important because it made the problem computable (Lang's vitriolic article notwithstanding).
  The Taniyama-Shimura-Weil conjecture was proven by Wiles, Taylor and others in the late 1990s in the context of the proof of the Fermat conjecture & $2-$tori $T^2 = E$.
  (Ending a campaign that lasted some 350 years!)

LM2: Automorphic Varieties

- The elliptic case is too simple to provide the true conceptual picture. It is also too simple to cover most of interesting physical applications.
- The real story involves several deep ideas and conjectures that link these ideas:
  1. One idea is Grothendieck's notion of motives in the 1960s. This concept is currently not understood but some aspects are under control.
  2. A second, independent, idea was Langlands' reciprocity conjecture, also from the 1960s, proposing a link between modular (automorphic) forms and Galois representations.
  3. These two ideas turn out to connected by the insight that Grothendieck's motives can be viewed as geometric structures that carry Galois representations.
     (They form a Tannakian category.)

LM2: Automorphic Varieties

- Upshot:

$$\text{Motives} \quad \longleftrightarrow \quad \begin{array}{c}\text{Galois}\\ \text{representations}\end{array} \quad \longleftrightarrow \quad \begin{array}{c}\text{Automorphic}\\ \text{forms}\end{array}$$

All this is conjectural, but the Galois interpretation of both Grothendieck's motives and of automorphic froms raises the hope that

G+L: *All motives are automorphic.*

Today this is often called the <u>Langlands reciprocity conjecture</u>.

No proof is in sight, although progress is being made and Fields medal worthy steps have been taken.

LM2: Automorphic Varieties

- The difficulty: no direct construction of automorphic forms from motives, or vice versa.
  Instead, the relation is constructed by associating functions to each of the different concepts:

$$\begin{aligned} \text{Motivic L} - \text{functions}: &\qquad L(M,s) \\ \text{Galois L} - \text{functions}: &\qquad L(\rho,s) \\ \text{Automorphic L} - \text{functions}: &\qquad L(\Phi,s), \end{aligned}$$

  where $s \in \mathbb{C}$.

- Functions are good, but to establish relations

$$L(M,s) \;=\; L(\Phi,s)$$

  remains quite nontrivial (even in the case of modular forms).
  (For elliptic curves this is where Weil's experimental observation was important.)

LM2: Automorphic Varieties

- On the nature of L-functions:
  L-functions are arithmetic objects, associated to $X/\mathbb{F}_p$ for finite fields $\mathbb{F}_p = \{0, 1, ..., p-1\}$ for all primes p.
  (Think of this as probing the geometry in a discrete way.)
- The arithmetic origin of L-functions and geometric automorphic forms is a longer story which will be hidden here as much as possible.

  The magic of the Weil conjectures makes this possible because it relates these arithmetic objects to the cohomology groups $H^*(X)$ of the variety.
  (Proven by Grothendieck and Deligne)

LM2: Automorphic varieties

- Key point:
  In general the cohomology groups are of large rank, leading to automorphic forms of high rank groups: $H^r(X)$ maybe automorphic, but in general is not an irreducible object.
- Needed are subgroups

$$H(M) \subset H^r(X_n)$$

  spanned by some irreducible "geometric atoms" M (called irreducible motives).
- Problem:
  No known universal construction of motives for arbitrary smooth (CY) varieties.

LM2: Automorphic Varieties

- Adopt instead a constructive strategy:
  consider special classes of Calabi-Yau varieties and construct certain types of automorphic motives originating from these manifolds. RS hep-th/0603 + hep-th/0812; Kadir-Lynker-RS hep-th/1012

- A universal type of CY motive: RS hep-th/0812
  $\Omega-$motives M_Ω are associated to the holomorphic $n-$forms of CY varieties:

$$H(M_\Omega) \subset H^n(X_n)$$

  (defined by an orbit of the representation of the Galois group $\mathrm{Gal}(K_X/\mathbb{Q})$ of a particular number field K_X constructed from the variety X).

LM2: Automorphic Varieties

- Modular $\Omega-$motives of CYs lead to forms $f(M_\Omega, q)$ of weight

$$w(f(M_\Omega, q)) \;=\; \dim_{\mathbb{C}} X + 1$$

for some congruence group $\Gamma_0(N) \subset \mathrm{SL}(2, \mathbb{Z})$. Such functions $f(\tau), \tau \in \mathcal{H}$ are highly constrained by their transformation behavior:

$$\Gamma_0(N) \;\ni A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \qquad f(A\tau) \;=\; (c\tau + d)^w f(\tau).$$

- Difficulty: no machine to compute the level N from geometry.
- In general $\Omega-$motives are automorphic, associated to $\mathrm{GL}(n)$, $n > 2$. RS hep-th/0603 + hep-th/0812

LM2: String automorphic spacetime

String theoretic of Langlands+Grothendieck:

- Whenever physics provides automorphic forms we can ask whether we can construct geometric structures/motives that induce these forms.

- First application: an emergent spacetime program.
  Consider modular forms defined on the string worldsheet and construct Calabi-Yau varieties (via motives) induced by those automorphic forms. RS hep-th/0603 + hep-th/0812

  Explicit examples in all physically relevant dimensions raise the hope for a relation of the type

$$\mathcal{A}_{\mathrm{CFT}_{\mathrm{N}=2}} \quad = \quad \mathcal{A}(\mathcal{M}_{\mathrm{CYF}})$$

  for spaces of automorphic forms associated to worldsheet CFTs and spacetime motives.

Automorphic black holes

Second application: motivic black hole entropy:
Given that both black holes and string compactification varieties are determined by automorphic forms (at least sometimes) we can ask the natural

Question:
What, if anything, do automorphic black holes encode about automorphic geometry (motives)?

The goal of this talk is to explore this question in the context of an explicit class of models.

CHL Models

- Concrete framework: CHL_N models Chaudhuri-Hockney-Lykken 95

$$\mathrm{CHL}_N : \quad \mathrm{Het}(T^6/\mathbb{Z}_N) \;\cong\; \mathrm{IIA}(K3 \times T^2/\mathbb{Z}_N).$$

- Duality group: $G_N \;=\; \Gamma_1(N) \times O(6, r_N - 6, \mathbb{Z})$, with $\Gamma_1(N) \subset \mathrm{SL}(2,\mathbb{Z})$ defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cong \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mathrm{mod\ N}.$$

- Electric and magnetic charges of rank r_N: (Q_e, Q_m)
- T-duality invariants: $Q_e^2,\ Q_m^2,\ Q_e \cdot Q_m$
- S-duality invariant: $Q_e^2 Q_m^2 - (Q_e Q_m)^2$

CHL Models

- Charge conjugate variables of the Siegel upper halfplane:

$$(\tau, \sigma, \rho) \;\cong\; Z \;=\; \begin{pmatrix} \tau & \rho \\ \rho & \sigma \end{pmatrix} \;\in\; \mathcal{H}_2$$

- Automorphic forms for CHL_N black holes are Siegel modular form of genus 2

$$\Phi^N \;\in\; S_w(\Gamma_0^{(2)}(N))$$

of weight

$$w = \left\lceil \frac{24}{N+1} \right\rceil - 2.$$

(For CHL_N: $w \in [1, 10]$)

- The level N is determined by the congruence subgroups

$$\Gamma_0^2(N) \;\subset\; \mathrm{Sp}(4, \mathbb{Z})$$

that are acting on the Siegel upper halfplane $\mathcal{H}_2$.

CHL Black Hole Entropy via Siegel

- The Hecke congruence group is defined as

$$\Gamma_0^{(2)}(N) \;=\; \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(4,\mathbb{Z}) \;\middle|\; C \equiv 0 (\mathrm{mod}\ N) \right\}$$

- For $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(4,\mathbb{Z})$ the form Φ^N transforms as

$$\Phi^N(MZ) \;=\; \det(CZ+D)^w \Phi^N(Z).$$

- The Fourier coefficients of the Siegel form partition function

$$\frac{1}{\Phi^N(\tau,\sigma,\rho)} \;=\; \sum_{k,\ell,m} d^N(k,\ell,m) q^k r^\ell s^m$$

with $q = e^{2\pi i\tau}$, $r = e^{2\pi i\sigma}$, $s = e^{2\pi i\rho}$ and

$$(k,\ell,m) \;=\; \left(\frac{1}{2}Q_e^2, \frac{1}{2}Q_m^2, Q_e Q_m \right).$$

CHL Black Hole Entropy via Siegel

The degeneracies $d^N(Q_e, Q_m)$ lead to the microscopic entropy

$$S^N_{\mathrm{micro}}(Q_e, Q_m) \;=\; \ln \tilde{d}^N(Q_e, Q_m).$$

in terms of multiplicities $\tilde{d}^N$ related to the d^N.

The Question Reformulated

The basic question can now be formulated in a more concrete way:

> *Are black hole Siegel modular forms associated to motives?*

To answer this question one has to know something about Siegel modular motives.

There are no theorems here, but it is conjectured that the motives that support Siegel modular forms should be of such a structure that their cohomological representation is of a specific type.

For genus 2 forms of weight w:

$$H(M_\Phi) \;=\; H^{2w-3,0} \oplus H^{w-1,w-2} \oplus H^{w-2,w-1} \oplus H^{0,2w-3}.$$

This looks good since rank 4 cohomology is certainly compatible with CYs.

The killer

Note though: this looks Calabi-Yau-ish for $w = 3$ only

$$H(M_\Phi) \;=\; H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}.$$

Problem:
The weight of the CHL_N black hole Siegel forms ranges between

$$w \;\in\; [1, 10].$$

But: the Siegel modular forms of CHL_N are not of the most general type – they are lifts of other forms.

The Maaß-Skoruppa lift

Key structure: CHL_N black hole Siegel forms $\Phi^N(\tau,\sigma,\rho)$ can be constructed from ordinary modular forms $f^N(\tau)$

$$S_{w+2}(\Gamma_1(N)) \ni f^N(\tau) \quad \xrightarrow{\mathrm{MS}} \quad \Phi^N(\tau,\sigma,\rho) \in S_w(\Gamma_0^{(2)}(N))$$

via a 2-step construction.

<u>1st Step: the Skoruppa lift.</u>

Construct Jacobi forms via a map that lifts ordinary cusp forms f_{w+2} to 2-variable forms on $\mathcal{H}\times\mathbb{C}$, where $\mathcal{H}$ is the upper half-plane

$$S_{w+2}(\Gamma_1(N)) \ni f^N_{w+2}(\tau) \quad \xrightarrow{\mathrm{SL}} \quad \varphi^N_w(\tau,\sigma) \in J_w$$

with $\Gamma_1(N) \subset \mathrm{SL}(2,\mathbb{Z})$ as defined earlier.

The Maaß-Skoruppa lift

The Skoruppa lift is constructed via a special form

$$K(\tau,\sigma) \;=\; \frac{\vartheta_1(\tau,\sigma)}{\eta^3(\tau)},$$

where $\vartheta_1(\tau,\sigma)$ is a particular Jacobi form and $\eta(\tau)$ is the Dedekind function:

$$\vartheta_1(\tau,\sigma) = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{1}{8}(2n+1)^2} r^{n+\frac{1}{2}}, \quad \eta(\tau) = q^{1/24} \prod_{n\in\mathbb{N}} (1-q^n),$$

with $q = e^{2\pi i\tau},\ r = e^{2\pi i\sigma}$

The Skoruppa lift is defined as

$$f^N_{w+2} \;\longrightarrow\; \varphi^N_w \;:=\; K^2 f^N_{w+2}$$

The Maaß-Skoruppa lift

The resulting Jacobi form can be expanded as

$$\varphi_w^N(q,r) \;=\; \sum_{k,\ell} c^N(k,\ell) q^k r^\ell = \sum_{k,\ell} c^N(4k-\ell^2) q^k r^\ell.$$

<u>2nd Step: the Maaß lift</u>

For any (weak) Jacobi form $\varphi_w^N(\tau,\sigma)$ one can define the additive lift as

$$\Phi_w^N(\tau,\sigma,\rho) \;=\; \sum_{k,\ell,m} g^N(k,\ell,m) e^{2\pi i(k\tau+\ell\sigma+m\rho)},$$

where χ is the Legendre symbol (± 1) with

$$g^N(k,\ell,m) \;=\; \sum_{d|(k,\ell,m)} \chi(d) d^{w-1} c^N\left(\frac{k\ell}{d^2}, \frac{m}{d}\right).$$

The Maaß-Skoruppa lift

Combining the Skoruppa lift from ordinary modular forms to Jacobi forms

$$f^N(q) \xrightarrow{\mathrm{SL}} \varphi^N(q,r)$$

with the Maaß lift

$$\varphi^N(q,r) \xrightarrow{\mathrm{ML}} \Phi^N(q,r,s)$$

leads to the Maaß-Skoruppa lift of ordinary modular forms to Siegel modular forms of genus 2

$$f^N(q) \xrightarrow{\mathrm{SL}} \varphi^N(q,r) \xrightarrow{\mathrm{ML}} \Phi^N(q,r,s).$$

The Maaß-Skoruppa lift

Key property of the CHL_N Maaß-Skoruppa lift:

The only ingredient of the construction of Φ_N that is sensitive to the structure of the CHL_N model is the ordinary modular form

$$f^N_{w+2}(\tau) \ \in \ S_{w+2}(\Gamma_1(N))$$

determined by the symmetry group $\mathbb{Z}_N$ of the CHL_N model.

Calling this form the Maaß-Skoruppa root of the Siegel modular form Φ^N, we can ask a better, more precise

Question:

What geometric/motivic information is contained in the Maaß-Skoruppa roots of the CHL_N black hole Siegel forms, if any:

$$f^N_{w+2}(\tau) \ \stackrel{?}{=} \ f(M,q).$$

Motivic modular forms

To answer this question we have to recall something about motivic modular forms.

- The $\Omega-$motives M_Ω with

$$H(M_\Omega) \subset H^n(X_n)$$

  are the only universal motives associated to arbitrary Calabi-Yau manifolds in any dimension.

- It is therefore natural to ask about the modular properties of these objects. If the $\Omega-$motive is modular, the weight $w(f)$ of the associated motivic modular form $f(M_\Omega, q)$ is given by

$$w(f(M_\Omega, q)) = \dim_{\mathbb{C}} X + 1.$$

f^N is not naturally motivic

Considering the weights

$$w + 2 \;=\; \left\lceil \frac{24}{N+1} \right\rceil$$

of the Maaß-Skoruppa lifts shows that these forms have weights in the range

$$w + 2 \in \{3, ..., 12\},$$

in general incompatible with the weight of CY threefold.

The weight obstruction to a direct geometric interpretation leads to the following

A further reduction

Question.
Can the black hole modular form given by the Maaß-Skoruppa root

$$f^N(q) \in S_{w+2}(\Gamma_1(N)), \qquad w+2 = \left\lceil \frac{24}{N+1} \right\rceil,$$

be constructed in terms of simpler forms that admit a motivic/geometric interpretation?

The answer to this is yes, and it involves two types of further lifts, beyond the Skoruppa lift and the Maaß lift.

The reason that <u>two types</u> of lifts are necessary comes from the fact that the Maaß-Skoruppa roots $f^N_{w+2} \in S_{w+2}(\Gamma_0(N))$ for the CHL_N models decompose into two distinct classes of forms.

A further reduction

These two distinct classes are distinguished by their symmetries:

- Class I: Modular forms f^N without complex multiplication
- Class II: Modular forms f^N with complex multiplication

Complex multiplication is a number theoretic symmetry which is reflected in the property that the Fourier expansion

$$f(q) = \sum_n a_n q^n$$

is sparse: for forms with complex multiplication the a_n vanish for roughly half of the integers n.

Class I elliptic lifts

Result I.
For each Maaß-Skoruppa root $f^N(q) \in S_{w+2}(\Gamma_1(N))$ of type I (without complex multiplication) there exists a modular cusp form of weight two

$$f_2^{(\widetilde{N})}(q) \in S_2(\Gamma_1(\widetilde{N}),$$

of some level

$$\widetilde{N} = \widetilde{N}(N)$$

such that the Maaß-Skoruppa root is given by

$$f^N(q) = f_2^{\widetilde{N}}(q^{1/m})^m,$$

where

$$m = \frac{1}{2}\left\lceil \frac{24}{N+1} \right\rceil.$$

Class I elliptic lifts

The geometric/motivic interpretation is then obtained by elliptic motives such that

$$H^1(M) \;=\; H^1(E)$$

since for each form $f_2^{\widetilde{N}}(q) \in S_2(\Gamma_1(\widetilde{N}))$ of this type there exists an elliptic curve $E_{\widetilde{N}}$ of conductor $\widetilde{N}$ such that

$$f_2^{\widetilde{N}}(q) \;=\; f(E_{\widetilde{N}}, q).$$

Example of class I reduction

Consider the case of CHL_3 with Maaß-Skoruppa root

$$f^3(q) = \eta(q)^6 \eta(q^3)^6 \in S_6\left(\Gamma_1(3)\right).$$

For this black hole root the elliptic form of weight two is given by

$$f_2^3(q) = \eta(q^3)^2 \eta(q^9)^2$$

with

$$m = \frac{1}{2}\left\lceil \frac{24}{4} \right\rceil = 3 \quad \text{and} \quad \widetilde{3} = 27.$$

Furthermore there exists an elliptic curve E_{27} of conductor $\widetilde{N} = 27$ such that

$$f_2^3(q) \;=\; f_2(E_{27}, q).$$

Class II reduction

The forms of type II admit complex multiplication, i.e. they are admit a symmetry.

- One reflection of this symmetry is that the modular forms $f^N(q)$ are more sparse than generally possible: their $q-$expansions
$$f^N(q) \;=\; \sum_n a_n q^n$$
vanish at certain prime exponents $p \in \mathbb{N}$.
- The class of primes for which this happens depends on the symmetry algebra, which can be viewed as the ring of algebraic integers
$$\mathcal{O}_K \;\subset\; K = \mathbb{Q}(\sqrt{-D})$$
of some imaginary quadratic field K.

Class II reduction

- The key consequence of CM by some imaginary field K is that the modular form is determined by a so-called Hecke character Ψ_K associated to K. These are maps defined on the set $\mathcal{I}_{\mathfrak{m}}$ of (fractional) ideals prime to an integral ideal $\mathfrak{m} \subset \mathcal{O}_K$

$$\Psi_K : \quad \mathcal{I}_{\mathfrak{m}} \longrightarrow \mathbb{C}$$

- Class II reduction is best described in terms of the L-series associated to modular forms $f(q)$:

$$f(q) = \sum_n a_n q^n \quad \longrightarrow \quad L(f,s) \;=\; \sum_n \frac{a_n}{n^s}.$$

Class II reduction

- Associated to Hecke characters Ψ_K are Hecke L-series

$$L(\Psi, s) \;=\; \sum_n \frac{b_n(\Psi)}{n^s},$$

where the coefficients b_n are determined by the ideals of the field K

$$b_n(\Psi) \;=\; \sum_{\substack{(\mathfrak{a},\mathfrak{m})=1 \\ \mathrm{N}\mathfrak{a}=n}} \Psi(\mathfrak{a}).$$

Result II.
For each CHL_N model of class II there exists an algebraic Hecke character Ψ^N such that the L-series $L(f^N, s)$ of the Maaß-Skoruppa roots f^N are given by

$$L(f^N, s) \;=\; L(\Psi^N, s).$$

Example of class II reduction

Example:
For CHL_N with $N = 4$ the Maaß-Skoruppa root is given by

$$f^{\mathrm{B},4}(\tau) \;=\; \eta(\tau)^4\eta(2\tau)^2\eta(4\tau)^4 \;\in\; S_5(\Gamma_0(4), \chi_{-1})$$

This modular form can be constructed in terms of an algebraic Hecke character Ψ_4 associated to the Gauss field

$$K \;=\; \mathbb{Q}(\sqrt{-1})$$

with

$$L(f^{\mathrm{B},4}, s) \;=\; L(\Psi_4^4, s).$$

The character itself Ψ_4 is elliptic, associated to the elliptic curve E_{32} of conductor 32

$$L(E_{32}, s) \;=\; L(\Psi_4, s).$$

Black holes as motivic probes

The result:
All Maaß-Skoruppa roots f^N of the CHL_N black hole Siegel modular forms Φ^N can be constructed from modular cusp forms of weight 2 associated to elliptic curves E.

This means that the compactification manifold of the CHL models

$$X = \left(K3xT^2\right)/\mathbb{Z}_N$$

is not fully probed by the Jatkar-Sen and Govindarajan-Krishna black holes.

More sensitive probes?

The Siegel modular forms constructed by Jatkar-Sen and Govindarajan-Krishna apply only to black holes with special charges that satisfy the torsion constraint

$$t = gcd\{Q_e^i Q_m^j - Q_e^j Q_m^i | i, j = 1, ..., n\} = 1$$

It would be interesting to check whether black holes with more general charges, leading to

$$t > 1$$

lead to more refined probes

$$t > 1 \quad \rightarrow \quad \Phi_t^N(\tau, \sigma, \rho) \quad \rightarrow \quad f_t^{BH}(q)$$

More sensitive probes?

Antoniadis-Hohenegger09:

There are further mixed couplings $\sim F_-^2 R_+^2$ that have been shown to provide corrections to the black hole entropy for $\mathrm{Het}(T^6) \cong \mathrm{IIA}(K3 \times T^2)$.

Perhaps these couplings provide more motivic information.

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