ENTANGLEMENT & HOLOGRAPHY

Mark Van Raamsdonk

Motivation: - AdS/CFT provides non-perturbative definition of certain quantum gravity theories

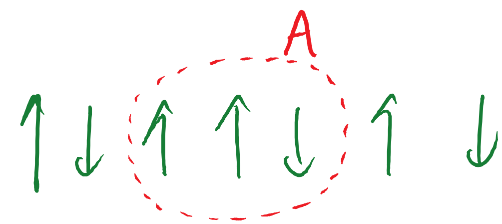
- Important question: how + why do spacetime/gravity emerge from CFT physics

- Recent work: measures of entanglement in CFT provide direct window into dual spacetime
  - spacetime "emerges" by entangling CFT d.o.f.
  - gravitational dynamics from physics of entanglement.

ENTANGLEMENT IN QUANTUM MECHANICS

QM: $|\psi\rangle \in \mathcal{H}$ "pure state"

Often $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$



For $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, what is the "state" of subsystem A?

Naive: find $|\psi_A\rangle \in \mathcal{H}_A$ s.t. $\forall\, \mathcal{O}_A$,

$$\langle\psi|\mathcal{O}_A \otimes \mathbb{1}|\psi\rangle = \langle\psi_A|\mathcal{O}_A|\psi_A\rangle$$

NOT POSSIBLE for general $|\psi_A\rangle$

The Density Matrix

$$\text{Let } |\psi\rangle = \sum c_{iI}\, \overset{\text{basis of } \mathcal{H}_A}{|i\rangle} \otimes \overset{\text{basis of } \mathcal{H}_B}{|I\rangle}$$

$$\text{Then } \langle\psi|\, \mathcal{O}_A \otimes \mathbb{1}\, |\psi\rangle = \langle i|\, c^*_{iI}\, \mathcal{O}_A\, c_{jI}\, |j\rangle$$

$$= \mathrm{Tr}\left(\mathcal{O}_A\, \rho_A\right)$$

$$\rho_A \equiv \sum_J c_{jJ}\, c^*_{iJ}\, |j\rangle\langle i| \quad = \mathrm{Tr}_B\, |\psi\rangle\langle\psi|$$

(REDUCED) DENSITY MATRIX FOR SUBSYSTEM A.
(OPERATOR)

PROPERTIES OF ρ_A:

* $\rho_A : \mathcal{H}_A \to \mathcal{H}_A$ Hermitian $\rho_A = \rho_A^\dagger$

* ρ_A: non-negative eigenvalues (Homework: prove)
  $\{p_i\}$ $\sum p_i = 1$

* Can write: $\rho_A = \sum p_i |\psi_i\rangle\langle\psi_i|$ for orthogonal $|\psi_i\rangle$

- A is in a pure state $|\psi_i\rangle$ with classical probability p_i

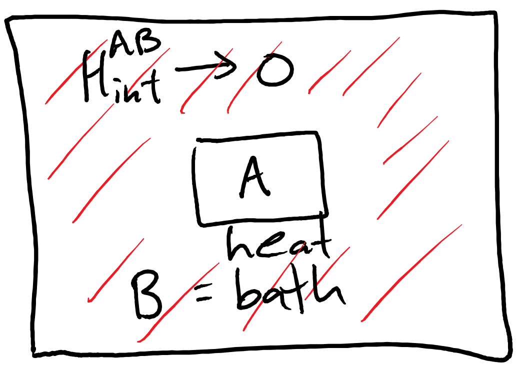
- For $\{p_i\} \neq \{1,0,0,\ldots\}$, A is in a MIXED STATE or ENSEMBLE familiar from quantum stat mech:

e.g. Microcanonical ensemble:

$$\rho_E = \sum_{i=1}^{n} \frac{1}{n} |E_i\rangle\langle E_i| \qquad E_i \in [E, E+dE]$$

Canonical ensemble: $|E_i\rangle$ w. probability $\propto e^{-\beta E_i}$

$$\rho_T = \frac{1}{Z} \sum_i e^{-\beta E_i} |E_i\rangle\langle E_i| = \frac{1}{Z} e^{-\beta H}$$



$$Z = \sum_i e^{-\beta E_i}$$

Aside: Same density matrix ρ_A can come from many different "PURIFICATIONS"

$$\rho_A = \sum p_i |\psi_i\rangle\langle\psi_i|$$

probabilities $\Rightarrow$ $\exists$ classical uncertainty about state.

· degree of uncertainty measured by ENTROPY

e.g.

$p_i = \frac{1}{n}$ want $S = \log(n)$

$i = 1 \ldots n$

to get EXTENSIVITY

$$\rho = \begin{pmatrix} \frac{1}{n} & & \\ & \ddots & \\ & & \frac{1}{n} \end{pmatrix}$$

$$S = \log(n)$$

$$= n \cdot \left\{ -\frac{1}{n} \log\left(\frac{1}{n}\right) \right\}$$

contribution of $-p_i \log p_i$ from each eigenvalue

Generally: $S[\rho] = \sum_i \{-p_i \log p_i\}$

$$= -\operatorname{tr}(\rho \log \rho)$$

$\boxed{\text{VON NEUMANN ENTROPY}}$

Pure state: $\{p_i\} = \{1, 0, 0, \dots\}$ $\quad S = 0$

HOMEWORK EXERCISE:

Maximize $S(\rho)$ subject to

$$\mathrm{tr}(\rho) = 1 \quad , \mathrm{tr}(\rho H) = E$$

ENTANGLEMENT

$$|\uparrow\rangle \otimes |\uparrow\rangle$$ NOT ENTANGLED

$$\frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle\right)$$ ENTANGLED

generally: $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$

A entangled with B $\Longleftrightarrow$

$$|\psi\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle$$

$$\{p_i\} \neq \{1, 0, 0, \dots\}$$

$$S(\rho_A) \neq 0$$

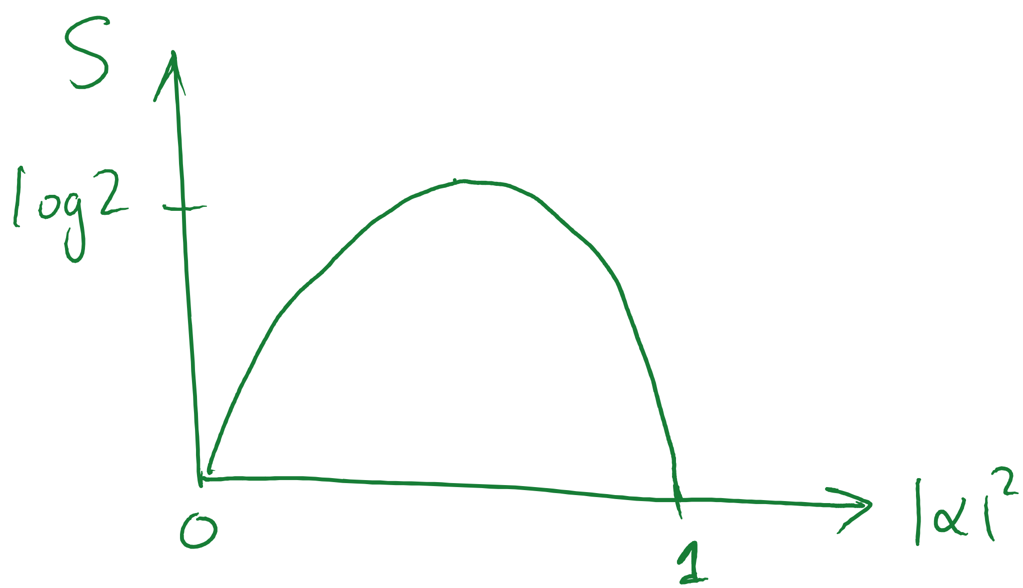
$S(\rho_A)$ quantifies the degree of entanglement "ENTANGLEMENT ENTROPY" e.g.

$$|\psi\rangle = \alpha|\uparrow\uparrow\rangle + \beta|\downarrow\downarrow\rangle$$

$$|\alpha|^2 + |\beta|^2 = 1$$

HOMEWORK EXERCISE: calculate ρ_A & S_A

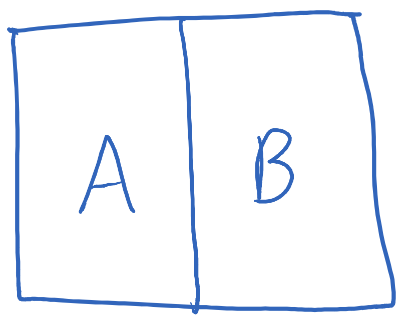
1st spin



PROPERTIES OF S

see: Nielsen & Chuang

1



for pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$

can write

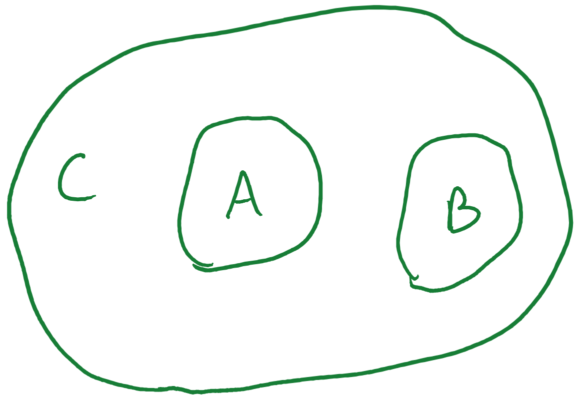
$$|\psi\rangle = \sum_i \sqrt{p_i}\,|\psi_i^A\rangle \otimes |\psi_i^B\rangle$$

SCHMIDT DECOMPOSITION

$\therefore$ spectrum of ρ_A = spectrum of ρ_B

$$\boxed{S_A = S_B}$$

SUBADDITIVITY



For general $\rho_{A\cup B}$:

$$|S_A - S_B| \leq S_{A\cup B}$$

follows from: $\boxed{S_A + S_B \geq S_{A\cup B}}$

ent of A w. rest — ent of B w. rest — ent of $A\cup B$ w. rest

equality iff

$$\rho_{AB} = \rho_A \otimes \rho_B$$

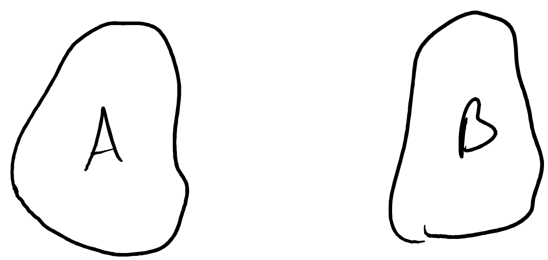
HOMEWORK: prove this

Define $I(A,B) \equiv S_A + S_B - S_{A\cup B}$

MUTUAL INFORMATION

measure of entanglement/correlations between A & B

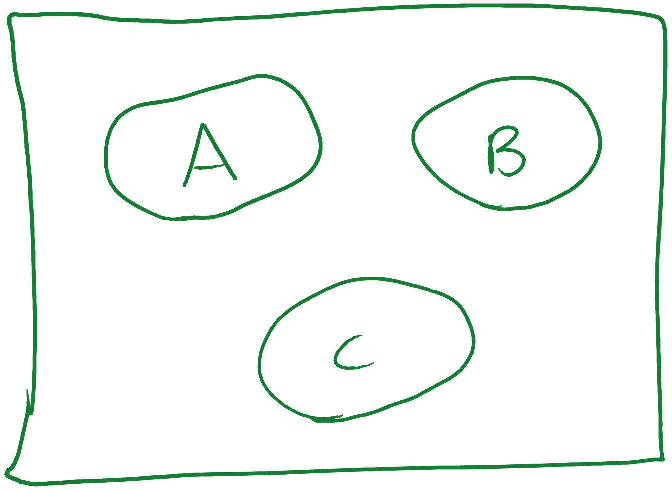
Mutual information provides upper bound on all correlations between A and B:



$$\frac{\langle O_A O_B \rangle - \langle O_A \rangle \langle O_B \rangle}{\langle |O_A|^2 \rangle^{\frac{1}{2}} \langle |O_B|^2 \rangle^{\frac{1}{2}}} \leq I(A,B)$$

O_A, O_B : any bounded operators in A, B

STRONG SUBADDITIVITY



$$\boxed{S_{A\cup C} + S_{B\cup C} \geq S_C + S_{A\cup B\cup C}}$$

HOMEWORK: find a simple proof
(+ publish, please)

ENTANGLEMENT IN QUANTUM FIELD THEORY

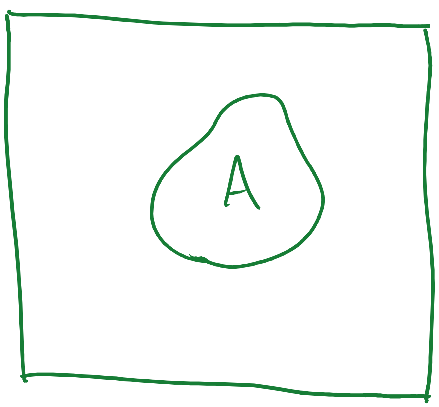
QFT: - many degrees of freedom

- many ways to decompose

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

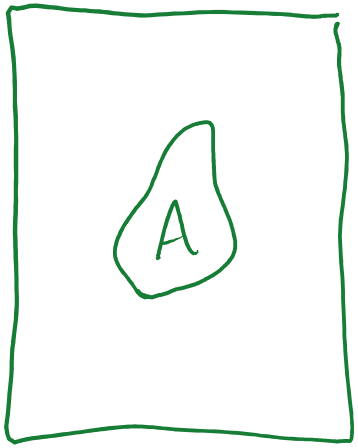
standard discussion:

take A = fields inside some spatial region



aside: could also take A = subset of d.o.f. in momentum space

e.g. single field theory mode

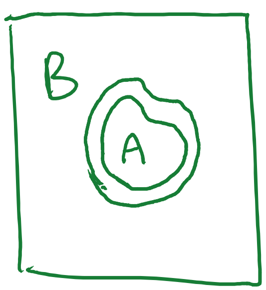


Spatial entanglement entropy DIVERGENT.

Leading divergence: $(\text{Area of } \partial A) \times \Lambda^{d-2}$

Can consider finite quantities instead:

① Look at mutual informations $S(A)+S(B)-S(A\cup B) \equiv I(A,B)$



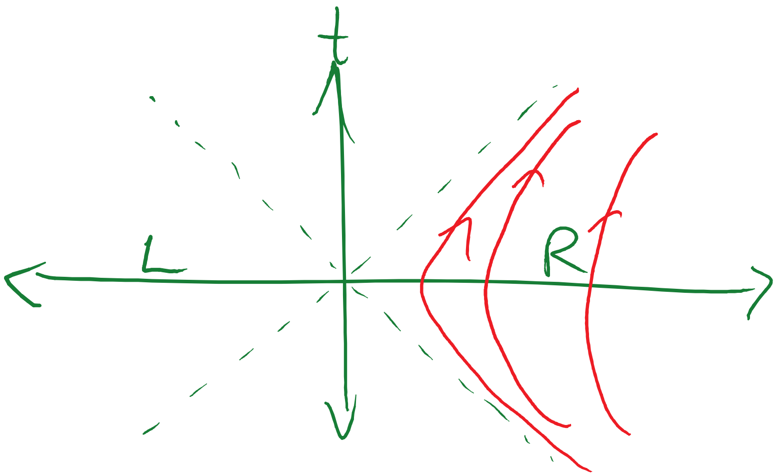
② Can choose B = points at distance $\geq \varepsilon$ from A $\longrightarrow$ gives regulated version of S

③ Can look at ΔS: difference between S for $|\psi\rangle$ and S for $|\text{vac}\rangle$

④ 2D: Can look at $\frac{d}{dL} S(L)$ interval of length L

Structure of entanglement in Minkowski space

vacuum:



Divide d.o.f. into L + R

Define H_R: generator of boosts

$|E_i^R\rangle$ eigenstates of H_R for QFT on right side

Then: $$|vac\rangle = \sum_i e^{-\beta E_i/2} |E_i^L\rangle \otimes |E_i^R\rangle$$

Unruh

Each mode on right entangled w. corresponding L mode.