Giovanni Ridolfi

Dipartimento di Fisica, Università di Genova and INFN Sezione di Genova
Via Dodecaneso 33, 16146 Genova, Italy

Three Lectures on Flavour Physics

This document contains a transcription of three lectures on flavour physics, given at the University of Cambridge (UK) in February, 2017 for PhD students in physics. There is no original content. Some of the material presented here, with minor modifications, has been published elsewhere, mainly in C. M. Becchi, G. Ridolfi, *An introduction to relativistic processes and the standard model of electroweak interactions*, Springer 2013, to which the reader is referred for notations and conventions. For up-to-date information on the present status of the comparison between theory and data, we refer the reader to the web page of the UTFit and CKMFitter collaborations, and to the Particle Data Group, where a complete bibliography can also be found.

1 Lecture One: Flavour

Let us consider the standard model with only one generation of quarks and leptons. Its lagrangian density is entirely specified by the assumption of local invariance with respect to the group $SU(2)_L \otimes U(1)_Y$, and by the following assignments of fermion fields to irreducible representations of the gauge group:

$$
\begin{aligned}
\psi_1 &= q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \sim (\mathbf{2}, 1/3) \\
\psi_2 &= u_R \sim (\mathbf{1}, 4/3) \\
\psi_3 &= d_R \sim (\mathbf{1}, -2/3) \\
\psi_4 &= \ell_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \sim (\mathbf{2}, -1) \\
\psi_5 &= e_R \sim (\mathbf{1}, -2).
\end{aligned}
$$

Here, the symbol $\sim$ means "transforms as"; the two numbers in brackets stand for the $SU(2)$ representation (**2** for the doublet, **1** for the singlet) and for the hypercharge quantum number $Y = 2(Q - T_3)$, respectively.

The $SU(2) \otimes U(1)$-invariant Lagrangian can be written in the following compact form:

$$
\mathcal{L}_{SU(2)\otimes U(1)} = \mathcal{L}_{\text{Yang-Mills}} + \sum_{k=1}^{5} \bar{\psi}_k \, i \not{D} \psi_k, \tag{1.1}
$$

where the sum runs over the five different irreducible representations of $SU(2)\otimes U(1)$ of fermions within one generation. Mass terms are forbidden by the gauge symmetry.

The Lagrangian density in eq. (1.1) is not yet suited to an accurate description of electroweak interactions, for two reasons:

1. the gauge symmetry is realized exactly, and all gauge vector bosons are massless, contrary to observations;

2. it is invariant under a large class of global transformations, most of which are not observed.

The way out of problem n. 1 is well known: the gauge symmetry must be spontaneously broken down to $U(1)_{\text{em}}$. The simplest way to do it is the minimal Higgs mechanism, whereby a doulbet of scalar fields is introduced with hypercharge $Y = 1$,

$$
\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \tag{1.2}
$$

and the term

$$
\mathcal{L}_{\text{Higgs}} = (D_\mu \phi)^\dagger D^\mu \phi - \mu^2 |\phi|^2 - \lambda |\phi|^4 \tag{1.3}
$$

is added to the gauge-invariant lagrangian. If $\mu^2 < 0$, spontaneous symmetry breaking is achieved by a non-zero vacuum expectation value of the scalar field

$$
\langle \phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}; \qquad v^2 = -\frac{\mu^2}{\lambda}. \tag{1.4}
$$

As a consequence, weak vector bosons acquire masses proportional to $v \sim 250$ GeV through gauge-invariant interactions with the scalar field. We will not elaborate on this issue any further here.

Rather, we concentrate on problem n. 2, namely the presence of a large, and largely undesirable, global symmetry in eq. (1.1). We note that the fermion fields within each representation can be multiplied by an arbitrary constant phase

$$\psi_k \to e^{i\phi_k}\psi_k \tag{1.5}$$

without affecting $\mathcal{L}_{SU(2)\otimes U(1)}$. This $[U(1)]^5$ global symmetry was not imposed: it is a consequence of the assumed local symmetry and of the renormalizability condition. For this reason, it is usually refferred to as an *accidental* symmetry. The five conserved currents corresponding to the global transformations (1.5) are

$$\begin{aligned}
J_1^\mu &= \bar{u}_L\gamma^\mu u_L + \bar{d}_L\gamma^\mu d_L \\
J_2^\mu &= \bar{u}_R\gamma^\mu u_R \\
J_3^\mu &= \bar{d}_R\gamma^\mu d_R \\
J_4^\mu &= \bar{\nu}_L\gamma^\mu \nu_L + \bar{e}_L\gamma^\mu e_L \\
J_5^\mu &= \bar{e}_R\gamma^\mu e_R .
\end{aligned}$$

It proves convenient to replace $J_1^\mu, \ldots, J_5^\mu$ by the following independent linear combinations:

$$\begin{aligned}
J_Y^\mu &= \sum_{k=1}^{5} \frac{Y_k}{2} J_k^\mu \\
J_b^\mu &= \frac{1}{3}(J_1^\mu + J_2^\mu + J_3^\mu) = \frac{1}{3}(\bar{u}\gamma^\mu u + \bar{d}\gamma^\mu d) \\
J_\ell^\mu &= J_4^\mu + J_5^\mu = \bar{\nu}_L\gamma^\mu\nu_L + \bar{e}\gamma^\mu e \\
J_{b5}^\mu &= -J_1^\mu + J_2^\mu + J_3^\mu = \bar{u}\gamma^\mu\gamma_5 u + \bar{d}\gamma^\mu\gamma_5 d. \\
J_{\ell 5}^\mu &= -J_4^\mu + J_5^\mu = \bar{\nu}_L\gamma^\mu\gamma_5\nu_L + \bar{e}\gamma^\mu\gamma_5 e
\end{aligned}$$

(we have used $\gamma_5\nu_L = -\nu_L$.) The current J_Y is the hypercharge current, which corresponds to a local invariance of the theory. The actual accidental symmetry is therefore $[U(1)]^4$, rather than $[U(1)]^5$.

The currents J_b and J_ℓ are immediately recognized to be the baryonic and leptonic number currents, respectively. The invariance of the Lagrangian under the corresponding global symmetries is welcome, because baryonic and leptonic number are known to be conserved to an extremely high accuracy. For example, the present bound on the proton lifetime is

$$\tau_p > 2.1 \cdot 10^{29}\ y. \tag{1.6}$$

The most accurate tests of lepton number conservation are provided by the following observables:

$$B(\mu \to e\gamma) \leq 1.2 \cdot 10^{-11}; \qquad B(\tau \to \mu\gamma) \leq 2.7 \cdot 10^{-6} \tag{1.7}$$

$$B(\mu \to 3e) \leq 1 \cdot 10^{-12} \tag{1.8}$$

$$\frac{\Gamma(\mu\ \mathrm{Ti} \to e\ \mathrm{Ti})}{\Gamma(\mu\ \mathrm{Ti} \to \mathrm{all})} \leq 4 \cdot 10^{-12}, \tag{1.9}$$

where B stands for the ratio between the rate of the indicated process and the total decay rate. On the other hand, experimental data are not compatible with the conservation of $J_{\ell 5}$ and J_{b5}, since they are incompatible with fermion mass terms. This part of the accidental symmetry must be explicitly broken.

We know that the structure of fermion fields outline above can be replicated a number n of times, with the only constraint (originating from the cancellation of the axial anomalies) that all five representations are replicated. This gives rise to the possible existence of families, or generations, of fermions. Experiments show the existence of three fermion families, which are distinguished on the basis of their masses. For this reason, the different fermion families are not distinguished one from the other in$\mathcal{L}_{SU(2)\otimes U(1)}$, which is therefore invariant under the group of global transformations

$$\psi_k \to U_k \psi_k, \tag{1.10}$$

where it is now understood that each ψ_k also carries a fermion generation index $f = 1, \ldots, n$, and U_k are constant, unitary $n \times n$ matrices in generation space. The group of accidental symmetries is therefore $[U(1)]^4 \otimes [SU(n)]^5$.

In order to build a realistic theory, this large global symmetry must be broken explicitly, without affecting gauge invariance and renormalizability. The solution to this puzzle arise in a natural way in the Standard Model, with the gauge symmetry spontaneously broken by the Higgs mechanism in its minimal realization. Indeed, it is easy to show that a Yukawa interaction term can be added to the Lagrangian density:

$$\begin{aligned} \mathcal{L}_{\rm Yukawa} \;\; = \;\; & -\left[\bar{q}_L\, h_{\rm U}\, u_R\, \tilde{\phi} + \tilde{\phi}^\dagger\, \bar{u}_R\, h_{\rm U}^\dagger\, q_L\right] \\ & -\left[\bar{q}_L\, h_{\rm D}\, d_R\, \phi + \phi^\dagger\, \bar{d}_R\, h_{\rm D}^\dagger\, q_L\,\right] \\ & -\left[\bar{\ell}_L\, h_{\rm L}\, e_R\, \phi + \phi^\dagger\, \bar{e}_R\, h_{\rm L}^\dagger\, \ell_L\right] \end{aligned} \tag{1.11}$$

where the scalar field $\tilde{\phi}$ is defined by

$$\tilde{\phi} = \epsilon \phi^* = \begin{pmatrix} \phi^{0*} \\ -\phi^- \end{pmatrix}, \tag{1.12}$$

and it can be shown to transform as a doublet under $SU(2)_L$. The matrices $h_{\rm U}$, $h_{\rm D}$, $h_{\rm L}$ are generic $n \times n$ constant complex matrices in the generation space. Generation indices are understood everywhere.

It should be noted that, because of accidental symmetries, Yukawa interactions cannot be generated by radiative corrections. This is the mechanism which keeps neutrinos massless, and protects fermion masses from receiving large radiative corrections.

The Yukawa lagrangian eq. (1.11) is manifestly Lorentz-invariant and renormalizable, since it only contains field operators of mass dimension 4 (or equivalently, the couplings $h_{\rm U}$, $h_{\rm D}$, $h_{\rm L}$ are dimensionless). It is also gauge invariant: invariance under $SU(2)_L$ is manifest, invariance under $U(1)_Y$ follows from the assignment $Y = 2(Q - T_3)$.

On the other hand, $\mathcal{L}_{\rm Yukawa}$ breaks the accidental symmetry (1.10) explicitly, as one can see by rewriting it as

$$\mathcal{L}_{\rm Yukawa} = -\left[\bar{\psi}_1\, h_{\rm U}\, \psi_2\, \tilde{\phi} + \bar{\psi}_1\, h_{\rm D}\, \psi_3\, \phi + \bar{\psi}_4\, h_{\rm L}\, \psi_5\, \phi\right] + \text{h.c.} \tag{1.13}$$

In particular, after spontaneous symmetry breaking, $\mathcal{L}_{\rm Yukawa}$ contains bilinear terms in the fermion fields which can eventually be interpreted as mass terms. Our first task is that of identifying the fermion fields with definite mass. To this purpose, we observe that after spontaneous symmetry breaking the bilinear term in $\mathcal{L}_{\rm Yukawa}$ is given by

$$\mathcal{L}^{(2)}_{\rm Yukawa} = -\frac{v}{\sqrt{2}} \sum_{f,g=1}^{n} \left[\bar{u}^f_L \, h^{fg}_{\rm U} \, u^g_R + \bar{d}^f_L \, h^{fg}_{\rm D} \, d^g_R + \bar{e}^f_L \, h^{fg}_{\rm L} \, e^g_R \right] + \text{h.c.} \tag{1.14}$$

The matrices $h_{\rm U,D,L}$ are not diagonal; however, a result in linear algebra called *singular value decomposition* guarantees that any complex (even rectangular) matrix h can be decomposed as

$$h = U \hat{h} V^\dagger, \tag{1.15}$$

where U, V are unitary matrices, and $\hat{h}$ is diagonal, with real non-negative diagonal entries. We may find such a decomposition for the Yukawa coupling matrices $h_{\rm U,D,L}$:

$$h_{\rm U} = U_{\rm U} \hat{h}_{\rm U} V^\dagger_{\rm U} \tag{1.16}$$
$$h_{\rm D} = U_{\rm D} \hat{h}_{\rm D} V^\dagger_{\rm D} \tag{1.17}$$
$$h_{\rm L} = U_{\rm L} \hat{h}_{\rm L} V^\dagger_{\rm L} \tag{1.18}$$

and redefine the fermion fields according to

$$u_L \to U_{\rm U} u_L, \quad u_R \to V_{\rm U} u_R \tag{1.19}$$
$$d_L \to U_{\rm D} d_L, \quad d_R \to V_{\rm D} d_R \tag{1.20}$$
$$e_L \to U_{\rm L} e_L, \quad e_R \to V_{\rm L} e_R \tag{1.21}$$

so that the bilinear part of eq. (1.11) now reads

$$\mathcal{L}^{(2)}_{\rm Yukawa} = -\frac{v}{\sqrt{2}} \sum_{f=1}^{n} \left[\bar{u}^f \, \hat{h}^{ff}_{\rm U} \, u^f + \bar{d}^f \, h^{ff}_{\rm D} \, d^f + \bar{e}^f \, h^{ff}_{\rm L} \, e^f \right]. \tag{1.22}$$

We can now identify the quark masses with

$$m^f_{\rm U} = \frac{v \hat{h}^{ff}_{\rm U}}{\sqrt{2}}, \qquad m^f_{\rm D} = \frac{v \hat{h}^{ff}_{\rm D}}{\sqrt{2}}, \qquad m^f_{\rm E} = \frac{v \hat{h}^{ff}_{\rm E}}{\sqrt{2}}. \tag{1.23}$$

We must now figure out how the rest of the lagrangian is modified by the transformations eqs. (1.19,1.20,1.21). Since the matrices $U_{\rm U,D,L}, V_{\rm U,D,L}$ are constant in space-time, eqs. (1.19,1.20,1.21) obviously leave the free quark Lagrangian unchanged. They also leave unchanged neutral-current interaction terms, because of the universality of the fermion couplings of different families to the photon and the Z. This is an important result: no flavour-mixing terms arise in the neutral-current part of the electroweak interactions, consistently with observations.

Thanks to the absence of right-handed neutrino fields, the lepton charged-current interaction term is also unaffected by the transformation (1.21); indeed, we may transform the left-handed neutrino field as the left-handed charged leptons, i.e.

$$\nu_L \to U_L \nu_L, \tag{1.24}$$

so that

$$\mathcal{L}_c^{\rm lepton} = \frac{g}{\sqrt{2}}\, W_\mu^+ \bar\nu_L\, \gamma^\mu\, e_L + {\rm h.c.} \;\; \rightarrow \;\; \frac{g}{\sqrt{2}}\, W_\mu^+ \bar\nu_L\, U_{\rm L}^\dagger \gamma^\mu\, U_{\rm L}\, e_L + {\rm h.c.}$$
$$= \frac{g}{\sqrt{2}}\, W_\mu^+ \bar\nu_L\, \gamma^\mu\, e_L + {\rm h.c.} \tag{1.25}$$

since U_L is unitary. In the Standard Model with massless neutrinos, no flavour mixing arises in the leptonic sector.

The only term in $\mathcal{L}_{SU(2)\otimes U(1)}$ which is affected by the transformations in eqs. (1.19,1.20) is the charged-current quark interaction, because the up and down components of the same left-handed $SU(2)$ doublet are transformed in different ways. Indeed, we find

$$\mathcal{L}_c^{\rm quark} = \frac{g}{\sqrt{2}}\, W_\mu^+ \bar u_L\, \gamma^\mu\, d_L + {\rm h.c.} \;\; \rightarrow \;\; \frac{g}{\sqrt{2}}\, W_\mu^+ \bar u_L\, U_{\rm U}^\dagger \gamma^\mu\, U_{\rm D}\, d_L + {\rm h.c.}$$
$$= \frac{g}{\sqrt{2}}\, W_\mu^+ \bar u_L\, \gamma^\mu\, V\, d_L^g + {\rm h.c.} \tag{1.26}$$

where we have defined

$$V = U_{\rm U}^\dagger V_{\rm D}. \tag{1.27}$$

The matrix V is usually called the *Cabibbo-Kobayashi-Maskawa* (CKM) matrix. It is a unitary matrix, and its unitarity guarantees the suppression of flavour-changing neutral currents. The elements of V are fundamental parameters of the standard model Lagrangian, on the same footing as masses and gauge couplings, and must be extracted from experiments.

Finally, the Yukawa lagrangian eq. (1.11) is transformed as follows:

$$\begin{aligned}
&\mathcal{L}_{\rm Yukawa} \\
= \;& -\bar u_L U_{\rm U} \hat h_{\rm U} V_{\rm U}^\dagger u_R \phi^{0*} + \bar d_L U_{\rm U} \hat h_{\rm U} V_{\rm U}^\dagger u_R \phi^- - \bar u_L U_{\rm D} \hat h_{\rm D} V_{\rm D}^\dagger d_R \phi^+ - \bar d_L U_{\rm D} \hat h_{\rm D} V_{\rm D}^\dagger d_R \phi^0 \\
& -\bar\nu_L U_{\rm L} \hat h_{\rm L} V_{\rm L}^\dagger e_R \phi^+ - \bar e_L U_{\rm L} \hat h_{\rm L} V_{\rm L}^\dagger e_R \phi^0 + {\rm h.c.}. \\
\rightarrow \;& -\bar u_L \hat h_{\rm U} u_R \phi^{0*} + \bar d_L V^\dagger \hat h_{\rm U} u_R \phi^- - \bar u_L \hat h_{\rm D} V d_R \phi^+ - \bar d_L \hat h_{\rm D} d_R \phi^0 \\
& -\bar\nu_L \hat h_{\rm L} e_R \phi^+ - \bar e_L \hat h_{\rm L} e_R \phi^0 + {\rm h.c.}.
\end{aligned} \tag{1.28}$$

Notice that the CKM matrix appears here only in terms involving the charged scalar component, which is in fact a non-physical would-be Goldstone bosons; it is set to zero in the unitary gauge.

We now determine the number of independent parameters in the CKM matrix. A generic $n \times n$ unitary matrix depends on n^2 independent real parameters (the easiest way to see this is writing V as the exponential of i times a hermitian matrix.) Some (n_A) of them can be thought of as rotation angles in the n-dimensional space of generations, and they are as many as the coordinate planes in n dimensions:

$$n_A = \begin{pmatrix} n \\ 2 \end{pmatrix} = \frac{1}{2} n(n-1). \tag{1.29}$$

The remaining

$$\hat n_P = n^2 - n_A = \frac{1}{2} n(n+1) \tag{1.30}$$

parameters are complex phases. Some of them can be eliminated from the Lagrangian density by redefining the left-handed quark fields as

$$u_L^f \to e^{i\alpha_f}\, u_L^f; \qquad d_L^g \to e^{i\beta_g}\, d_L^g, \tag{1.31}$$

with α_f, β_g real constants. Indeed, the transformations eq. (1.31) are symmetry transformations for the whole standard model Lagrangian except $\mathcal{L}_c^{\rm hadr}$, and therefore amount to a redefinition of the CKM matrix:

$$V_{fg} \to e^{i(\beta_g - \alpha_f)}\, V_{fg}. \tag{1.32}$$

The $2n$ constants α_f, β_g can be chosen so that $2n-1$ phases are eliminated from the matrix V, since there are $2n-1$ independent differences $\beta_g - \alpha_f$. The number of really independent complex phases in V is therefore

$$n_P = \hat{n}_P - (2n-1) = \frac{1}{2}(n-1)(n-2). \tag{1.33}$$

Observe that, with one or two fermion families, the CKM matrix can be made real. The first case with non-trivial phases is $n = 3$, which corresponds to $n_P = 1$. In the standard model with three fermion families, the CKM matrix has four independent parameters: three rotation angles and one complex phase. In the general case, the total number of independent parameters in the CKM matrix is

$$n_A + n_P = (n-1)^2. \tag{1.34}$$

The presence of complex coupling constants implies violation of the CP symmetry. CP violation phenomena in weak interactions were first observed around 1964 in the $K^0 - \overline{K}^0$ system; the existence of a third quark generation may therefore be considered as a prediction of the standard model, confirmed by the discovery of the b and t quarks.

The elements of the CKM matrix are fundamental parameters of the Standard Model: their values are not predicted by the theory, and must be extracted from experiment. For example, it was soon recognized, by the study of β decays nd of the decays of strange baryons, that $V_{ud} \simeq \cos\theta_c$ and $V_{us} \simeq \cos\theta_c$, where θ_c, the Cabibbo angle, is rather small: $\sin\theta_C = 0.22$. The mixing between the first and the third generation is even smaller: decays of B mesons into final states with no charmed particles are extremely rare, and were observed only recently. This results in a small value of $|V_{ub}|$, of the order of $\sin^3\theta_c$. These simple considerations suggest that a convenient way of parametrizing the CKM matrix is an expansion in powers of $\lambda = \sin\theta_c$, as suggested by L. Wolfenstein. One finds

$$V = \begin{bmatrix} 1-\lambda^2/2 & \lambda & \lambda^3 A(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{bmatrix} + O(\lambda^4). \tag{1.35}$$

The parameters A, ρ, η turn out to be of order 1.

As mentioned above, the unitarity of V is a crucial ingredient of the Standard Model, because it guarantees the absence of flavour-changing neutral current weak processes, which are observed to be very rare. Conversely, any signal of non-unitarity of the CKM matrix can in principle be interpreted as a deviation from the Standard Model. For this reason, experimental tests of unitarity are of great relevance, and have been perfomed by experimentalists to a great degree of accuracy. It is convenient to define the combinations

$$\xi_i^{\alpha\beta} = V_{i\alpha} V_{i\beta}^*, \tag{1.36}$$

where $i = u, c, t$ and $\alpha, \beta = d, s, b$. Unitarity of V corresponds to the six relationships

$$\sum_i \xi_i^{\alpha\beta} = \delta^{\alpha\beta}. \tag{1.37}$$

For $\alpha = \beta$, this equations are real, while the three phase-dependent relations

$$\sum_i \xi_i^{\alpha\beta} = 0; \qquad \alpha \neq \beta \tag{1.38}$$

can be represented as triangles in the complex plane. Note that the CKM entries $V_{i\alpha}$ are phase-convention-dependent quantities, because they can be modified by phase transformations of the fermion fields, but the relative phases of $\xi_i^{\alpha\beta}$ for fixed α, β are not; hence, a generic phase transformation of the quark fields has the effect of rotating rigidly each of the unitarity triangles (1.38), but does not affect its sides and internal angles.

It turns out that, among the three triangles in eq. (1.38), only one has all three sides of comparable sizes, namely the one corresponding to $\alpha = d, \beta = b$. This is immediately seen by inspection of eq. (1.35): the ds triangle has two sides of order λ and one of order λ^5, while the sb triangle has two sides of order λ^2 and one of order λ^4, and therefore are nearly degenerate. On the contrary, the sides of the db triangle are all of order λ^3, which is the reason why it is often referred to as *the* unitarity triangle. It is customary to rewrite the corresponding unitarity relation as

$$\frac{\xi_u^{db}}{\xi_c^{db}} + \frac{\xi_t^{db}}{\xi_c^{db}} + 1 = 0 \tag{1.39}$$

and to define

$$\bar\rho + i\bar\eta = -\frac{\xi_u^{db}}{\xi_c^{db}}, \tag{1.40}$$

so that the unitarity triangle has one side between 0 and 1 along the real axis in the $\bar\rho, \bar\eta$ complex plane, and the opposite vertex in $\bar\rho + i\bar\eta$. The parameters $\bar\rho, \bar\eta$ are related ro ρ, η in the Wolfenstein parametrization by

$$\bar\rho + i\bar\eta = \left(1 - \frac{\lambda^2}{2}\right)(\rho + i\eta). \tag{1.41}$$

The present status of the experimental tests of the unitarity triangle is shown in fig. 1. As one can see, the CKM picture of flavor mixing appears to be consistent with observations. In the next lecture we shall study in some detail how one of these constraints is imposed on the entries of the CKM matrix.

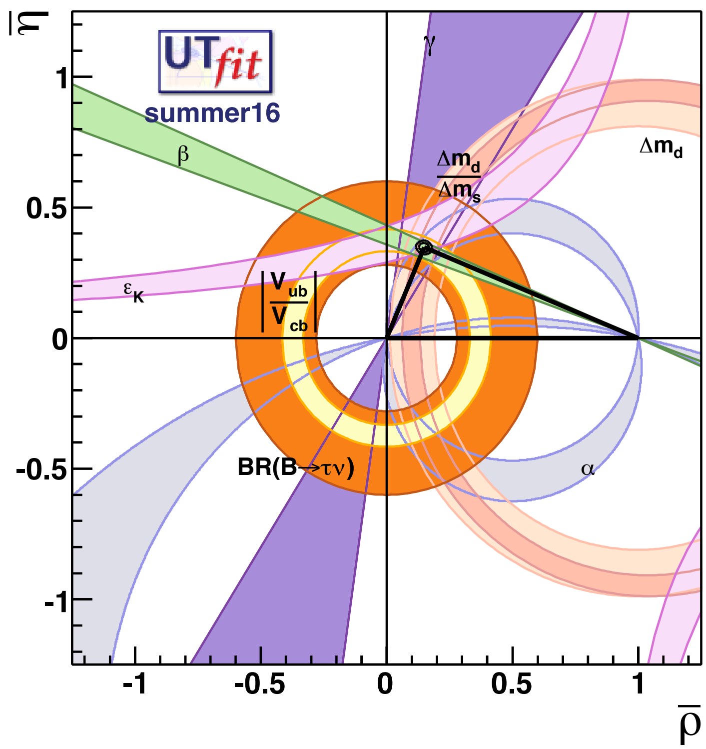

Figure 1: Present constraints on the unitarity triangle.

2 Lecture Two: An example

One interesting example of the experimental constraints on the CKM matrix entries is provided by the $K^0 - \overline{K}^0$ system. The K^0 meson is a pseudoscalar, spin-0 particle with a mass of about 490 MeV and a definite value of the strangeness quantum number $S = +1$.

Let us begin by reviewing the general properties of $K^0 - \overline{K}^0$ systems. Since K^0 mesons are produced by strong interaction processes (for example, $\pi^- + p \to K^0 + \Lambda^0$), at the initial time $t = 0$ the K meson is in a quantum state with a definite value of the strangeness quantum number S ($S = +1$ for K^0 and $S = -1$ for $\overline{K}^0$). K^0 mesons are stable with respect to strong interactions, because of strangeness conservation, but weak interactions induce vaious decay modes; furthermore, the transition $K^0 \to \overline{K}^0$ is also allowed, since strangeness is not conserved by the weak hamiltonian. It is therefore appropriate to introduce a two-dimensional subspace of the full state space, spanned by the two states $|K^0\rangle$ and $|\overline{K}^0\rangle$:

$$|K^0\rangle \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |\overline{K}^0\rangle \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{2.1}$$

The Schrödinger equation for a generic state in this subspace,

$$|\Psi(t)\rangle = \Psi_1(t)|K^0\rangle + \Psi_2(t)|\overline{K}^0\rangle = \begin{bmatrix} \Psi_1(t) \\ \Psi_2(t) \end{bmatrix}. \tag{2.2}$$

has the form

$$i\frac{\partial}{\partial t}|\Psi(t)\rangle = H^{\text{eff}}|\Psi(t)\rangle, \tag{2.3}$$

where the effective hamiltonian H^{eff} is a generic 2×2 matrix, in general non-hermitian in order to account for decay processes outside the subspace. The stationary states (that is, states with definite mass and lifetime) are linear combinations of $|K^0\rangle$ and $|\overline{K}^0\rangle$ that diagonalize the matrix H^{eff}. It is always possible to decompose the effective hamiltonian as

$$H^{\text{eff}} = M - \frac{i}{2}\Gamma, \tag{2.4}$$

where M and Γ are both hermitian matrices:

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{22} \end{bmatrix}; \quad \Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{12}^* & \Gamma_{22} \end{bmatrix}. \tag{2.5}$$

It can be shown that CPT invariance implies

$$M_{11} = M_{22} \equiv m_{\text{K}}; \qquad \Gamma_{11} = \Gamma_{22} \equiv \gamma. \tag{2.6}$$

We now diagonalize the matrix H^{eff}. The two eigenvalues are

$$M_{\text{S,L}} - \frac{i}{2}\Gamma_{\text{S,L}} = m_{\text{K}} - \frac{i}{2}\gamma \pm R, \tag{2.7}$$

where

$$R = \sqrt{\left(M_{12} - \frac{i}{2}\Gamma_{12}\right)\left(M_{12}^* - \frac{i}{2}\Gamma_{12}^*\right)}. \tag{2.8}$$

Therefore, mass and lifetime differences between the two eigenstates are

$$\Delta M = M_{\text{S}} - M_{\text{L}} = 2\text{Re}\, R \tag{2.9}$$

$$\Delta\Gamma = \Gamma_{\text{S}} - \Gamma_{\text{L}} = -4\text{Im}\, R. \tag{2.10}$$

The measured values for masses and lifetimes are (PDG 2016)

$$m_{\text{K}} = \frac{M_{\text{S}} + M_{\text{L}}}{2} = 497.611 \pm 0.013 \text{ MeV} \tag{2.11}$$

$$\Delta M = (-0.5289 \pm 0.0010) \times 10^{10}\ s^{-1} \simeq -3.53 \times 10^{-6} \text{ eV} \tag{2.12}$$

$$\Gamma_{\text{L}} = 1.93 \times 10^7 \text{ s}^{-1} \tag{2.13}$$

$$\Gamma_{\text{S}} = 1.12 \times 10^{10} \text{ s}^{-1}. \tag{2.14}$$

Notice that $\Gamma_{\text{L}} \ll \Gamma_{\text{S}}$, and that

$$\Delta M \simeq -\frac{1}{2}\Delta\Gamma. \tag{2.15}$$

The two eigenvectors are given by

$$|K_{\text{S}}\rangle = p|K^0\rangle + q|\overline{K}^0\rangle \tag{2.16}$$

$$|K_{\text{L}}\rangle = p|K^0\rangle - q|\overline{K}^0\rangle, \tag{2.17}$$

where

$$\frac{q}{p} = \sqrt{\frac{M_{12}^* - \frac{i}{2}\Gamma_{12}^*}{M_{12} - \frac{i}{2}\Gamma_{12}}}; \qquad |p|^2 + |q|^2 = 1. \tag{2.18}$$

Equations (2.18) fix q and p up to a common phase. The indices L (for *long*) and S (for *short*) reflect the fact that the two eigenstates have very different lifetimes.

We are now ready to solve the problem of time evolution of K^0 meson states. Integrating eq. (2.3), one immediately obtains

$$|K_{\rm S}(t)\rangle = e^{-i\left(M_{\rm S} - \frac{i}{2}\Gamma_{\rm S}\right)t}|K_{\rm S}(0)\rangle \tag{2.19}$$

$$|K_{\rm L}(t)\rangle = e^{-i\left(M_{\rm L} - \frac{i}{2}\Gamma_{\rm L}\right)t}|K_{\rm L}(0)\rangle. \tag{2.20}$$

The time evolution of a generic state $|\Psi(t)\rangle$ is obtained by expanding $|\Psi(0)\rangle$ on the $|K_{\rm S}\rangle, |K_{\rm L}\rangle$ basis and then using eqs. (2.19-2.20). For example, consider a beam composed of $|K^0\rangle$ mesons only, produced at $t = 0$. From eqs. (2.16-2.17),

$$|\Psi(0)\rangle = |K^0\rangle = \frac{1}{2p}\left(|K_{\rm S}(0)\rangle + |K_{\rm L}(0)\rangle\right), \tag{2.21}$$

which in turn implies

$$|\Psi(t)\rangle = \frac{1}{2p}\left[|K_{\rm S}(t)\rangle + |K_{\rm L}(t)\rangle\right] = f_+(t)|K^0\rangle + \frac{q}{p}f_-(t)|\overline{K}^0\rangle, \tag{2.22}$$

where

$$f_\pm(t) = \frac{1}{2}\left[e^{-i\left(M_{\rm S} - \frac{i}{2}\Gamma_{\rm S}\right)t} \pm e^{-i\left(M_{\rm L} - \frac{i}{2}\Gamma_{\rm L}\right)t}\right]. \tag{2.23}$$

The probability of finding a K^0 in the beam after a time t is proportional to

$$|\langle K^0|\Psi(t)\rangle|^2 = |f_+(t)|^2 = \frac{1}{4}\left[e^{-\Gamma_{\rm S}t} + e^{-\Gamma_{\rm L}t} + 2e^{-\frac{\Gamma_{\rm S}+\Gamma_{\rm L}}{2}t}\cos(\Delta M t)\right] \tag{2.24}$$

and analogously the probability of finding a $\overline{K}^0$ is proportional to

$$|\langle\overline{K}^0|\Psi(t)\rangle|^2 = \left|\frac{q}{p}\right|^2 |f_-(t)|^2 = \left|\frac{q}{p}\right|^2 \frac{1}{4}\left[e^{-\Gamma_{\rm S}t} + e^{-\Gamma_{\rm L}t} - 2e^{-\frac{\Gamma_{\rm S}+\Gamma_{\rm L}}{2}t}\cos(\Delta M t)\right]. \tag{2.25}$$

The time-integrated fractions of K^0 and $\overline{K}^0$ in the beam are sometimes useful. We find

$$\frac{\int_0^{+\infty} dt\,|f_+(t)|^2}{\int_0^{+\infty} dt\left[|f_+(t)|^2 + |f_-(t)|^2\right]} = \frac{1}{2}\frac{2 + x^2 - y^2}{1 + x^2} \tag{2.26}$$

$$\frac{\int_0^{+\infty} dt\,|f_-(t)|^2}{\int_0^{+\infty} dt\left[|f_+(t)|^2 + |f_-(t)|^2\right]} = \frac{1}{2}\frac{x^2 + y^2}{1 + x^2}, \tag{2.27}$$

where

$$x = \frac{\Delta M}{\Gamma}; \qquad y = \frac{\Delta\Gamma}{2\Gamma} \tag{2.28}$$

and $\Gamma = (\Gamma_S + \Gamma_L)/2$. Both quantities in eqs. (2.26,2.27) are approximately equal to 1/2 in the case of K^0 mesons.

The mass difference ΔM can be determined using semileptonic decays of K^0 and $\overline{K}^0$. As shown above, a beam of K^0 mesons at time $t = 0$ will contain at time $t > 0$ both K^0 and $\overline{K}^0$ with probabilities given by eqs. (2.24) and (2.25) respectively. Neglecting for the moment CP-violation effects (which amounts to assuming $|q/p| = 1$) we find

$$\frac{N(K^0) - N(\overline{K}^0)}{N(K^0) + N(\overline{K}^0)} = \frac{2\cos(\Delta M t)}{e^{-\frac{\Delta\Gamma t}{2}} + e^{\frac{\Delta\Gamma t}{2}}}, \tag{2.29}$$

where $N(K^0)$ ($N(\overline{K}^0)$) denotes the number of K^0 ($\overline{K}^0$) mesons in the beam at the time t. These numbers can be determined experimentally by studying the semileptonic decays of K^0 mesons:

$$K^0 \rightarrow \pi^- e^+ \nu_e \tag{2.30}$$
$$\overline{K}^0 \rightarrow \pi^+ e^- \bar{\nu}_e. \tag{2.31}$$

The same processes with opposite sign of the lepton charges are forbidden. As a consequence, $N(K^0)$ is proportional to the number of observed *positrons*, and $N(\overline{K}^0)$ to the number of observed *electrons*, and the quantity in eq. (2.29) can be experimentally determined as a function of time.

We now turn to an analysis of CP violation. We define the CP transformation as

$$CP|K^0\rangle = e^{i\alpha}|\overline{K}^0\rangle \qquad CP|\overline{K}^0\rangle = e^{-i\alpha}|K^0\rangle, \tag{2.32}$$

where α is an arbitrary phase. We begin by proving the following, important result:

$|K_S\rangle$ and $|K_L\rangle$ are CP eigenstates $\Leftrightarrow \mathrm{Im}(M_{12}\Gamma_{12}^*) = 0$.

The proof is straightforward: let us assume that $|K_S\rangle$ and $|K_L\rangle$ are eigenstates of the CP operator, defined as in eq. (2.32), with eigenvalues $+1$ and -1 respectively:

$$CP|K_S\rangle = pe^{i\alpha}|\overline{K}^0\rangle + qe^{-i\alpha}|K^0\rangle = p|K^0\rangle + q|\overline{K}^0\rangle \tag{2.33}$$
$$CP|K_L\rangle = pe^{i\alpha}|\overline{K}^0\rangle - qe^{-i\alpha}|K^0\rangle = -p|K^0\rangle + q|\overline{K}^0\rangle\,. \tag{2.34}$$

Equations (2.33,2.34) give

$$\frac{q}{p} = e^{i\alpha}. \tag{2.35}$$

On the other hand, using eq. (2.18) we find

$$\left|\frac{q}{p}\right|^2 = \sqrt{\frac{|M_{12}|^2 + |\Gamma_{12}|^2/4 + \mathrm{Im}(M_{12}\Gamma_{12}^*)}{|M_{12}|^2 + |\Gamma_{12}|^2/4 - \mathrm{Im}(M_{12}\Gamma_{12}^*)}}. \tag{2.36}$$

The condition $|q/p|^2 = 1$ is therefore equivalent to

$$\mathrm{Im}(M_{12}\Gamma_{12}^*) = 0. \tag{2.37}$$

The argument can be reversed: if $\mathrm{Im}(M_{12}\Gamma_{12}^*) = 0$, then $q/p = e^{i\beta}$, and $|K_\mathrm{S}\rangle$, $|K_\mathrm{L}\rangle$ are CP eigenstates, provided the CP operator is defined by $CP|K^0\rangle = e^{i\beta}|\overline{K}^0\rangle$.

Is it possible to decide experimentally whether the eigenstates of the effective hamiltonian H^eff are also CP eigenstates? To answer this question, we must consider the decays of K^0 mesons into two-pion states, $|\pi^0\pi^0\rangle$ or $|\pi^+\pi^-\rangle$, and three-pion states, $|\pi^0\pi^0\pi^0\rangle$ or $|\pi^+\pi^-\pi^0\rangle$. This is because two-pion states are CP eigenstates with eigenvalue $+1$, while three-pion states are predominantly in a CP eigenstate with eigenvalue -1. As a consequence, if K_L and K_S are also CP eigenstates, we expect K_S to decay mainly into two pions, and K_L only into three pions. Studying these processes, we will see that another mechanism of CP violation, independent of $K^0 - \overline{K}^0$ mixing, can take place.

We define the following ratios of decay amplitudes:

$$\eta_\pm = \frac{\mathcal{A}(K_\mathrm{L} \to \pi^+\pi^-)}{\mathcal{A}(K_\mathrm{S} \to \pi^+\pi^-)}; \qquad \eta_{00} = \frac{\mathcal{A}(K_\mathrm{L} \to \pi^0\pi^0)}{\mathcal{A}(K_\mathrm{S} \to \pi^0\pi^0)} \tag{2.38}$$

Clearly, both $\eta_\pm$ and η_{00} vanish if CP is conserved, since in that case K_L would be a CP eigenstate with eigenvalue -1, and could not decay into two pions[1]. So, $\eta_\pm$ and η_{00} are the appropriate quantities to investigate CP violation. Decomposing the two-pion states in terms of states with definite isotopic spin I and using the familiar Clebsch-Gordan decomposition we find

$$\begin{aligned} |\pi^+\pi^-\rangle &= \sqrt{\frac{2}{3}}\,|\pi\pi, I=0\rangle + \sqrt{\frac{1}{3}}\,|\pi\pi, I=2\rangle \\ |\pi^0\pi^0\rangle &= -\sqrt{\frac{1}{3}}\,|\pi\pi, I=0\rangle + \sqrt{\frac{2}{3}}\,|\pi\pi, I=2\rangle. \end{aligned} \tag{2.39}$$

where we have used the fact that the possible values of the total isospin of two-pion states are $I = 0, 1, 2$, but $I = 1$ is forbidden by Bose statistic. Hence

$$\eta_\pm = \frac{\epsilon_0 + \frac{\omega}{\sqrt{2}}\epsilon_2}{1 + \frac{\omega}{\sqrt{2}}}; \qquad \eta_{00} = \frac{\epsilon_0 - \sqrt{2}\omega\epsilon_2}{1 - \sqrt{2}\omega}, \tag{2.40}$$

where

$$\epsilon_{0,2} = \frac{\mathcal{A}(K_\mathrm{L} \to \pi\pi, I=0,2)}{\mathcal{A}(K_\mathrm{S} \to \pi\pi, I=0,2)}; \qquad \omega = \frac{\mathcal{A}(K_\mathrm{S} \to \pi\pi, I=2)}{\mathcal{A}(K_\mathrm{S} \to \pi\pi, I=0)}. \tag{2.41}$$

The ratio ω turns out to be rather small, as a manifestation of the so-called $\Delta I = 1/2$ selection rule. The present (PDG 2016) measured values of $|\eta_\pm|, |\eta_{00}|$ are

$$|\eta_\pm| = \sqrt{\frac{\Gamma(K_\mathrm{L} \to \pi^+\pi^-)}{\Gamma(K_\mathrm{S} \to \pi^+\pi^-)}} = (2.232 \pm 0.011) \times 10^{-3} \tag{2.42}$$

$$|\eta_{00}| = \sqrt{\frac{\Gamma(K_\mathrm{L} \to \pi^0\pi^0)}{\Gamma(K_\mathrm{S} \to \pi^0\pi^0)}} = (2.220 \pm 0.011) \times 10^{-3}, \tag{2.43}$$

[1] Observe that the reversed statement for three pion states is not true: K_S can decay into three pions even if CP is an exact symmetry, since three pion states contain a small component with $CP = +1$.

which show that violation of the CP symmetry is an effect of order 10^{-3}, to be compared with parity and charge-conjugation violation in weak interactions, which are instead maximal.

Let us assume for a moment that the term of the weak hamiltonian which is responsible for K decays into $\pi\pi$ does not violate the CP symmetry. In this case we would have

$$\mathcal{A}(K^0 \to \pi\pi) = e^{i\alpha} \mathcal{A}(\overline{K}^0 \to \pi\pi) \tag{2.44}$$

for both $I = 0$ and $I = 2$ (the phase factor $\exp(i\alpha)$ originates from the definition of the CP operator, eq. (2.32)). This implies

$$\epsilon_0 = \epsilon_2 = \frac{p\mathcal{A}(K^0 \to \pi\pi) - q\mathcal{A}(\overline{K}^0 \to \pi\pi)}{p\mathcal{A}(K^0 \to \pi\pi) + q\mathcal{A}(\overline{K}^0 \to \pi\pi)} = \frac{1 - \frac{q}{p}e^{i\alpha}}{1 + \frac{q}{p}e^{i\alpha}} \tag{2.45}$$

and therefore, recalling eq. (2.40),

$$\eta_\pm = \eta_{00} = \epsilon_0. \tag{2.46}$$

Present (PDG 2016) experimental data show a small deviation from this picture of CP violation: indeed

$$\left| \frac{\eta_{00}}{\eta_\pm} \right| = 0.9950 \pm 0.0007, \tag{2.47}$$

which shows that direct CP violation (that is, CP violation in K decays) is suppressed by another factor of 10^{-3} with respect to CP violation in the mixing $K^0 - \overline{K}^0$. For the moment we neglect this small effect.

Eq. (2.45) can be used to relate ϵ_0 with the effective hamiltonian $M - i\Gamma/2$, which we will eventually compute within a given theory. To this purpose, we define

$$M_{12} = |M_{12}|\, e^{i(\phi+\delta)}; \qquad \Gamma_{12} = |\Gamma_{12}|\, e^{i\phi}. \tag{2.48}$$

Using eqs. (2.18) we obtain

$$\frac{q}{p} = e^{-i\phi} \sqrt{\frac{|M_{12}|\, e^{-i\delta} - \frac{i}{2}\, |\Gamma_{12}|}{|M_{12}|\, e^{i\delta} - \frac{i}{2}\, |\Gamma_{12}|}} \tag{2.49}$$

Comparing eqs. (2.42-2.43) with eq. (2.45) we conclude that $|q/p| = 1 + O(10^{-3})$. Therefore, the phase mismatch δ must be close to 0 or π. Expanding in powers of $\sin\delta$ to first order we obtain

$$\frac{q}{p} = e^{-i\phi} \left(1 - \frac{i\, |M_{12}| \sin\delta}{|M_{12}| \cos\delta - \frac{i}{2}\, |\Gamma_{12}|} \right) + \mathcal{O}(\sin^2\delta), \tag{2.50}$$

where, to this order, $\cos\delta = \pm 1$. Furthermore

$$R = \sqrt{\left(M_{12} - \frac{i}{2}\Gamma_{12} \right) \left(M_{12}^* - \frac{i}{2}\Gamma_{12}^* \right)} = |M_{12}| \cos\delta - \frac{i}{2}\, |\Gamma_{12}| + \mathcal{O}(\sin^2\delta). \tag{2.51}$$

The sign of $\cos\delta$ can now be determined recalling eqs. (2.9,2.10), which give

$$\Delta M = 2\, |M_{12}| \cos\delta + \mathcal{O}(\sin^2\delta), \qquad \Delta\Gamma = 2\, |\Gamma_{12}| + \mathcal{O}(\sin^2\delta). \tag{2.52}$$

Since experimentally $\Delta M < 0$, we conclude that $\cos\delta < 0$ and therefore δ is close to π. We may therefore rename $\delta \to \pi+\delta$ and expand around $\delta = 0$: $\sin(\pi+\delta) = -\sin\delta \simeq -\delta$, $\cos(\pi+\delta) \simeq -1$. Frmo eqs. (2.45,2.50) we obtain

$$\epsilon_0 = \frac{1-\frac{q}{p}e^{i\alpha}}{1+\frac{q}{p}e^{i\alpha}} = \frac{i}{2}\frac{|M_{12}|\,\delta}{|M_{12}|+\frac{i}{2}\,|\Gamma_{12}|} + \mathcal{O}(\delta^2), \tag{2.53}$$

where we have chosen the phase α in the definition of CP to be equal to the phase ϕ of Γ_{12}. Finally, the experimental fact that $\Delta M \simeq -\Delta\Gamma/2$, together with eq. (2.52) allow us to obtain the simple result

$$\epsilon_0 \simeq \frac{i\delta}{1+i} + \mathcal{O}(\delta^2) = \frac{\delta}{\sqrt{2}}\,e^{i\pi/4} + \mathcal{O}(\delta^2). \tag{2.54}$$

Notice that the value of $\pi/4$ we have obtained for the phase of ϵ_0 (and therefore, approximately, of $\eta_\pm$ and η_{00}) is an *experimental* value; it follows from the fact that CP violation effects are so small, which allowed us to neglect terms of order δ^2, and from the experimental observation that $\Delta M \simeq -\Delta\Gamma/2$ for kaons. This is not the case, for example, for B^0 mesons, for which $\Delta M \gg \Delta\Gamma$.

As mentioned above, experiments show small deviations from the equality $\eta_\pm = \eta_{00}$, which signal CP violation in K decays. In order to account for this small effect, we rewrite $\eta_\pm, \eta_{00}$ in the form

$$\eta_\pm = \epsilon_0 + \frac{1}{1+\omega/\sqrt{2}}\frac{\omega}{\sqrt{2}}(\epsilon_2-\epsilon_0) = \epsilon_K + \epsilon' + O(\omega^2) \tag{2.55}$$

$$\eta_{00} = \epsilon_0 - \frac{2}{1-\omega\sqrt{2}}\frac{\omega}{\sqrt{2}}(\epsilon_2-\epsilon_0) = \epsilon_K - 2\epsilon' + O(\omega^2), \tag{2.56}$$

where we have used the widely adopted notation

$$\epsilon_K \equiv \epsilon_0; \qquad \epsilon' \equiv \frac{\omega}{\sqrt{2}}(\epsilon_2-\epsilon_0) \tag{2.57}$$

and we have neglected terms of order ω^2, since the $\Delta I = 1/2$ rule is rather accurate in this context.

The quantity ϵ' signals direct CP violation. It can be extracted from measurements of the quantity

$$D = \frac{B(K_{\rm L}\to\pi^+\pi^-)/B(K_{\rm S}\to\pi^+\pi^-)}{B(K_{\rm L}\to\pi^0\pi^0)/B(K_{\rm S}\to\pi^0\pi^0)} = \left|\frac{\eta_\pm}{\eta_{00}}\right|^2 \simeq 1 + 6{\rm Re}\frac{\epsilon'}{\epsilon_K}. \tag{2.58}$$

The present (PDG 2016) world data average gives

$${\rm Re}\,\frac{\epsilon'}{\epsilon_K} = (1.66\pm 0.7)\times 10^{-3}. \tag{2.59}$$

2.1 $K^0 - \overline{K}^0$ mixing in the standard model

The transition $K^0 \to \overline{K}^0$ is a neutral current process where strangeness changes by two units; such process can take place within the standard model, but they involve diagrams with at least one loop and the exchange of two W bosons. The relevant diagram is shown in fig. 2. The corresponding amplitude can be interpreted as an effective lagrangian term for processes

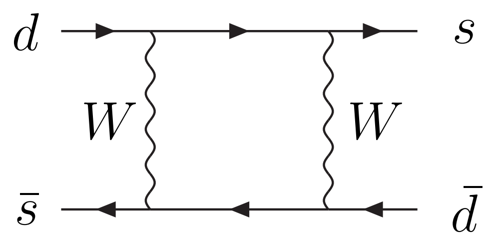

Figure 2: One loop diagram for $\Delta S = 2$ transitions in the standard model.

involving $d\bar{s} \to s\bar{d}$ transitions. In the limit of zero external momenta one finds

$$\mathcal{L}_{\rm eff}^{\Delta s=2} = \frac{G_F^2 m_{\rm W}^2}{64\pi^2} \left[\bar{s}\gamma^\mu(1-\gamma_5)d\right]^2 \sum_{i=u,c,t} \sum_{j=u,c,t} \xi_i \xi_j F(x_i, x_j) + \text{h.c.}, \tag{2.60}$$

where

$$\xi_i = V_{is} V_{id}^*, \qquad x_i = \frac{m_i^2}{m_{\rm W}^2} \tag{2.61}$$

and

$$F(x_i, x_j) = x_i x_j \frac{f(x_i) - f(x_j)}{x_i - x_j}; \qquad f(x) = \frac{x^2 - 8x + 4}{(x-1)^2} \log x + \frac{3}{x-1} \tag{2.62}$$

$$F(x_i, x_i) = x_i^2 f'(x_i) = x_i \frac{6x_i^2 \log x_i + x_i^3 - 12x_i^2 + 15x_i - 4}{(x_i - 1)^3}. \tag{2.63}$$

Although straightforward, the calculation is rather complicated. Note that in the unitary gauge the W propagators behave as constants in the large momentum limit, and therefore the diagram is divergent by power counting. However, in the limit of large loop momenta all quark masses can be neglected, and the integrand tends to be proportional to the sum $\sum_i \xi_i \sum_i \xi_j$, which is zero by unitarity of the CKM matrix. For the same reason, the amplitude would be zero if the up-type quarks were degenerate in mass.

We first compute the $K_{\rm L} K_{\rm S}$ mass difference

$$\Delta M \simeq 2\,|M_{12}|\,. \tag{2.64}$$

Using the standard Lorentz-invariant state normalization

$$\langle K^0|K^0\rangle = (2\pi)^3\delta^3(0)2m_K = \mathcal{V}2m_K, \tag{2.65}$$

where $\mathcal{V}$ is the volume of three-dimensional space, we find

$$M_{12} = \frac{1}{2m_K\mathcal{V}}\int d^3x\langle K^0|\left(-\mathcal{L}_{\text{eff}}^{\Delta s=2}(x)\right)|\overline{K}^0\rangle\ . \tag{2.66}$$

It is easy to show that the integrand is constant, and thus a factor of $\mathcal{V}$ is generated that cancels the $\mathcal{V}$ in the denominator. We get

$$M_{12} = -\frac{1}{2m_K}\langle K^0|\mathcal{L}_{\text{eff}}^{\Delta s=2}|\overline{K}^0\rangle\ , \tag{2.67}$$

and therefore

$$\Delta M \simeq 2\,|M_{12}| = \frac{G_F^2 m_{\text{W}}^2}{64\pi^2 m_K}\left|\sum_{i,j}\xi_i\xi_j F(x_i,x_j)\right|\langle K^0|\left[\bar{s}\gamma^\mu(1-\gamma_5)d\right]^2|\overline{K}^0\rangle. \tag{2.68}$$

A numerical estimate of ΔM is considerably simplified if one takes into account the numerical values of x_i and ξ_i. Current (PDG 2016) estimates give

$$x_u = \frac{m_u^2}{m_{\text{W}}^2} \sim 10^{-10}; \qquad x_c = \frac{m_c^2}{m_{\text{W}}^2} \simeq 2.5\cdot 10^{-4}; \qquad x_t = \frac{m_t^2}{m_{\text{W}}^2} \simeq 4.6. \tag{2.69}$$

Thus, for $i,j=u,c$ we have $x_i \ll 1, x_j \ll 1$, and

$$F(x_i,x_j) = \frac{4x_ix_j}{x_i-x_j}\log\frac{x_i}{x_j} + O(x_i^2,x_j^2); \qquad F(x_i,x_i) = 4x_i + O(x_i^2). \tag{2.70}$$

Since

$$\xi_u \simeq -\xi_c = \lambda \tag{2.71}$$

(see eq. (1.35)), we conclude that the dominant contribution from u,c circulating in the loop is the one proportional to $\xi_c^2F(x_c,x_c) \sim 4\lambda^2x_c^2$. The contributions from top quarks are proportional to either

$$F(x_t,x_i) = x_i\left[f(x_t) - 4\log x_i\right] + O(x_i^2) \tag{2.72}$$

or

$$F(x_t,x_t) \sim 10. \tag{2.73}$$

The only contribution which is not suppressed by light quark masses is therefore the one proportional to $\xi_t^2 \sim A^4\lambda^10$, with $A \sim 0.8$. This term is however negligible with respect to the analogous contribution from charm quarks. Indeed

$$\frac{\xi_t^2F(x_t,x_t)}{\xi_c^2F(x_c,x_c)} \simeq \frac{A^4\lambda^{10}}{\lambda^2}\frac{10x_t}{4x_c} \sim 0.05. \tag{2.74}$$

So

$$\Delta M \simeq 2\,|M_{12}| = \frac{G_F^2 \xi_c^2 mc^2}{16\pi^2 m_K} \langle K^0|\,[\bar{s}\gamma^\mu(1-\gamma_5)d]^2\,|\overline{K}^0\rangle. \tag{2.75}$$

The matrix element in eq. (2.75) depends on the strong interaction dynamics at energies of the order of the K^0 mass, and therefore it cannot be computed in a perturbative framework. It is usually parametrized as

$$\langle K^0|\,[\bar{s}\gamma^\mu(1-\gamma_5)d]^2\,|\overline{K}^0\rangle = \frac{8}{3} f_K^2 m_K^2 B_K, \tag{2.76}$$

where $f_K \simeq 1.23 f_\pi \simeq 114$ MeV is the K decay constant, extracted from the measured decay rate for the process $K^+ \to \mu^+\nu_\mu$, and B_K is a parameter which is expected to be of order 1 on the basis of flavor symmetry considerations and lattice calculations. We have therefore

$$\Delta M \simeq \frac{G_F^2}{6\pi^2}\, \xi_c^2 m_c^2\, f_K^2 m_K B_K\,. \tag{2.77}$$

Equation (2.77) allows us to estimate of the charm quark mass (we recall that the $K_{\rm L} K_{\rm S}$ mass difference was measured before the charm quark discovery). For $B_K = 1$ we find

$$m_c \sim 1.6 \text{ GeV}, \tag{2.78}$$

which is remarkably close to current estimates.

We now turn to the relationship between the CP violation parameter ϵ_0 and the CKM matrix entries. We have

$$\epsilon_0 = \frac{\delta}{\sqrt{2}}\, e^{i\pi/4} + \mathcal{O}(\delta^2), \tag{2.79}$$

where δ is the difference between the complex phases of M_{12} and Γ_{12}. We may choose the phase in the definition of the K^0 state such that Γ_{12} is real (this is the case in the Wolfenstein parametrization of the CKM matrix.) In this case

$$\text{Im}\,(M_{12}\Gamma_{12}^*) = \Gamma_{12}\text{Im}\,M_{12} = \Gamma_{12}|M_{12}|\sin\delta \simeq \Gamma_{12}|M_{12}|\delta. \tag{2.80}$$

We have therefore

$$|\epsilon_0| \simeq \frac{\delta}{\sqrt{2}} \simeq \frac{1}{\sqrt{2}}\left|\frac{\text{Im}M_{12}}{M_{12}}\right| \simeq \frac{1}{\sqrt{2}}\left|\frac{\text{Im}M_{12}}{\text{Re}M_{12}}\right|. \tag{2.81}$$

The dependence on the hadronic matrix element $\langle K^0|\,[\bar{s}\gamma^\mu(1-\gamma_5)d]^2\,|\overline{K}^0\rangle$ cancels in the ratio, and we are left with

$$|\epsilon_0| \simeq \frac{1}{\sqrt{2}}\frac{\text{Im}\Phi}{\text{Re}\Phi}, \tag{2.82}$$

where

$$\Phi = \sum_{i,j} \xi_i \xi_j F(x_i, x_j). \tag{2.83}$$

In this case we are no longer allowed to neglect the contributions from top quarks in the loop, since it is the only ones which carry a dependence on the complex phase of the CKM matrix.

Taking into account the numerical values of quark masses and CKM entries we find, in the Wolfenstein parametrization,

$$|\epsilon_0| \simeq \frac{1}{\sqrt{2}} \frac{F(x_t, x_t)}{4x_c} \frac{\mathrm{Im}\, \xi_t^2}{\xi_c^2} = \frac{1}{\sqrt{2}} \frac{F(x_t, x_t)}{4x_c} A^2 \lambda^3 \eta (1-\rho). \tag{2.84}$$

Recalling that

$$\bar{\rho} + i\bar{\eta} = \left(1 - \frac{\lambda^2}{2}\right)(\rho + i\eta), \tag{2.85}$$

we see from eq. (2.84) that a measurement of $|\epsilon_0| \simeq |\epsilon_K|$ corresponds to an allowed region in the $\bar{\rho}, \bar{\eta}$ complex plane given by

$$\bar{\eta}\left(1 - \frac{\bar{\rho}}{1 - \frac{\lambda^2}{2}}\right) = K \pm \Delta K, \tag{2.86}$$

where K is a function of various measured quantities: λ, A, the charm and top quark masses, the W mass and ϵ_K itself. One can see that this is in fact appoximately the shape of the constraint marked ϵ_K in fig. 1.

3 Lecture Three: Neutrinos

In the original formulation of the standard model, neutrinos are assumed to be massless. This assumption has its historical motivation in the direct experimental upper bounds on neutrino masses:

$$m_{\nu_e} \leq 3 \text{ eV}; \qquad m_{\nu_\mu} \leq 0.19 \text{ MeV}; \qquad m_{\nu_\tau} \leq 18.2 \text{ MeV}, \tag{3.1}$$

Today we know that neutrino masses are in fact non-zero; nonetheless, the approximation $m_\nu \ll m_f$, where f is any fermion in the standard model spectrum, is extremely accurate for most applications. However, it is mandatory to discuss the possible ways to introduce neutrino mass terms.

The absence of neutrino mass terms in the standard model is related to the absence of right-handed components for the neutrino fields, which belong to the singlet representation of $SU(2)$, and would have zero hypercharge. One may nevertheless assume that right-handed neutrinos do exist. This assumption brings us outside the standard model, and has far-reaching consequences. We restrict ourselves to the case of only one lepton generation, and we introduce a right-handed neutrino through the term

$$\bar{\nu}_R \, i \not{D} \nu_R \equiv \bar{\nu}_R \, i \not{\partial} \, \nu_R. \tag{3.2}$$

In the presence of a right-handed neutrino field, a Dirac mass term is generated through the Higgs mechanism by a Yukawa coupling similar to that of up-type quarks:

$$- h_{\mathrm{N}} \left[\bar{\ell}_L \tilde{\phi} \nu_R + \bar{\nu}_R \tilde{\phi}^\dagger \ell_L\right], \tag{3.3}$$

which develops a mass term

$$- m \left(\bar{\nu}_L \nu_R + \bar{\nu}_R \, \nu_L\right); \qquad m = \frac{h_{\mathrm{N}} v}{\sqrt{2}} \tag{3.4}$$

after spontaneous symmetry breaking, in analogy with the case of quarks. As observed in Lecture One, the Yukawa coupling in eq. (3.3) cannot be generated by radiative corrections, because it breaks explicitly the global accidental symmetry

$$\nu_R \to e^{i\phi}\nu_R \tag{3.5}$$

of the kinetic term eq. (3.2), the only other term in the Lagrangian density where ν_R appears.

From the experimental bounds in eq. (3.1), we conclude that the constant $h_{\rm N}$ must be smaller than the corresponding constants for charged leptons by several order of magnitudes. For example

$$\frac{h_{\rm N}}{h_e} = \frac{m}{m_e} \sim 10^{-6}. \tag{3.6}$$

This large hierarchy seems rather unnatural, since mass differences within the other $SU(2)$ doublets are much smaller:

$$\frac{m_u}{m_d} \sim 1; \qquad \frac{m_c}{m_s} \sim 10; \qquad \frac{m_t}{m_b} \sim 40. \tag{3.7}$$

There are however additional mass terms that can be included in the case of neutrinos. Because of its transformation properties with respect to gauge transformations, a right-handed neutrinos also admit a Majorana mass term

$$-\frac{1}{2}\, M\, \left(\bar{\nu}_R^c\, \nu_R + \bar{\nu}_R\, \nu_R^c\right). \tag{3.8}$$

where

$$\nu_R^c = \mathcal{C}\bar{\nu}_R^T = i\gamma_2\gamma_0\bar{\nu}_R^T, \tag{3.9}$$

the charge-conjugated spinor. This possibility is not shared by any other fermion field in the standard model, because of the limitations imposed by gauge invariance. In particular, it is not possible to build a Majorana mass term for left-handed neutrinos by means of renormalizable terms in the Lagrangian density. The Majorana mass parameter M, contrary to the Dirac mass m, can assume arbitrarily large values, since no extra symmetry is recovered in the limit $M = 0$. Furthermore, Majorana mass terms violate lepton number conservation; thus, we must assume that M is large enough, in order that lepton number violation effects, typically suppressed by inverse powers of M, are compatible with observations. It is natural to assume that M is of the order of the energy scale characteristic of the unknown phenomena (e.g. the effects of grand unification) experienced by right-handed neutrinos.

The most general neutrino mass term can therefore be written in the form

$$\mathcal{L}_{\nu\,\rm mass} = -\frac{1}{2}\,(\bar{\nu}_L^c\;\bar{\nu}_R)\begin{pmatrix} 0 & m \\ m & M \end{pmatrix}\begin{pmatrix} \nu_L \\ \nu_R^c \end{pmatrix} + \text{h.c.}, \tag{3.10}$$

where we have used $\bar{\nu}_L^c\,\nu_R^c = \bar{\nu}_R\,\nu_L$. The mass matrix in eq. (3.10) can be written in the diagonal Majorana form

$$\mathcal{L}_{\nu\,\rm mass} = -\frac{1}{2}\,(\bar{\nu}_1^c\;\bar{\nu}_2)\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \nu_1 \\ \nu_2^c \end{pmatrix} + \text{h.c.}, \tag{3.11}$$

with

$$\nu_1 = i(\cos\theta\ \nu_L - \sin\theta\ \nu_R^c); \qquad \nu_2 = \cos\theta\ \nu_R + \sin\theta\ \nu_L^c; \qquad \tan 2\theta = \frac{2m}{M} \tag{3.12}$$

and

$$m_1 = \frac{1}{2}\left(\sqrt{M^2+4m^2} - M\right); \qquad m_2 = \frac{1}{2}\left(\sqrt{M^2+4m^2} + M\right). \tag{3.13}$$

For $m \ll M$

$$\theta \simeq \frac{m}{M}; \qquad m_1 \simeq \frac{m^2}{M}; \qquad m_2 \simeq M; \qquad \nu_1 \simeq i\nu_L; \qquad \nu_2 \simeq \nu_R. \tag{3.14}$$

This mechanism, usually called the *see-saw* mechanism, provides a natural explanation of the observed smallness of neutrino masses: one of the two mass eigenstates in the neutrino sector is extremely heavy, and has no observable effects on physics at the weak scale, while the other one has a mass which is suppressed with respect to typical fermion masses m by a factor m/M. In this way, light neutrinos arise without the need of assuming unnaturally small values of the Yukawa couplings.

The see-saw mechanism can be generalized to the case of n different species of left-handed neutrinos and an undetermined number k of right-handed neutrinos. For simplicity, we will consider the case $k = n$, when there are as many right-handed as left-handed neutrinos. In this case, the Yukawa interaction introduced in Lecture One must be modified as follows:

$$\mathcal{L}_{\rm Yukawa}^{\rm lept} = -\left[\bar{\ell}_L \phi h_{\rm L} e_R + \bar{e}_R \phi^\dagger h_{\rm L}^\dagger \ell_L\right] - \left[\bar{\ell}_L \tilde{\phi} h_{\rm N} \nu_R + \bar{\nu}_R \tilde{\phi}^\dagger h_{\rm N}^\dagger \ell_L\right], \tag{3.15}$$

where we have introduced an array ${\nu_R}^\alpha$; $\alpha = 1, \ldots, n$, and $h_{\rm N}$ is a generic complex constant matrix. The neutrino mass term takes the form

$$\mathcal{L}_{\nu\,\rm mass} = -\frac{1}{2}\,(\bar{\nu}_L^c\ \bar{\nu}_R) \begin{pmatrix} 0 & \frac{v}{\sqrt{2}} h_{\rm N} \\ \frac{v}{\sqrt{2}} h_{\rm N} & M \end{pmatrix} \begin{pmatrix} \nu_L \\ \nu_R^c \end{pmatrix} + \text{h.c.}, \tag{3.16}$$

where now the entries of the mass matrix are $n \times n$ blocks, and M is a matrix in lepton flavour space which, without loss of generality, can be chosen real, diagonal and positive.

If the eigenvalues of M are much larger than $|v h_{\rm N}^{ij}|$, the (Majorana) mass terms for light neutrinos are

$$-\frac{1}{2}\left(\mu_{ij} \bar{\nu}_{Li}^c \nu_{Lj} + \mu_{ji}^* \bar{\nu}_{Li} \nu_{Lj}^c\right), \tag{3.17}$$

where the indices i, j are lepton flavour indices; the light neutrino fields ν_{Li} only approximately coincide with neutrinos with definite leptonic flavour. One finds that to a very good approximation

$$\mu \simeq \frac{v^2}{2}\left(h_{\rm N}\right)^T M^{-1} h_{\rm N} = U^\dagger \hat{\mu} U^*, \tag{3.18}$$

where $\hat{\mu}$ is a diagonal real matrix and the unitary matrix U is such that

$$\mu\mu^\dagger = U^\dagger \hat{\mu}^2 U. \tag{3.19}$$

The matrix U, usually refferret to as the Pontecorvo-Maki-Nagakawa-Sakata (PMNS) matrix, is identified up to three phases associated with the ν'^{i}_{Li}'s, which can be freely chosen due to the lepton flavour conservation property of the electroweak Lagrangian. Hence, U depends on six parameters: three angles and three complex phases. It is the leptonic analogous of the CKM matrix, and gives rise to the lepton flavour mixing (and, possibly, CP violation). It is usually parametrized as

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{i\delta} \\ -s_{12}c_{23} - c_{12}c_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13}e^{i\delta} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13}e^{i\delta} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha} & 0 \\ 0 & 0 & e^{i\beta} \end{pmatrix} \tag{3.20}$$

where $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$.

Neutrinos are produced in weak-interaction processes with a definite flavour: for example, β decays of nuclei in the Sun produce electron neutrinos. Denoting flavour eigenstates by Greek indices, and mass eigenstates by Latin indices, we have

$$|\nu_\alpha\rangle = \sum_{i=1}^{n} U^*_{\alpha i}\,|\nu_i\rangle. \tag{3.21}$$

Let us consider a neutrino beam of definite flavour, produced at the origin $L = 0$. Each definite-mass component of the beam propagates at the distance L as

$$|\nu_i(L)\rangle = e^{ip_iL}\,|\nu_i(0)\rangle, \tag{3.22}$$

where

$$p_i = \sqrt{E^2 - \hat{\mu}_i^2} \simeq E - \frac{\hat{\mu}_i^2}{2E}, \tag{3.23}$$

since neutrinos are almost massless. Hence,

$$|\nu_\alpha(L)\rangle \simeq e^{iEL}\sum_{i=1}^{n} U^*_{\alpha i}\,\exp\left(-i\frac{\hat{\mu}_i^2}{2E}L\right)\,|\nu_i(0)\rangle, \tag{3.24}$$

where E is the energy of the beam, which is assumed monochromatic for the time being.

The probability amplitude of observing the flavour β at a distance L from the source is given by

$$\begin{aligned} \langle\nu_\beta|\nu_\alpha(L)\rangle &\simeq e^{iEL}\sum_{i=1}^{n} U^*_{\alpha i}\,\exp\left(-i\frac{\hat{\mu}_i^2}{2E}L\right)\,\sum_{j=1}^{n} U_{\beta j}\,\langle\nu_j|\nu_i\rangle \\ &= e^{iEL}\sum_{i=1}^{n}\xi_i^{\alpha\beta}\,e^{-i\epsilon_iL}, \end{aligned} \tag{3.25}$$

where we have defined

$$\xi_i^{\alpha\beta} = U^*_{\alpha i}\,U_{\beta i}; \qquad \epsilon_i = \frac{\hat{\mu}_i^2}{2E}. \tag{3.26}$$

The corresponding probability is given by

$$
\begin{aligned}
P_{\alpha\beta}(L) &= \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i^{\alpha\beta}\xi_j^{*\alpha\beta} e^{i(\epsilon_j-\epsilon_i)L} \\
&= \delta_{\alpha\beta} - 4\sum_{i=1}^{n}\sum_{j=i+1}^{n} \mathrm{Re}\left(\xi_i^{\alpha\beta}\,\xi_j^{*\alpha\beta}\right)\sin^2\frac{1}{2}(\epsilon_j-\epsilon_i)L \\
&\quad -2\sum_{i=1}^{n}\sum_{j=i+1}^{n} \mathrm{Im}\left(\xi_i^{\alpha\beta}\,\xi_j^{*\alpha\beta}\right)\sin(\epsilon_j-\epsilon_i)L,
\end{aligned}
\tag{3.27}
$$

where we have used the unitarity of U. Observe that $P_{\alpha\beta}$ is unchanged if one replaces $U \to U^*$ and $\alpha \leftrightarrow \beta$:

$$
P(\nu_\alpha \to \nu_\beta; U^*) = P(\nu_\beta \to \nu_\alpha; U). \tag{3.28}
$$

On the other hand, CPT invariance implies

$$
P(\nu_\beta \to \nu_\alpha; U) = P(\bar{\nu}_\alpha \to \bar{\nu}_\beta; U). \tag{3.29}
$$

Hence,

$$
P(\nu_\alpha \to \nu_\beta; U^*) = P(\bar{\nu}_\alpha \to \bar{\nu}_\beta; U), \tag{3.30}
$$

or in other words neutrino oscillation probabilities can only differ from anti-neutrino oscillation probabilities if $U \neq U^*$, which is also a condition for CP violation.

In real situations eq. (3.27) requires important corrections for three reasons. First, neutrino beams are not monochromatic. This implies that the particles in the beams are not associated with plane waves as in eq. (3.22). Rather, their states must be described by wave packets whose space extension is approximately $\frac{1}{\Delta E}$, ΔE being the energy resolution of the beam. The components of the wave packets associated with different mass eigenvalues move along the beam with different velocities:

$$
|v_i - v_j| \sim \frac{|\epsilon_i - \epsilon_j|}{E}, \tag{3.31}
$$

and hence different components cease overlapping after a distance $D \sim E/(\Delta E|\epsilon_i - \epsilon_j|)$. More precisely, the exponential in the first line of eq. (3.27) is replaced by

$$
e^{i(\epsilon_j-\epsilon_i)L} \to e^{i(\epsilon_j-\epsilon_i)L} e^{-\frac{(\epsilon_j-\epsilon_i)^2(\Delta E)^2L^2}{8E^2}} \underset{L\to\infty}{\longrightarrow} \delta_{ij} \tag{3.32}
$$

and therefore neutrino oscillations are damped after a distance which is approximately equal to the oscillation length times $\frac{E}{\Delta E}$. In some instances, e.g. the case of athmospheric neutrinos, $\frac{\Delta E}{E}$ is of order one; in such cases, what is observed is not oscillations, but a continuous monotonic transition between

$$
P_{\alpha\beta}(0) = \delta_{\alpha\beta} \qquad \text{and} \qquad P_{\alpha\beta}(\infty) = \sum_i |\xi_i^{\alpha\beta}|^2. \tag{3.33}
$$

In other cases, e.g. in the case of solar neutrinos with energy of order 10 MeV, $\frac{\Delta E}{E}$ is relatively small, but the observer-source distance L is statistically distributed over millions of kilometers.

In these situations, oscillations average to zero, and what is in fact measured is $P_{\alpha\beta}(\infty)$ anywhere. Finally, eq. (3.27) was obtained under the assumption that neutrino propagation takes place in empty space; in principle, there might be sizable corrections due to the interaction of neutrinos with matter.

In the case of solar neutrinos, which are produced in the electron flavour state ($\alpha = e$) and detected in the same flavour state, one finds a flux reduction factor

$$P_{ee}(\infty) = \sum_i |U_{ei}|^4 = 0.58 \pm 0.07 \tag{3.34}$$

in the 1 MeV energy region. A different flux reduction, $P_{ee}(\infty) \simeq 0.3$, is instead measured for solar neutrinos in the 10 MeV energy region. This difference cannot be explained by eq. (3.33), which is manifestly independent of the neutrino energy.

An elegant explanation of this effect is based on the possibility that oscillations in matter be different from those in empty space. This might look surprising since neutrinos interact very weakly. However, it has been suggested that in certain conditions of neutrino energy and electron density, and for certain values of the relevant neutrino squared mass differences, a resonance mechanism can take place, the so-called the Mikheyev-Smirnov-Wolfenstein (MSW) mechanism, which may modify the first (electron) line of the PMNS matrix setting in particular $U_{e2} = 1$ in matter.[2] In these conditions, electrons would be created in the Sun in a mass eigenstate, and the beam would remain in the same mass eigenstate also emerging from the Sun into the vacuum.[3] In this situation one would find a flux reduction factor equal to the vacuum value of $|U_{e2}|^2$, which might fairly well be close to 1/3. Given the solar electron density and the neutrino energy, the resonance hypothesis favours a squared mass difference $\Delta m^2_\odot \sim 7 \cdot 10^{-5}$ eV^2.

An important source of experimental information on neutrinos is the study of the multi-GeV atmospheric neutrinos produced by the interactions of primary cosmic rays with the atmosphere. One observes a reduction by a factor about two in the muon neutrino flux when the neutrino azimutal angle varies between zero (particles coming from above) and 180° (particles from below), and hence L varies between few times 10 km and $2 \cdot 10^4$ km. One observes about five damped oscillations corresponding to an L of 13000 km, the diameter of the earth; for an average energy of about 8 GeV, eq. (3.27) gives $\Delta m^2 \sim 3 \cdot 10^{-3}$ (eV)^2.

The choice of the ordering of the mass eigenstates is, of course, arbitrary. The parametrization of the PMNS mixing matrix given in eq. (3.20) is motivated by the fact that the solar problem seems to involve two mass eigenstates, which are conventionally identified with the first two eigenstates:

$$\Delta m^2_\odot = \Delta m^2_{21}. \tag{3.35}$$

On the other hand, since most of the atmospheric neutrinos ($\sim 2/3$) are μ neutrinos, it is natural to identify the mass difference Δm^2_A measured in atmospheric neutrino experiments with Δm^2_{31}. Because $\Delta m^2_A \gg \Delta m^2_\odot$, we conclude that $\Delta m^2_{31} \simeq \Delta m^2_{32}$.

Further important experiments originate from the anti-neutrino flux generated by the nuclear power stations, which can be measured at distances of few kilometers, and from long baseline

[2] In the case of solar neutrinos the matter particles are electrons which have different forward scattering amplitudes with the neutrinos of different species due to the presence of charged current interactions.

[3] This is a consequence of the quantum mechanical version of the adiabatic theorem.

experiments based on high-energy artificial (anti)-neutrino beams, which will be crucial in order to detect possible CP violating phases in the PMNS matrix.

The analyses of the anti-neutrino flux generated by the nuclear power stations at distances of the order of one kilometer can put into evidence oscillations in the electron anti-neutrino survival probability $P_{ee}(L)$ corresponding to $\Delta m^2_{31} \simeq \Delta m^2_{32}$. Indeed, using eq. (3.27) and eq. (3.20), one has

$$P_{ee}(L) = 1 - \sin^2(2\theta_{13}) \sin^2 \frac{\Delta m^2_{31} L}{4E}, \tag{3.36}$$

and for $E \sim 3$ MeV, the tipical average value of antineutrino energy, one has an oscillation length $L \sim 600$ m and a damping length few times larger. Recently, a tiny effect has been detected, which can be interpreted in terms of a small, but non zero, value of $\sin^2\theta_{13} \simeq 0.025$.