

2014 BUSSTEPP LECTURES

Physics Beyond the Standard Model

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November 11, 2014

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Acknowledgments

I thank my collaborators and many lecturers who have provided inspiration over the years; some of their excellent notes can be found in the references. The references are intended to provide an entry point into the literature, and so are necessarily incomplete; I apologize to those left out. For errors and comments, please contact me by e-mail at the address on the front page.

1 Avant propos

On the one hand, giving 4 lectures on physics beyond the Standard Model (BSM) is an impossible task, because there is so much that one could cover. On the other hand, it is easy, because it can be summed up in the single phrase: There must be something, but we don't know what it is! This is, of course, tremendously exciting. It has to be said that we also know a great deal about what BSM physics *isn't*, because of myriad experimental and theoretical constraints. Whether this is to be considered good news or bad news is somewhat subjective. At least it keeps us honest.

My aim in these lectures is to give a flavour of the field of BSM physics today, with an attempt to focus on aspects which are not so easy to find in textbooks. I have tried to make the lectures introductory and to dumb them down as much as possible. I apologize in advance to those who feel that their intelligence is being insulted.

A number of shorter derivations are left as exercises. These are numbered and indicated by '(exercise n)' where they appear.

2 Notation and conventions

As usual, $\hbar = c = 1$, and our metric is mostly-minus: $\eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$.

We also need to decide on conventions for fermions. This is somewhat involved, and I don't have the time to do it properly. Let me at least lay down the rules of the game. I assume that the reader is familiar with Dirac (4-component) fermions, for which a set of gamma matrices, satisfying $\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$, is

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad (2.1)$$

where $\sigma^\mu = (1, \sigma^i)$, $\bar{\sigma}^\mu = (1, -\sigma^i)$, and σ^i are the usual 2 x 2 Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.2)$$

Hence,

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (2.3)$$

Dirac fermions are fine for QED and QCD, but they are not what appears in the SM.¹ A Dirac fermion carries a 4-dimensional, *reducible* representation (henceforth ‘rep’) of the Lorentz group $SO(3,1)$.² We can, therefore, assign fermions to carry either of the 2, 2-d irreducible reps (henceforth ‘irreps’) that are carried by a 4-d Dirac fermion. These are called Weyl fermions. The 2 2-d irreps are inequivalent. We’ll let one of the irreps be carried by a 2-component object ψ_α , with $\alpha \in \{1,2\}$, and the other be carried by a different 2-component object $\bar{\chi}^{\dot{\alpha}}$, with $\dot{\alpha} \in \{1,2\}$. We put a bar on $\bar{\chi}$ and a dot on $\dot{\alpha}$ to indicate that they are different sorts of object to ψ and α , respectively. We also put one index upstairs and one index downstairs, for reasons that will become clear.

These two representations are conjugate, meaning that the complex conjugate of one is equivalent (via a similarity transformation) to the other. We thus make the definitions

$$\bar{\psi}_{\dot{\alpha}} \equiv (\psi_\alpha)^*, \quad \chi^\alpha \equiv (\bar{\chi}^{\dot{\alpha}})^*. \quad (2.4)$$

We now introduce conventional definitions for raising and lowering indices, *viz.*

$$\psi_\alpha \equiv \epsilon_{\alpha\beta} \psi^\beta, \quad \psi^\beta \equiv \epsilon^{\beta\gamma} \psi_\gamma, \quad (2.5)$$

$$\bar{\psi}_{\dot{\alpha}} \equiv \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\psi}^{\dot{\beta}}, \quad \bar{\psi}^{\dot{\beta}} \equiv \epsilon^{\dot{\beta}\dot{\gamma}} \bar{\psi}_{\dot{\gamma}}, \quad (2.6)$$

where $\epsilon_{\alpha\beta} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Note that these imply, *e.g.*, $\epsilon_{\alpha\beta} \epsilon^{\beta\gamma} = \delta_\alpha^\gamma$, so $\epsilon^{\alpha\beta} \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Similarly, the above relations imply $\epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}}$.

The point of all this notation (for which you can thank Van der Waerden), is that, just like for 4-vectors in relativity, anything with all undotted and dotted indices (separately) contracted upstairs and downstairs in pairwise fashion is a Lorentz invariant. (Look in a QFT book for a proof.)

This means that given just a single Weyl spinor, ψ_α we can write a mass term in the lagrangian of the form $\psi^\alpha \psi_\alpha$ (to which we should add the Hermitian conjugate to be sure that the action is real). This is called a Majorana mass term, and we can only write it if ψ transforms in a real rep of any internal symmetry group (because only the product of two real reps contains a singlet rep). If we have two Weyl spinors, ψ_α and χ_α we can write a mass term of the form $\psi^\alpha \chi_\alpha + \text{h. c.}$ and this will contain a singlet if ψ and χ transform under conjugate reps of any internal symmetry. This is called a Dirac mass term.

To save drowning in a sea of indices, it is useful to define $\psi^\alpha \chi_\alpha \equiv \psi\chi$ and $\bar{\psi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}} \equiv \bar{\psi}\bar{\chi}$. But note that, *e.g.*, $\psi_\alpha \chi^\alpha = -\psi\chi$ (exercise 1). Fortunately, since the fermions take Grassman number values, we do have that $\psi\chi = \chi\psi$ (exercise 2).

¹A gauge theory containing Dirac fermions is called *vector-like*; otherwise it is called *chiral*.

²One way to see that it is reducible is to consider the complex form of the Lie algebra which is isomorphic to the complex form of $SU(2) \times SU(2)$. (A complex form is obtained by taking the original Lie algebra, which is a real vector space, and promoting the coefficients in linear superpositions from \mathbb{R} to \mathbb{C} .) The Dirac fermion corresponds to the $(2,1) \oplus (1,2)$ rep of $SU(2) \times SU(2)$, which is reducible. Note that while $(2,1) \oplus (1,2)$ is a unitary rep of $SU(2) \times SU(2)$, the Dirac fermion does not carry a unitary rep of $SO(3,1)$, because of factors of i that appear in taking the different real forms. This is why things like $\bar{\psi}\psi$ appear rather than $\psi^\dagger\psi$.

Going back to 4-component language, we can write a Dirac fermion as $\Psi = \begin{pmatrix} \psi_\alpha & \bar{\chi}^{\dot{\alpha}} \end{pmatrix}^T$.

A Majorana fermion can be written as $\Psi = \begin{pmatrix} \psi_\alpha & \bar{\psi}^{\dot{\alpha}} \end{pmatrix}^T$, such that $\Psi = \Psi^c$ (exercise 3).

Lorentz-invariant terms can also be formed using the same rules with the objects $(\sigma^\mu)_{\alpha\dot{\alpha}}$ and $(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}$, which appear in γ^μ .³ So we may write terms $i\chi^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu \bar{\chi}^{\dot{\alpha}} \equiv i\chi \sigma^\mu \partial_\mu \bar{\chi}$ and $i\bar{\psi}_{\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\alpha} \partial_\mu \psi_\alpha \equiv i\bar{\psi} \bar{\sigma}^\mu \partial_\mu \psi$ and indeed the usual Dirac kinetic term produces the sum of these (exercise 4).

Since we can get from one rep to the other by taking the complex conjugate, we can, w. l. o. g. assign all fermions to just one irrep, which we take to be the undotted one.

3 The Standard Model

To go *beyond* the Standard Model (SM), we first must know something about the SM itself. We define the SM as a gauge quantum field theory with gauge symmetry $SU(3) \times SU(2) \times U(1)$, together with matter fields comprising 15 Weyl fermions and one complex scalar, carrying irreps of $SU(3) \times SU(2) \times U(1)$. The fermions consist of 3 copies (the different families or flavours or generations) of 5 fields $\psi \in \{q, u^c, d^c, l, e^c\}$ carrying reps of $SU(3) \times SU(2) \times U(1)$ as listed in Table 1. The scalar field, H , carries the $(1, 2, -\frac{1}{2})$ rep of $SU(3) \times SU(2) \times U(1)$. The final part of the definition is that the lagrangian should contain all terms up to dimension four, such that it is renormalizable.

Field	$SU(3)_c$	$SU(2)_L$	$U(1)_Y$
q	$\mathbf{3}$	$\mathbf{2}$	$+\frac{1}{6}$
u^c	$\bar{\mathbf{3}}$	$\mathbf{1}$	$-\frac{2}{3}$
d^c	$\bar{\mathbf{3}}$	$\mathbf{1}$	$+\frac{1}{3}$
l	$\mathbf{1}$	$\mathbf{2}$	$-\frac{1}{2}$
e^c	$\mathbf{1}$	$\mathbf{1}$	$+1$

Table 1. Fermion fields of the SM and their $SU(3) \times SU(2) \times U(1)$ representations.

To see how elegant the SM is, we note that we can write the lagrangian on a single line (just). It is, schematically,

$$\mathcal{L} = i\bar{\psi}_i \bar{\sigma}^\mu D_\mu \psi_i - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \lambda^{ij} \psi_i \psi_j H^{(c)} + \text{h. c.} + |D_\mu H|^2 - V(H), \quad (3.1)$$

where i, j label the different families and a labels the different gauge fields. You might counter that the way I choose to write it is arbitrary (and indeed the paradox of “the greatest integer which cannot be described in fewer than twenty words” is brought to mind). So let’s explore it in more detail.

³Again, this is just group theory: a 4-vector corresponds to the $(2, 2)$ irrep of $SO(3, 1)$, and we can make something that transforms like a $(2, 2)$ by taking the (tensor) product of $(1, 2)$ and $(2, 1)$ irreps.

3.1 The gauge sector

The lagrangian is

$$\mathcal{L} = i\bar{\psi}_i \bar{\sigma}^\mu D_\mu \psi_i - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu}, \quad (3.2)$$

with 5 fermion irreps $\psi \in \{q, u^c, d^c, l, e^c\}$ and 3 copies of each, corresponding to the 3 families. There are really 12 gauge fields: 8 in an adjoint of $SU(3)$, 3 in an adjoint of $SU(2)$, and 1 for $U(1)$. The covariant derivative D_μ contains the gauge couplings $g_s, g,$ and g' , with the gauge group generators in the appropriate reps. The fermion terms are invariant under a $U(3)^5$ global symmetry. (Exercise 5: show this. What is the global symmetry when the gauge couplings are switched off?) This part of the SM has been tested at the per mille level via charge universality, gauge boson interactions (*e.g.* $\sigma(e^+e^- \rightarrow W^+W^-)$). Exercise 6: draw the Feynman diagrams that contribute at leading order to $\sigma(e^+e^- \rightarrow W^+W^-)$ in the SM).

3.2 The flavour sector

The lagrangian is

$$\mathcal{L} = \lambda^u q H^c u^c + \lambda^d q H d^c + \lambda^e l H e^c + \text{h. c.} \quad (3.3)$$

The λ^i are 3×3 complex matrices (in family space) and so it appears that there are a lot of free parameters, and also the possibility of CP violation (since a CP transformation is equivalent to interchanging the λ^i s with their complex conjugates. Exercise 7). But not all of these parameters are physical. This follows from the fact that we are free to do unitary rotations of the different fields without changing other terms in the lagrangian. So, for example, there is a basis in which we can write (exercise 8)

$$\lambda^u q H^c u^c + \lambda^d V q H d^c + \lambda^e l H e^c + \text{h. c.} \quad (3.4)$$

where now all the λ^i are diagonal, and V is a 3×3 unitary matrix, called the CKM matrix. Since this is the only off-diagonal object in the lagrangian, it must contain all the information about mixing of flavours in the SM. Very roughly, we find that

$$V \sim \begin{pmatrix} 1 & \lambda & \lambda^3 \\ \lambda & 1 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix}, \quad (3.5)$$

where $\lambda \sim 0.2$.

To actually count the number of physical parameters, we need to be a bit more careful. In the quark sector, the Yukawa terms in (3.3) break the $U(3)^3$ symmetry down to the $U(1)_B$ corresponding to baryon number conservation. In the lepton sector, $U(3)^2$ is broken to $U(1)_{L_e} \times U(1)_{L_\mu} \times U(1)_{L_\tau}$, corresponding to conservation of individual lepton family numbers. There is also the overall $U(1)_Y$ which is conserved.

Knowing the pattern of symmetry breaking, we can count the number of physical parameters in, *e. g.*, the quark sector. There are two complex 3×3 matrices in the quark

sector, with a total of 18 real and 18 imaginary parameters (equivalently, there are 18 complex phases). But many of these parameters are unphysical, in the sense that they can be removed using the unbroken ‘symmetries’.⁴ An $N \times N$ unitary matrix has N^2 parameters, of which $\frac{N(N-1)}{2}$ are real and $\frac{N(N+1)}{2}$ are imaginary (for an Hermitian matrix, it is the other way around. Exercise 9: Prove these results.) So in the quark sector, with $U(3)^3$ broken to $U(1)_B$, we have 9 unbroken real parameters and 17 phases, meaning that there are $18 - 9 = 9$ physical real parameters and $18 - 17 = 1$ physical phase. If we now refer to (3.4), we see that 6 of the real parameters are the quark masses, so there must be 3 physical angles in the CKM matrix, and a single phase. This phase is a source of CP -violation.

The flavour sector of the SM is tested at the per cent level, in many experiments (observations of rare processes). As we shall see below, this puts constraints on new physics that are much more stringent than one might naïvely guess.

3.3 The Higgs sector

The lagrangian is

$$\mathcal{L} = -\mu^2 H^\dagger H - \lambda(H^\dagger H)^2. \quad (3.6)$$

Until recently, this sector was hardly tested at all. But now, with the discovery of the Higgs boson, it is being probed directly at the ten per cent level.

So, ugly or not, the SM does an implausibly good job of describing the data, reaching the per mille level in individual measurements and with an overall fit (to hundreds of measurements) that cannot be denied: the SM is undoubtedly correct, at least in the regime in which we are currently probing it.

4 Miracles of the SM

So far we have written down a definition of the SM, and argued that it gives a compelling explanation of current data. This approach has a major deficiency, which is that, in simply writing it down, we completely overlook the heroic (and sometimes tragic and comedic) struggles of our forefathers over many decades to arrive at it.

On the one hand, this is exactly the sort of youthful disrespect for one’s elders that leads to great advances in physics, and should be encouraged as much as possible. But on the other hand, it means that we miss certain features of the SM that are very special, and are vital clues in our quest for the form of physics BSM. Let us discuss some of them.

4.1 Flavour

Let’s look in more detail at the flavour structure (for introductory lectures on this topic, see [1, 2]). We have already argued that there is a basis in which we can write the Yukawa couplings as (in a matrix notation for flavour)

$$q\lambda^u H^c u^c + q\lambda^d V H d^c + l\lambda^e H e^c + \text{h. c.} \quad (4.1)$$

⁴They are not really symmetries, because they do not leave the lagrangian invariant!

To go from here to the mass basis after EWSB, all we need to do is a rotation, $d \rightarrow V^\dagger d$, of the d quarks in $q = (u \ d)^T$. This has the effect of making the gauge interactions non-diagonal. In particular, we find charged current interactions involving the W^\pm of the form

$$g(\bar{u}\sigma \cdot W^+ V^\dagger d + \bar{d}V\sigma \cdot W^- u). \quad (4.2)$$

The neutral current interactions involving the Z remain diagonal, however:

$$-g\bar{d}V\sigma \cdot W^3 V^\dagger d = -g\bar{d}\sigma \cdot W^3 d, \quad (4.3)$$

since $VV^\dagger = 1$. This is a key feature of the SM, which was motivated by the experimental absence of FCNC. In fact, the absence of FCNC in the SM (at tree-level – we discuss loop interactions below) also extends to the other neutral currents in the SM, *viz.* those involving gluons, photons, or Higgs bosons. For the gluons and photons, this arises simply because the corresponding gauge symmetries are unbroken in the vacuum, and the couplings are universal. More prosaically, the coupling matrices are proportional to the identity matrix, and so are diagonal in any basis. For the Higgs boson, it arises because the couplings of the Higgs to fermion are diagonal in the mass basis. This is simply because (in unitary gauge), $H = \begin{pmatrix} 0 \\ v + h \end{pmatrix}$ and so we are diagonalizing the same matrix for the fermion masses as for the couplings of the fermions to the Higgs boson. But note that this would no longer necessarily be true in a theory with more than 1 Higgs doublet, where there are extra Yukawa coupling matrices in general, and the EW VEV is shared between the different doublets (Exercise 10: Show explicitly that \exists tree-level FCNC in a 2 Higgs doublet model. How is this avoided in the MSSM?).

The absence of tree-level FCNC mediated by Z bosons is also a special feature of the SM. It arises because all fields carrying the same irrep of the unbroken $SU(3)_c \times U(1)_Q$ symmetry (which can mix with each other) also carry the same irrep of the broken $SU(3)_c \times SU(2)_L \times U(1)_Y$ symmetry. If this were not the case, the different irreps would, in general, have different couplings to the Z . The matrix of couplings to the Z would then be diagonal in the interaction basis, but not proportional to the identity matrix. The matrix would then acquire off-diagonal entries when we rotate to the mass basis. For example, the d, s are both colour anti-triplets with charge $-\frac{1}{3}$, and can mix, but they also all come from $SU(2)$ doublets with hypercharge $+\frac{1}{6}$, so there are no FCNC. (Exercise 11: before the charm quark was invented, it was thought that the s quark lived in an $SU(2)$ singlet. Show that this leads to tree-level FCNC.)

There are also remarkable suppressions of loop level processes. Consider, for example the diagram contributing to the process $b \rightarrow s\gamma$ in Fig. 1. The internal quark can be any one of u, c, t and so the amplitude is proportional to

$$\sum_{i \in \{u, c, t\}} V_{ib} V_{is}^* f\left(\frac{m_i^2}{m_W^2}\right) \quad (4.4)$$

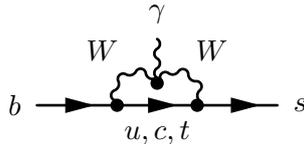


Figure 1. Diagram contributing to $b \rightarrow s\gamma$.

where f is some function obtained by doing the loop integral.⁵ Now, suppose we do a Maclaurin expansion of f . The first term in the sum then vanishes, by unitarity of the CKM matrix. At the next order, we have terms that go like $\frac{m_u^2}{m_W^2}$ (which is tiny), $\frac{m_c^2}{m_W^2}$ (which is small), and $\frac{m_t^2}{m_W^2}$ (which is certainly not small, but whose contribution is suppressed by $V_{ib}V_{is}^*$; the same is true at higher order in $\frac{m_t^2}{m_W^2}$). (Exercise 12: show that the latter 2 contributions are roughly the same size for the analogous process $s \rightarrow d\gamma$. Is this true for all processes?) We thus find that these loop diagrams feature a GIM [3] suppression. This suppression is in addition to the factor of $(\frac{1}{4\pi})^2$ that comes from the fact that we have to do a loop integral: $\int \frac{d^4p}{(2\pi)^4} f(p^2) = (\frac{1}{4\pi})^2 \int x dx f(x)$ (exercise 13: show this). Overall, the SM contribution is of size $\frac{1}{(4\pi)^2} \frac{m_c^2}{m_W^2} \frac{1}{m_W^2}$ compared to a generic new physics contribution with mass scale Λ and $O(1)$ couplings of $\frac{1}{\Lambda^2}$. Thus the latter are greatly enhanced and the bounds on Λ are typically way above m_W (given that the SM contributions give a good fit to the data). In fact, they reach as high as 10^5 GeV or so.

4.2 CP-violation

The peculiar structure of the flavour sector in the SM, and the fact that CP -violation resides in the CKM matrix, imply that there are also suppressions of CP -violating processes, which do not occur in generic models BSM. To see this, note that if there had only been two generations of quarks in the SM, there would be no physical CP -violating parameter in the flavour sector (exercise 14). This means that any process which violates CP in the SM must involve all 3 quark generations. For similar reasons, CP -violation cannot occur if any of the masses are degenerate in either the up or down sector, or if any of the 3 mixing angles is 0 or $\frac{\pi}{2}$: all of these situations increase the symmetry in the quark sector and result in no physical phase. But many of the SM quark masses are roughly degenerate, and many mixings in (3.5) are small, so again there is a huge suppression. For the mixings, for example, we get a factor of $\lambda^6 \sim 10^{-3}$.

Again, these properties do not hold for generic BSM physics, and so the constraints thereon are strong.

4.3 Electroweak precision tests and custodial symmetry

There is also a SM suppression in electroweak precision tests which is not generic. To see it, consider the Higgs sector. The Higgs is a complex $SU(2)$ doublet, and so there are four

⁵Note that these loop integrals are finite. This must be the case, because they generate operators in the low-energy effective lagrangian with four fermions, for which there are no counterterms available in the renormalizable SM.

real fields. The kinetic terms therefore have an $O(4)$ symmetry. Let us now consider how this symmetry gets broken when we switch on the various couplings.

One of the miracles of group theory is that the Lie algebra of the group $O(4)$ is the same as that of the group $SU(2) \times SU(2)$ (exercise 15). So the Higgs fields can be thought of as carrying 2 $SU(2)$ symmetries, rather than the single $SU(2)_L$ of the standard model. It is usual to call the other symmetry $SU(2)_R$, so the Higgs carries a $(2, 2)$ rep (exercise 16) of $SU(2)_L \times SU(2)_R$. Now, when we switch on the $SU(2)_L$ gauge coupling g , we still have global symmetry $SU(2)_L$ (because the gauge symmetry includes constant gauge transformations, which are the same as the global ones) and we still have global symmetry $SU(2)_R$, because this factor is independent of $SU(2)_L$. So the full $SU(2)_L \times SU(2)_R$ remains unbroken.

What is more, this $SU(2)_L \times SU(2)_R$ is also unbroken when we switch on the Higgs potential, because $V(H)$ is only a function of $|H|^2 = h_1^2 + h_2^2 + h_3^2 + h_4^2$, which is manifestly invariant under $O(4)$.

The Yukawa couplings do break $SU(2)_L \times SU(2)_R$,⁶ as does the coupling to the Z (which couples to the combination $T_L^3 + T_R^3$). So the correct statement is that the SM is invariant under $SU(2)_L \times SU(2)_R$ in the limit that $\lambda^t = g' = 0$.

When the Higgs gets a VEV, the $SU(2)_L \times SU(2)_R$ is broken to the diagonal $SU(2)_V$ combination of the 2 original $SU(2)$ s (exercise 17). This (approximate) symmetry is called ‘custodial $SU(2)$ ’. So what? Consider the lagrangian for the gauge bosons after EWSB (the proper way to do this is using an effective field theory, which we’ll discuss later). The symmetry is just the $U(1)$ of electromagnetism and so we should write the most general lagrangian consistent with this. At quadratic level, in momentum space, we have

$$\mathcal{L} = \Pi_{+-} W^+ W^- + \Pi_{33} W^3 W^3 + \Pi_{3B} W^3 B + \Pi_{BB} B B, \quad (4.5)$$

where $\Pi_{ab}(p^2)$ are functions of momentum which are generated by the currents to which the W and B couple: $\Pi_{ab} \sim \langle J_a J_b \rangle$. At low energies, we may Maclaurin expand $\Pi(p^2) = \Pi(0) + p^2 \Pi'(0) + \dots$. Consider the combination $\Pi_{+-}(0) - \Pi_{33}(0)$, which is proportional to something called the ‘ T -parameter’, and which is evidently related to the W - and Z -boson masses. Its parts are each generated from the product of two W currents, each of which transforms as a $(3, 1)$ of $SU(2)_L \times SU(2)_R$, or a 3 of $SU(2)_V$. The product of two 3 s decomposes as $3 \times 3 = 1 + 3 + 5$ (exercise 18). The particular combination $\Pi_{+-}(0) - \Pi_{33}(0)$ is symmetric in the two indices and traceless, so transforms as the 5 of $SU(2)_V$. But since $SU(2)_V$ is a symmetry of the vacuum, only singlets of $SU(2)_V$ can have non-vanishing VEVs. This implies that $T = 0$, which in turn implies a definite relation for, say, $\frac{m_W}{m_Z}$.

At this point, you might be wondering why I have gone through such an arcane derivation of a result that could have easily be obtained directly by plugging the Higgs VEV into the SM lagrangian and computing the masses. The point is that our derivation applies not only to the SM, but extends to any theory with $SU(2)_L \times SU(2)_R$ symmetry that is broken to $SU(2)_V$ in the vacuum. Any such theory will naturally come out with the right

⁶A technical point: if $\lambda^u = \lambda^d$, then we can group u^c and d^c into an $SU(2)_R$ doublet, and $SU(2)_L \times SU(2)_R$ is restored.

value for the measured T parameter. Conversely, BSM theories which do not feature this symmetry are likely to be ruled out. As examples, a model with extra Higgs scalar states that get a VEV will have problems. For example, if we add a Higgs triplet, then there is no approximate $SU(2)_L \times SU(2)_R$. Even if we add only an additional Higgs doublet, we will get in to trouble, because the theory remains $SU(2)_L \times SU(2)_R$ symmetric, but $SU(2)_V$ is now broken in the vacuum. One can easily see this using the $O(4)$ language: a single complex Higgs doublet (which can be thought of as a scalar field with values in \mathbb{R}^4 breaks the group of $O(4)$ orthogonal transformations of \mathbb{R}^4 down to $O(3)$ (which is locally equivalent to $SU(2)_V$); a second complex Higgs doublet will break this even further to $O(2) \simeq U(1)$, which is just electromagnetism. As a result, our proof above does not go through, because the vacuum is no longer $SU(2)_V$ symmetric, and T contains a singlet under the surviving $U(1)$ (exercise 19).

4.4 Accidental symmetries and proton decay

The last miracle of the SM that I want to mention is that it has accidental symmetries. These are symmetries of the lagrangian that are not put in by *fiat*, but arise accidentally from the field and other symmetry restrictions, and the insistence on renormalizability.

A simple example of an accidental symmetry is parity in QED. The most general, Lorentz-invariant, renormalizable lagrangian for electromagnetism coupled to a Dirac fermion Ψ may be written as

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + iaF_{\mu\nu}\tilde{F}^{\mu\nu} + i\bar{\Psi}\not{D}\Psi + \bar{\Psi}(m + i\gamma^5 m_5)\Psi, \quad (4.6)$$

where both the term involving $\tilde{F}^{\mu\nu} \equiv \epsilon^{\mu\nu\sigma\rho}F_{\sigma\rho}$ and the term involving γ^5 naïvely violate parity (exercise 20). However, the former term is a total derivative (exercise 21) and so does not contribute to physics at any order in perturbation theory (we will discuss important non-perturbative contributions from such terms when we discuss the axion later on). The latter term can be removed by a chiral rotation $\psi \rightarrow e^{i\alpha\gamma^5}\psi$ to leave a parity-invariant theory with fermion mass $\sqrt{m^2 + m_5^2}$. So we find that the lagrangian is invariant under parity, even though we did not require this in the first place. The same is true of charge conjugation symmetry. Note that if we had not insisted on renormalizability, we could write dimension-six terms like $\bar{\Psi}\gamma^\mu\gamma^5\Psi\bar{\Psi}\gamma_\mu\Psi$, which do violate parity (exercise 22: show this violates P). We will explore these when we discuss effective field theories later on.

As we already alluded to above, the SM lagrangian is accidentally invariant under a $U(1)_B$ baryon number symmetry (an overall rephasing of all quarks) and three $U(1)$ lepton number symmetries, corresponding to individual rephasings of the three different lepton families (which contains an overall lepton number symmetry $U(1)_L$ as the diagonal subgroup). Either $U(1)_B$ or $U(1)_L$ symmetry, together with Lorentz invariance, prevents the proton from decaying. Indeed, a putative final state must (by Lorentz invariance, which implies the fermion number is conserved mod 2) contain an odd number of fermions lighter than the proton. The only such states carry lepton number but not baryon number, whereas the proton carries baryon number but not lepton number.

Again, once we allow higher dimension operators, we will find that lepton and baryon number are violated (by operators of dimension five or six, respectively), meaning that the proton can decay. Similarly, generic theories of physics BSM will violate them and hence will be subject to strong constraints.

(Exercise 23: consider just the Higgs sector coupled to the W -boson. Show that $SU(2)_L \times SU(2)_R$ is an accidental symmetry, and find a dimension-six operator that violates it.)

4.5 What isn't explained

We have now seen that the SM does a wonderful job of explaining a lot of data, and it does so by means of a delicate structure that is not preserved by generic BSM theories. This means that it is hard to write down BSM models that are consistent with all the data.

But we are impelled to write down models by the fact that there are, by now, also plenty of data that the SM patently cannot describe. These include:

- (i) *neutrino masses and mixings*

No term in (3.3) yields these.

- (ii) *the presence of non-baryonic, cold dark matter*

Dark matter is neutral, colourless, non-baryonic, and massive. The only such particles in the SM are neutrinos, but these are too light, making instead warm dark matter.

- (iii) *the presence of scale-invariant, Gaussian, and apparently acausal density perturbations, consistent with a period of inflation at early times*

- (iv) *the observed abundance of matter over anti-matter*

I have yet to meet my *alter ego* and indeed there would appear to be a general predominance of matter over anti-matter in the Universe. Note, moreover, that inflation would destroy any asymmetry imposed as an initial condition.

All of these have been established beyond reasonable doubt, in many cases overwhelmingly so. In addition, there are many unexplained features of nature that lead us to believe that there must be physics BSM:

- (i) *the inability to describe physics at planckian scales*

Contrary to what you might read in the New Scientist, general relativity makes perfect sense as a theory of quantum gravity up to planckian scales (as an effective field theory, to be described below), but beyond that we need a theory of quantum gravity, such as string theory.

- (ii) *the hierarchy between the observed cosmological constant and other scales*

The measured energy density associated with the accelerated expansion of the Universe is $(10^{-3}\text{eV})^4$, but receives contributions of size $(\text{GeV})^4$, $(\text{TeV})^4$, $\mathcal{E}c$. from QCD, weak scale physics, $\mathcal{E}c$. Such a small value appears necessary to support intelligent life (and you and me), but how is it achieved?

(iii) *the hierarchy between the weak and other presumed scales*

As above, but now the question is how to get a TeV from, *e.g.* the Planck scale. Here, it is less clear that such a small value is a *sine qua non* for us to be here worrying about it in the first place.

(iv) *the comparable values of matter, radiation, and vacuum energy densities today*

These 3 scale in vastly different ways during the Universe's evolution, so why are they roughly the same today?

(v) *the structure in fermion masses and mixings*

There is a hierarchical structure in these (as described, *e. g.*, in [4]), but in the SM they are just free parameters. So why do they exhibit structure?

(vi) *the smallness of measured electric dipole moments*

These violate CP , and are at least 10^{-10} smaller than a naïve guess of $O(1)$ for the corresponding parameter, which is an angle $\in [0, 2\pi]$; why? No anthropic argument is known, by the way.

(vii) *the comparable size of the 3 gauge couplings*

These are all $O(1)$ and different, but not *so* different; why?

(viii) *the quantization of electric charges*

The SM contains the gauge group $U(1)_Y$, for which any charge is allowed. Why do we find integer multiples of $\frac{1}{3}$ for the electric charge, rather than, *e. g.* $\sqrt{2}$ or π ?

(ix) *the number of fermion families*

Why 3? As Rabi said about the muon, 'Who ordered that?'

(x) *the number of spacetime dimensions*

Why 4? Why 3+1 for that matter?

An explanation (together with an experimental confirmation, of course) of any one of these would surely merit a Nobel prize, not least because they are such deep issues, but also because it seems so hard to extend the SM in such a way as to furnish a solution, without contravening some other experimental test.

I will certainly not provide definitive answers to any of them in these lectures, or even address any of them in detail. Rather, I aim to introduce you to one or two of the ideas that have been proposed, in order to give you a flavour of the way that the game is played. But before I do that, I introduce a framework that enables us to discuss many of the issues in BSM physics in a generic, model-independent way. This requires us to learn a little about effective field theory.

5 Beyond the SM - Effective field theory

Once we go BSM, it seems like an infinity of possibilities opens up – we could write down any lagrangian we like. Fortunately, we have a good starting point, since we know that we must reproduce the SM in some limit.

Even better, we can make things very concrete by making one assumption. Let us suppose that any new physics is rather heavy. This is indicated experimentally by the fact that observed deviations from the SM are small, but it is not the only possibility. New physics could instead be very light, but also very weakly coupled to us. We'll see an example of this later on, when we mention large extra dimensions.

With this assumption in hand, we can analyse physics BSM in a completely general way using methods of effective field theory (EFT).

5.1 Effective field theory

To motivate the EFT idea (about which we shall be scandalously brief; for more thorough treatments, see [5–8]), suppose we start with the renormalizable SM, and consider only energies and momenta well below the weak scale, $\sim 10^2$ GeV. We can never produce W , Z , or h bosons on-shell and so we can simply do the path integral with respect to these fields (we ‘integrate them out’, to use the vernacular). At tree-level, this just corresponds to replacing the fields using their classical equations of motion, and expanding $\frac{-1}{q^2 - m_W^2} = \frac{1}{m_W^2} + \frac{q^2}{m_W^4} + \dots$. It is already clear that our expansion breaks down for momenta comparable to m_W , so that the theory is naturally equipped with a cut-off scale. We will be left with a path integral for the light fields, but with a complicated lagrangian that is non-local in space and time. But since we are only interested in low energies and momenta, we can expand in powers of the spacetime derivatives (and the fields) to obtain an infinite series of *local* lagrangian operators, which become less and less important as we go down in (energy-)momentum. We can use this theory to make predictions to any desired order in the momentum expansion simply by retaining sufficiently many terms. Many of these operators are, of course, non-renormalizable, but this is unimportant since the theory is naturally equipped with a UV cut-off $\sim 10^2$ GeV, beyond which we know that it breaks down. So we can simply cut off our loop integrals there and never have to worry about divergences. We call such a theory an EFT. The rules for making an EFT are exactly the same as those for making a QFT, except that we no longer insist on renormalizability. Instead, we specify the fields and the symmetries, write down all the possible operators, and accept that the theory will come equipped with a cut-off, Λ , beyond which the expansion breaks down.

So let us, in this way, now imagine that the SM itself is really just an effective, low-energy description of some more complete BSM theory. Thus, the fields and the (gauge) symmetries of the theory are exactly the same as in the SM, but we no longer insist on renormalizability. For operators up to dimension 4, we simply recover the SM. But at dimensions higher than 4, we obtain new operators, with new physical effects. As a striking example of these, we expect that the accidental baryon and lepton number symmetries of the SM will be violated at some order in the expansion, and protons will decay.⁷

We don't know what the BSM theory actually is yet, and so when we write down the EFT, we should allow the coefficients of the operators in the expansion to be arbitrary. Since there are infinitely many such operators, and infinitely many coefficients, one might worry that this means that predictivity is lost – it seems that we need to make infinitely many

⁷Let us hope that we can finish the lecture before they do so!

measurements (to fix all the coefficients) before we can make predictions. But this is not true, once we truncate the theory at a given order in the operator/momentum expansion: at any given finite order, the number of coefficients is finite and so we can eventually make predictions for observables, with a finite precision that is fixed by the truncation of the momentum expansion.

Before we discuss the specific operators that arise in the SM and their physical effects, it is useful to make a few technical points about EFTs.

The first point is that, while we don't know the actual values of the coefficients, we can estimate their size using dimensional analysis, since we expect the expansion to break down at energies of order the cut-off, Λ . So the natural size of coefficients is typically just an $O(1)$ number in units of Λ .

The second point is that the operators of a given dimension form a vector space, and so it is useful to choose a basis for these. This is not so straightforward as it sounds (and indeed, there are still disputes about it in the literature from time to time), because of equivalences between operators. In particular, any two operators that are equal up to a total divergence may be considered equal (since they give the same contribution at any order in perturbation theory), as may operators that differ by terms that vanish when the equations of motion hold, because such pieces can be removed by a field redefinition in the path integral (see, *e. g.*, [9]).

The third point concerns loop effects. We previously argued that operators of higher and higher dimension give smaller and smaller contributions to low energy processes. This is true for tree-level diagrams, but it is not obviously true when we insert these operators into loops, and integrate over all loop momenta up to the cut-off Λ . Apparently then, all higher-dimension operators are unsuppressed in loop diagrams. This looks like a disaster because it appears that we cannot truncate the expansion when we include loops. But we are saved by the fact that the only effect of such diagrams, once we expand them in powers of the external momenta, is to generate corrections to lower dimensional operators. So all these loops do is to correct the coefficients of other operators. This suggests that there should exist a regularization scheme in which these corrections are already taken into account, which will be far more convenient than regulating with a hard cut-off. The 'right' scheme is dimensional regularization, because then Λ s don't appear in the numerators of the loop amplitudes. The upshot is that if you ever have to compute a loop diagram in EFT, you should do it using dimensional regularization.⁸

A fourth point: if non-renormalizable EFTs make sense, why did we ever insist on renormalizability? Well, a renormalizable theory can now be thought of as a special case of a non-renormalizable theory, in which we take the cut-off to be very large. Then, the operators with dimension greater than 4 become completely negligible (hence they are known as 'irrelevant' operators in the jargon), operators with dimension =4 stay the same (hence they are 'marginal'), and operators with dimension <4 dominate (and are called 'relevant'). So if we observe physics that is well described by a renormalizable theory, we should conclude that the cut-off (at which new physics presumably appears) is rather far

⁸For an example of what happens if you don't, see [10].

away.

Finally, we can see that there is a big problem with relevant operators in EFTs. Consider for example, the mass of a scalar field in an EFT with cut-off Λ . By the arguments above, our estimate for the size of the mass (which is just a dimensionful operator coefficient) is $m \sim \Lambda$. But then the EFT is not of much use, because the particle can never be produced in the regime of validity of the EFT. So, unless there is some dynamical mechanism or tuning that makes the mass rather smaller than our estimate, the EFT does not make sense. The same is true for any relevant operator, and this leads to the hierarchy problems of the SM that we discuss in more detail below.

Now we know vaguely what an EFT is, and how to do computations using one, we can discuss the extension of the SM to an EFT and the operators that arise, starting with the most relevant ones.

5.2 $D = 0$: the cosmological constant

We have avoided mentioning it up to now, but clearly a constant term (which has dimension 0) is consistent with the symmetries of the SM. It has no effect until the SM is coupled to gravity, whereupon it causes the Universe to accelerate. On the one hand, this looks like good news, because the Universe is observed to accelerate. On the other hand, this is bad news because our estimate of the size of this operator coefficient (the operator is 1) is Λ^4 , while the observed energy density is around $(10^{-3} \text{ eV})^4$. But the cut-off of the SM had better not be 10^{-3} eV , because if it were then we could certainly not use it to make predictions at LHC energies of several TeV. So either dynamics or a tuning makes the constant small. If we consider the Planck scale to be to be a real physical cut-off, then we need to tune at the level of 1 part in 10^{120} . It is fair to say, that despite $O(10^{120})$ papers having been written on the subject, no satisfactory dynamical solution has been suggested hitherto. An alternative is to argue that we live in a multiverse in which the constant takes many different values in different corners, and we happen to live in one which is conducive to life. Indeed, it has been argued [11] that if the constant were much larger and positive, structure could never form, while if it were too large and negative, the Universe would recollapse before life could appear. The flavour-of-the-month as regards how the multiverse itself arises is by a process of eternal inflation in string theory.

5.3 $D = 2$: the Higgs mass parameter

The only other relevant operator in the SM is the Higgs mass parameter, which sets the weak scale. As above, the natural size for this is Λ . But we measure $v \sim 10^2 \text{ GeV}$, leaving us with 2 options: either the natural cut-off of the SM is not far above the weak scale (in which case we can hope to see evidence for this, in the form of new physics, at the LHC) or the cut-off is much larger, and the weak scale is tuned, perhaps once again by anthropics.

5.4 $D = 4$: marginal operators

We have discussed these already in the context of the renormalizable SM, and there is nothing to add here.

5.5 $D = 5$: neutrino masses and mixings

Now things get more interesting. There is precisely one operator at $D = 5$, namely $\frac{\lambda^l}{\Lambda}(lH)^2$, where λ^l is a dimensionless 3×3 matrix in flavour space. Note that this operator violates the individual and total lepton numbers; moreover, it gives masses to neutrinos after EWSB (exercise 24), just as we observe. So, one might argue that it is no surprise that neutrino masses have been observed, since they represent the *leading* deviation from the SM, in terms of the operator expansion. Given the observed 10^{-3} eV^2 mass-squared differences of the neutrinos, we estimate $\Lambda \sim 10^{14} \text{ GeV}$. Thus, one could also argue that while neutrino masses are undeniable evidence for physics BSM, they are also evidence that the SM is valid up to energy scales that are way, way beyond the reach of conceivable future colliders.

Even so, it is worthwhile to consider what theory might replace the EFT at Λ to give a UV completion, extending the regime of validity.

In fact there is an extremely simple possibility, which is to add to the SM a new fermion, ν^c , that is a singlet under $SU(3) \times SU(2) \times U(1)$. In fact we need at least 2 of these to generate the two observed neutrino mass-squared differences, and it seems plausible that there are 3 – one for each SM family.

We may then replace the $D = 5$ operator with the renormalizable Yukawa term $\lambda^\nu lH^c \nu^c$ (which is a Dirac mass term for neutrinos after EWSB), along with the Majorana mass term $m^\nu \nu^c \nu^c$. This leads to the so-called ‘see-saw’ mechanism, about which you may have heard. (Exercise 25: Count the mixing angles and phases in the lepton sector, in the presence of either or both of these terms.) (Exercise 26: How is λ^l related to λ^ν and m^ν ?)

Finally, note that the other neutrino mass eigenstates in this renormalizable model need not be heavy. Indeed, they could be much lighter, but very weakly coupled to SM states.

5.6 $D = 6$: trouble at t’mill

Once we get to $D = 6$, a whole slew of operators appear. These include operators that violate baryon and lepton number, such as $\frac{qqql}{\Lambda^2}$ and $\frac{u^c u^c d^c e^c}{\Lambda^2}$ (exercise 27: check these are invariants), and which cause the proton to decay via $p \rightarrow e^+ \pi^0$. We can estimate a lower bound on Λ from the experimental bounds on the proton lifetime, $\tau_p > 10^{33} \text{ yr}$, as follows. The decay rate (which comes from the amplitude squared) is proportional to $\frac{1}{\Lambda^4}$ and the remaining dimensions must be supplied by phase space, giving a factor of m_p^5 . Plugging in the numbers (exercise 28), we get $\Lambda > 10^{15} \text{ GeV}$. Again, the implication is that new physics either respects baryon or lepton number, or is a long way away.

There are also operators that give corrections to flavour-changing processes that are highly suppressed in the SM, for reasons already given. As an example, the operator $(s^c d)(d^c s)/\Lambda^2$ contributes to Kaon mixing and measurements of Δm_K and ϵ_K yield a bound of $\Lambda > 10^5 \text{ TeV}$.

6 Grand Unification

We now turn to discuss in more detail some of the hints for physics BSM. Perhaps the most compelling of these is the apparent unification of gauge couplings.

As you know, one consequence of renormalization in QFT is that the parameters of the theory must be interpreted as being dependent on the scale at which the theory is probed. The QCD coupling and g get smaller as the energy scale goes up, while g' gets larger. Remarkably, if one extrapolates far enough, one finds that all three couplings are nearly⁹ equal¹⁰ at a very high scale, *c.* 10^{15} GeV. Could it be that, just as electromagnetism and the weak force become the unified electroweak force at the 100 GeV scale, all three forces become unified at 10^{15} GeV?

The fact that the couplings seem to become equal is a hint that we could try to make all three groups in $SU(3) \times SU(2) \times U(1)$ subgroups of one big group, with a single coupling constant. We need a group with rank at least 4 (exercise 29: why?), where the rank is the maximal number of commuting generators in a basis for the Lie algebra. The group $SU(5)$ is an obvious contender (exercise 30: prove that $SU(N)$ has rank $N-1$). How does $SU(3) \times SU(2) \times U(1)$ fit into $SU(5)$? Consider $SU(5)$ in terms of its defining rep: 5×5 unitary matrices with unit determinant acting on 5-d vectors. We can get an $SU(3)$ subgroup by considering the upper-left 3×3 block and we can get an independent $SU(2)$ subgroup from the lower right 2×2 block. There is one more Hermitian, traceless generator that is orthogonal to the generators of these two subgroups: it is $T = \sqrt{\frac{3}{5}} \text{diag}(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{2}, \frac{1}{2})$, in the usual normalization. Our goal will be to try to identify this with the hypercharge $U(1)_Y$ in the SM. To do so, we first have to work out how the SM fermions fit into reps of $SU(5)$.

Before going further, let's do a bit of basic $SU(N)$ representation theory. The defining, or *fundamental*, representation is an N -dimensional vector, α^i , acted on by $N \times N$ matrices. We can write the action as $\alpha^i \rightarrow U_j^i \alpha^j$, with the indices i, j enumerating the N components. Given this rep, we can immediately find another (at least for $N > 2$) by taking the complex conjugate. (Exercise 31: prove that the 2-d irrep of $SU(2)$ is pseudo-real.) This is called the *antifundamental* rep. It is convenient to denote an object which transforms according to the antifundamental with a downstairs index, β_i . Why? The conjugate of $\alpha^i \rightarrow U_j^i \alpha^j$ is $\alpha^{*i} \rightarrow U_j^{*i} \alpha^{*j} = U_i^{\dagger j} \alpha^{*j}$. So if we define things that transform according to the conjugate with a downstairs index, we can write $\beta_i \rightarrow U_i^{\dagger j} \beta_j$. The beauty of this is that $\alpha^i \beta_i \rightarrow \alpha^j U_j^i U_i^{\dagger k} \beta_k = \alpha^j \delta_j^k \beta_k = \alpha^k \beta_k$, where we used $UU^\dagger = 1$. Thus when we contract an upstairs index with a downstairs index, we get a singlet. This is, of course, much like what happens with μ, α , and $\dot{\alpha}$ indices for Lorentz transformations. Note that the Kronecker delta, δ_j^k , naturally has one up index and one down and it transforms as $\delta_i^j \rightarrow U_k^i \delta_j^k U_l^{\dagger j}$. But $UU^\dagger = 1 \implies \delta_i^l \rightarrow \delta_i^l$ and so we call δ_i^l an *invariant tensor* of $SU(N)$. Note, furthermore, that there is a second invariant tensor, namely $\epsilon_{ijk\dots}$ (or $e^{ijk\dots}$), the totally antisymmetric tensor with N indices. Its invariance follows from the relation $\det U = 1$.

These two invariant tensors allow us to find all the irreps $SU(N)$ from (tensor) products of fundamental and antifundamental irreps. The key observation is that tensors which are symmetric or antisymmetric in their indices remain symmetric or antisymmetric under the

⁹Nearly enough to be impressive, but not quite. The discrepancy is resolved in the MSSM, however.

¹⁰At the moment, this is an trivial statement: the normalization of g' is arbitrary and can always be chosen to make all three couplings meet at the same point. But we will soon be able to give real meaning to it.

group action (exercise 32), so cannot transform into one another. So to reduce a generic product rep into irreps, one can start by symmetrizing or antisymmetrizing the indices. This doesn't complete the process, because one can also contract indices using either of the invariant tensors, which also produces objects which only transform among themselves (exercise 33).

Let's see how it works for some simple examples. Start with $SU(2)$, which is locally equivalent to $SO(3)$ and whose representation theory is known to the man on the Clapham omnibus as 'addition of angular momenta in quantum mechanics'. The fundamental rep is a 2-vector (a.k.a. spin-half); call it α^j . Via the invariant tensor ϵ_{ij} this can also be thought of as an object with a downstairs index, *viz.* $\epsilon_{ij}\alpha^j$, reflecting the fact that the doublet and anti-doublet are *equivalent* representations. So all tensors can be thought of as having indices upstairs, and it remains only to symmetrize (or antisymmetrize). Take the product of two doublets for example. We decompose $\alpha^i\beta^j = \frac{1}{2}(\alpha^{(i}\beta^{j)} + \alpha^{[i}\beta^{j]})$, where we have explicitly (anti)symmetrized the indices. The symmetric object is a triplet irrep (it has (11), (22), and (12) components), while the antisymmetric object is a singlet (having only a [12] component). We write this decomposition as $2 \times 2 = 3 + 1$, where we label the irreps by their dimensions.¹¹

The representation theory of $SU(3)$ is not much harder. The fundamental is a triplet and the anti-triplet is inequivalent.¹² The product of two triplets contains a symmetric sextuplet and an antisymmetric part containing three states. We can use the invariant tensor ϵ_{ijk} to write the latter as $\epsilon_{ijk}\alpha^{[i}\beta^{j]}$, meaning that it is equivalent to an object with one index downstairs, *viz.* an anti-triplet. Thus the decomposition is $3 \times 3 = 6 + \bar{3}$. On the other hand, we cannot symmetrize the product of a 3 and a $\bar{3}$, because the indices are of different type. The only thing we can do is to separate out a singlet obtained by contracting the two indices with the invariant tensor δ_j^i . Thus $3 \times \bar{3} = 8 + 1$ (exercise 34: decompose $\alpha^i\beta_j$ explicitly). The 8 is the adjoint rep. Again, the man on the Clapham omnibus calls this 'the eightfold way'.

For $SU(5)$, things are much the same. The only reps we shall need are the smallest ones, namely the (anti)fundamental $5(\bar{5})$ and the 10 which is obtained from the antisymmetric product of two 5s.

Now let's get back to grand unified theories. We'll try to do the dumbest thing imaginable which is to try to fit some of the SM particles into the fundamental five-dimensional representation of $SU(5)$. I hope you can see that, under $SU(5) \rightarrow SU(3) \times SU(2) \times U(1)$, this breaks up into a piece (the first three entries of the vector) that transform like the fundamental (triplet) rep of $SU(3)$ and the singlet of $SU(2)$ and a piece (the last two entries of the vector) which is a singlet of $SU(3)$ and a doublet of $SU(2)$. For this to work the last two entries would have to correspond to l (since this is the only SM multiplet which is a singlet of $SU(3)$ and a doublet of $SU(2)$), in which case the hypercharge must be fixed to be $Y = -\sqrt{\frac{5}{3}}T$. Then the hypercharge of the first three entries is $+\frac{1}{3}$. This is just what we

¹¹One has to be careful doing this: $SO(5)$ for example, has 2, inequivalent 30-dimensional irreps [12].

¹²It is inequivalent, because we cannot convert one to the other using ϵ_{ij} , which has been replaced by ϵ_{ijk} .

need for d^c , except that d^c is a colour anti-triplet rather than a triplet. But we can fix it up by instead identifying $Y = +\sqrt{\frac{5}{3}}T$ and then identifying (d^c, l) with the *anti-fundamental* rep of $SU(5)$.

What about the other SM fermions? The next smallest rep of $SU(5)$ is ten dimensional. It can be formed by taking the product of two fundamentals and then keeping only the antisymmetric part of the product. But since we now know that under $SU(5) \rightarrow SU(3) \times SU(2) \times U(1)$, $5 \rightarrow (3, 1, -\frac{1}{3}) + (1, 2, +\frac{1}{2})$, you can immediately deduce (exercise 35) that $10 \rightarrow (3, 2, +\frac{1}{6}) + (\bar{3}, 1, -\frac{2}{3}) + (1, 1, +1)$. These are precisely q, u^c , and e^c .

That things fit in this way is nothing short of miraculous. Let's now justify our statement about the couplings meeting at the high scale. The $SU(5)$ covariant derivative is

$$D_\mu = \partial_\mu + ig_{\text{GUT}} A_\mu \supset ig_{\text{GUT}} \left(W_\mu^3 T^3 + i\sqrt{\frac{3}{5}} Y B_\mu \right), \quad (6.1)$$

so unification predicts that $\tan \theta_W = \frac{g}{g'} = \sqrt{\frac{3}{5}} \implies \sin^2 \theta_W = \frac{3}{8}$. This is the relation which is observed to hold good (very nearly) at the unification scale, $\Lambda \sim 10^{15}$ GeV. This scale is very high, which is bad news for testing unification. But it is just as well, since it is clear that baryon and lepton number cannot be symmetries of this model (exercise 36: why not?). So protons decay, and indeed Λ is right around the bound therefrom. When GUTs were first put forward, this led to high hopes that protons would be observed to decay, and many experiments were carried out. So far, no dice.

There is another GUT which is based on the group $SO(10)$. This is perhaps even more remarkable, in that the fifteen states of a single SM generation fit into a 16 dimensional rep (it is in fact a spinor) of $SO(10)$. You might be thinking that this doesn't look so good, but — wait for it — the sixteenth state is a SM gauge singlet and plays the rôle of ν^c .

At this point, it is worthwhile to pause and to reflect on just how much has been explained. It is already a surprise in the SM that the 3 gauge couplings are remotely comparable in size at low energy. After all, we know only that they should be at most $O(1)$ for perturbativity; tiny values for one or more of them would seem to be fine. Unification explains not only why they are comparable, but also gives a precise prediction for them which seems, very nearly, to hold. Even more, the predicted scale of unification is exactly where it needs to be: somewhat below the (ultimate?) Planck scale, but just above the bound implied by the proton lifetime. Finally, the SM multiplets fit precisely into the smallest multiplets of the GUT groups, with no missing or extra states, and charge quantization is explained, because hypercharge lies in an underlying non-Abelian group. How can this not be correct, you might wonder?

(Exercise 37: show that a scalar particle in an adjoint of $SU(5)$ can achieve the required breaking of $SU(5) \rightarrow SU(3) \times SU(2) \times U(1)$.)

7 Approaches to the hierarchy problem

In a nutshell, the electroweak hierarchy problem is to explain how the weak scale, ~ 100 GeV, emerges from a more fundamental theory with (presumably) much higher scales,

like m_{GUT} , m_P or even the indicated neutrino mass scale $\sim 10^{14}$ GeV. This would be a problem even in a classical theory. In QFT, it is even worse, because quantum corrections will typically generate all operators at low energy that are not forbidden by symmetries, with size set by the cut-off.¹³ To see the problem, consider a lagrangian with two scalar fields, ϕ and Φ , with masses m and $M(\gg m)$ respectively. Loops of Φ fields will generate corrections of order M to m . So to keep the mass of ϕ light, we need to very carefully tune these corrections against an $O(M)$ bare mass for ϕ , in order for a small mass m to emerge at low energies. More generally, in any theory with a heavy mass scale, we expect quantum corrections to lift light masses up to the heavy scale, unless some delicate mechanism prevents it.

It is worth pointing out that we do know of mechanisms by which particles can remain light in the presence of heavier scales. For example, because of chiral symmetry, quantum corrections to a light Dirac fermion mass m must be proportional to m .

To see this, let us suppose m is actually a field. We can then assign it a charge such that the chiral symmetry is restored. Explicitly, $(q, q^c) \rightarrow e^{i\alpha}(q, q^c)$ and $m \rightarrow e^{-2i\alpha}m$.

Now consider the quantum corrections to m , δm . δm must transform in the same way as m under the chiral symmetry, and so covariance implies that the expression for δm (which cannot involve the fields q and q^c) must be of the form

$$\delta m = mf(|m|^2), \tag{7.1}$$

where f is regular at the origin. This tells us immediately that δm is small when m is small.

This type of argument is a powerful one, of very general applicability, and so we pause to examine it further. To make it, we take some parameter of the theory and observe that the theory has some enhanced symmetry (here chiral symmetry) if we allow that parameter to transform in a certain way. We can use this to work out how the parameter must appear in various expressions, by insisting that the symmetry is respected. One way to think about this is to imagine that we are pretending that the parameter is just an additional field in the theory, and so we call it a *spurionic* field, or just a *spurion*. We'll use these ideas again later on.

In the case at hand, the argument explains how the electron mass, for example, can remain so small in the presence of other heavier scales. But it doesn't explain why the electron mass is so small in the first place!

There is also a mechanism by which scalar particles can be light. Indeed, we know that Goldstone bosons are massless, because of symmetry. If we have an approximate symmetry of this type, then we end up with a naturally light *pseudo*-Goldstone boson.

7.1 SUSY

Another way to make a scalar field (like the Higgs) light is to tie its mass to the mass of a fermion (which, as we have just argued, can be naturally light). Miraculously, this can be

¹³This is sometimes called Gell-Mann's totalitarian principle: everything which is not forbidden is compulsory.

achieved by enlarging the Poincaré invariance of spacetime to a supersymmetry, with extra symmetry generators that take bosons into fermions and *vice versa*. This is covered in other lectures, and I make only 2 remarks here. The first is that supersymmetry is necessarily broken in Nature, and one still needs to explain how the supersymmetry breaking scale itself is generated in a natural way. This is not a huge problem, however, in that supersymmetry breaking (see, *e. g.* [13]) can (and probably does) take place in a *hidden* sector, and there are known ways to do it. The second remark is that supersymmetry only forces the masses of fermions and bosons to be the same: they do not necessarily have to be light. In particular, in the minimal supersymmetric extension of the SM (the MSSM), there is a mass term for the Higgs bosons, and one needs a mechanism to make this mass small. This is called the μ problem; again, ways to do it are known.

7.2 Large Extra Dimensions

Another way to solve the hierarchy problem is to suppose that there aren't any high scales. In particular, it may be that the Planck scale of gravity emerges somehow from much lower energy scales. One way to do this is to suppose that there are really $n > 3$ space dimensions, with the extra $n - 3$ dimensions having size R . If so, the effective 3 + 1 dimensional Planck constant is given in terms of the fundamental scale m of $n + 1$ -d gravity by $m_P^2 = m^{n-1} R^{n-3}$ (exercise 38). If $m \sim \text{TeV}$, the hierarchy problem goes away, but we find that the extra dimensions have radius $R \sim 10^{13}$ m for $n = 1$ (even most theorists would probably have noticed this!), $R \sim 10^{-3}$ m for $n = 2$ (the realization that gravity had only been tested down to comparable distances sparked a frenzy of tests in the last decade, and we have now got down to about 10 μm) and $R \sim 10^{-8}$ m for $n = 3$ (good luck testing this). Again, 2 remarks are in order. The first is that we again have not actually solved the hierarchy problem; we have turned it into the question of why the extra dimensions are so large (again, there are ideas for how to achieve this). The second is that such large extra dimensions give another example of physics BSM that cannot be described by an EFT. Why not? Theories with extra dimensions have towers of Kaluza-Klein excitations (like the vibrating modes of a guitar string), with masses in units of $1/R$. So these new states are extremely light, for the relevant values of R . For more details, see, *e. g.*, [14, 15].

7.3 Composite Higgs

In considering the problem of the weak scale hierarchy, it is worthwhile to note that there is a large hierarchy in physics that is well understood. The hierarchy in question is that between the mass of the proton, $m_p \sim \text{GeV}$, and higher scales. It is explained by the logarithmic running of the QCD coupling constant. This starts off small at high energies and slowly increases as we go down in energy. Eventually, it becomes large enough that QCD confines, creating a low physical scale by dimensional transmutation.

The example of QCD is relevant to our discussion for another reason. As well as confinement, the strong coupling regime of QCD leads to the breaking of the approximate chiral symmetries acting on light quarks. This breaking of chiral symmetry would have led to a perfectly acceptable and natural breaking of electroweak symmetry in the SM, even if there had been no Higgs at all!

To see this in more detail, let us consider the SM without a Higgs, and with only up and down quarks, for simplicity. The global symmetry of the quark sector (in the absence of EW interactions) is $SU(2)_L \times SU(2)_R \times U(1)_B$. When we switch on the EW interactions, we gauge an $SU(2)_L \times U(1)_Y$ subgroup of this, where $Y = T_R^3 + B/2$. Now, when the QCD coupling becomes strong, $SU(2)_L \times SU(2)_R \times U(1)_B$ gets spontaneously broken to $SU(2)_V \times U(1)_B$, resulting in 3 massless Goldstone bosons (the 3 pions of QCD). But these pions get eaten in the usual way by the EW gauge fields, since the EW gauge symmetry is broken from $SU(2)_L \times U(1)_Y$ to $U(1)_Q$, where $Q = T_L^3 + T_R^3 + B/2 = T_L^3 + Y$.

So the pattern of gauge symmetry breaking is exactly what we observe in Nature, and moreover, since the theory has the custodial symmetry we mentioned earlier, we are guaranteed to get the right ratio of W and Z boson masses.

There is one small problem however, which is that the absolute masses of the W and Z are way too small! How small? Well, the W , say, gets its mass from diagrams mixing it with the pions, through a vertex coming from the matrix element between the weak current and the pion

$$\langle 0 | J_\mu^+ | \pi^-(p) \rangle \equiv i \frac{f_\pi}{\sqrt{2}} p_\mu \quad (7.2)$$

and yields (exercise 39: show this, without worrying about factors of 2) $m_W = \frac{g f_\pi}{2} = 29$ MeV.

Evidently, this is a disaster, but it is easy to get from here to a more viable model, as follows. We simply imagine that there is another gauge group, called *technicolour*, which becomes strongly coupled by slow running of its coupling constant, but at a TeV rather than a GeV.¹⁴

Technicolour was a fantastic idea, but it doesn't work, not least because, contrary to recent experimental evidence, it does not predict a Higgs boson! But we were already fairly sure that Technicolour was wrong *before* the Higgs discovery, because of problems with other electroweak precision observables (and because of problems with flavour physics, which we discuss shortly). Particularly problematic was the so-called 'S-parameter', which in the EFT language of (4.5), can be written as Π'_{3B} , and which is believed to be too large in technicolour models.¹⁵ Now we know that Π_{3B} is non-zero in the vacuum (it gives the $W^3 B$ mass mixing) and so we know that, unlike for the T parameter, no global (*ergo* momentum-independent) symmetry can make S vanish.

However, it is worth observing that Π'_{3B} transforms as a triplet of $SU(2)_L$. $SU(2)_L$ is not a symmetry of the vacuum, because it is broken the presence of the electroweak vev, v , which is a doublet. This implies that (because for $SU(2)$ irreps, $2 \otimes 2 = 3 \oplus 1$) $S \propto v^2$.

(Note that we are really making a spurionic argument, again.)

¹⁴Note that the technicolour gauge group doesn't have to be $SU(3)$: the only desideratum is that it have the same pattern of chiral symmetry breaking as QCD.

¹⁵I say 'believed to be', because we cannot actually calculate these quantities in a strongly coupled gauge theory, except for gauge groups of large rank N , in which case they are of $O(N)$. Our best estimate for their size near $N \sim 1$ is therefore $O(1)$, compared to measured values of $O(0.3)$, the latter agreeing well with the SM.

To recap, we have argued that $S \propto v^2$. But S is dimensionless, so $S \propto v^2/f^2$, where f is some dimensionful scale of the EWSB dynamics. For technicolour, f is the techni-analogue of f_π , and $v \sim f$, so $S \sim 1$. But what if we could find a theory of strong dynamics (so as to solve the hierarchy problem in the same way as QCD) in which v was somewhat smaller than f , either by accident or design? This is where the composite Higgs enters the story.

To describe it in more detail, consider a different point of view. In QCD and technicolour, the pattern of global symmetry breaking is $SU(2)_L \times SU(2)_R \rightarrow SU(2)_V$ (times an overall $U(1)$ that we'll ignore for now). This is desirable because (i) we can embed the SM gauge group in it and get the right pattern of breaking and (ii) it naturally protects the value of the T parameter. Now, we have already seen that, at least locally, $SU(2) \simeq SO(3)$ and $SU(2) \times SU(2) \simeq SO(4)$. This means that we can also write the breaking as $SO(4) \rightarrow SO(3)$. The advantage of writing it this way is that we can easily see how to change the symmetries, while preserving the two desirable features just described: we can have any $G \rightarrow H$ where G contains $SO(4)$ and H contains $SO(3)$.

The first, obvious, extension is to consider $SO(5) \rightarrow SO(4)$ [16] (though other examples have been considered [17, 18]). The Lie algebra of $SO(n)$ is generated by antisymmetric, imaginary matrices and has dimension $\frac{n(n-1)}{2}$. This means that there are $10 - 6 = 4$ broken generators and so 4 Goldstone bosons. Moreover, it is easy to show (exercise 40) that those Goldstone bosons transform as a 4 of the unbroken $SO(4)$ subgroup. But we have already seen that a 4 of $SO(4)$ is a $(2, 2)$ of $SU(2) \times SU(2)$, which are exactly the charges of the SM Higgs doublet. To summarise: a strongly-coupled model with $SO(5) \rightarrow SO(4)$ produces a set of Goldstone bosons that have precisely the same charges as the SM Higgs doublet, which of course is what we observe!

Our excitement is tempered somewhat by the realization that the Higgs boson, with mass 125 GeV, looks nothing like a massless Goldstone boson. However, we know that the $SO(5)$ symmetry of our model cannot be exact, because Nature manifestly does not exhibit it. For one thing, the SM fermions cannot be arranged into degenerate multiplets of the unbroken $SO(4)$. For another, we know that an $SU(2) \times U(1)$ subgroup of $SO(5)$ is gauged, and this breaks $SO(5)$ by singling out certain generators.

So we know the symmetry can only be at best approximate, in which case the Higgs is at best an approximate, or *pseudo* Goldstone boson. In particular, it will acquire a potential, and non-derivative couplings to other particles, just like the SM Higgs.

All of this can be computed explicitly in an EFT formalism for the low-energy Goldstone bosons and gauge bosons (but just like in QCD, we cannot compute the EFT parameters themselves from the underlying strongly-coupled theory; we can only estimate their size using naïve dimensional analysis). Such an EFT is called a *non-linear sigma model*. Unfortunately, these are often written down in a haphazard way in the literature, leading to results that are not always correct. As an example, it is often claimed in the literature that the contribution to the Higgs potential coming from the gauge bosons in an $SO(5) \rightarrow SO(4)$ model is given by

$$V(h) = A(3g^2 + g'^2) \sin^2 \frac{h}{f}, \quad (7.3)$$

where A is some constant and f is the scale of $SO(5) \rightarrow SO(4)$ breaking (it is the analogue of f_π in QCD). In fact it is given by the more general expression

$$V(h) = 2A(3g^2 \cos^4 \frac{h}{2f} + g'^2 \sin^4 \frac{h}{2f}) + 2B(3g^2 \sin^4 \frac{h}{2f} + g'^2 \cos^4 \frac{h}{2f}), \quad (7.4)$$

which reduces (exercise 41) to the claimed expression only when $A = B$. In fact, there is a general result for $G \rightarrow H$, which is that one can write down one invariant term in the lagrangian for each real or pseudo-real irrep of H that appears in decomposing the adjoint irrep of G , but that one of these is an h -independent constant. So (exercise 42), for $SO(5) \rightarrow SO(4)$ the 10-d adjoint decomposes as $10 \rightarrow (3, 1) + (1, 3) + (2, 2)$ and there are two invariant lagrangian terms.^{16 17}

The problem with a potential like (7.3) is that its minimum is at $h = 0$, meaning no EWSB.¹⁸ Salvation comes in the form of the couplings of the strong sector to the SM fermions, which must also break $SO(5)$, and thus generate contributions to $V(h)$.

These couplings must be present, because we know that the Higgs (which is here part of the strongly coupled sector) couples to fermions (and gives them mass after EWSB). There are two ways in which we can imagine the couplings arising. The first is much like the SM Yukawa couplings, in that the strong sector couples to fermion bi-linears. Schematically,

$$\mathcal{L} \supset \frac{q\mathcal{O}_h u^c}{\Lambda^{d-1}} + \dots, \quad (7.5)$$

where \mathcal{O}_h is some operator in the strong sector of arbitrary dimension d with the right quantum numbers to couple to SM fermions.

However, to this EFT lagrangian we should also add other operators that are compatible with the symmetries of the theory. Amongst these are

$$\mathcal{L} \supset \frac{qqqq}{\Lambda^2} + \Lambda^{4-d'} \mathcal{O}_h^\dagger \mathcal{O}_h. \quad (7.6)$$

The first of these is responsible for flavour changing neutral currents; for these to be small enough, $\Lambda > 10^{3-5}$ TeV. But then, in order to get a mass as large as that of the top from the operator in (7.5), we need to choose d to be rather small: $d \lesssim 1.2 - 1.3$ [21]. Next, we need to worry about the second operator in (7.6). In order not to de-stabilize the hierarchy, its dimension, d' , had better be greater than four, rendering it irrelevant.¹⁹ So what is the problem? The limit in which $d \rightarrow 1$ corresponds to a free theory (for which

¹⁶I should remark that if we instead have $SO(5) \rightarrow O(4)$ (which is desirable to protect the rate for $Z \rightarrow b\bar{b}$ decays [19]), then $(3, 1) \oplus (1, 3)$ is an irrep of $O(4)$ [12], and we get only the single invariant in (7.3).

¹⁷I wish I had time to show you how to do all this properly, but I don't. If you want to figure it out for yourself, a good place to start to learn how to write down the EFT *comme il le faut* is [20]. Next, make the gauge coupling a spurion transforming as an adjoint of both G and a copy of the subgroup to be gauged (call it K). Finally, write $G \times K$ invariants built out of the gauge coupling spurion and the Goldstone bosons.

¹⁸This is true more generally, when we gauge a subgroup of the unbroken subgroup H .

¹⁹It is, perhaps, instructive to see how the hierarchy problem of the SM is cast in this language. There, \mathcal{O}_h corresponds to the Higgs field h , with dimension close to unity, whilst $\mathcal{O}_h^\dagger \mathcal{O}_h$ is the Higgs mass operator, with dimension close to 2.

the operator \mathcal{O}_h is just the Higgs field h), and in that limit $d' \rightarrow 2d \rightarrow 2$. So in order to have an acceptable theory, we need a theory containing a scalar operator \mathcal{O}_h (with the right charges) with a dimension that is close to the free limit, but such that the theory is nevertheless genuinely strongly-coupled, with the dimension of $\mathcal{O}_h^\dagger \mathcal{O}_h$ greater than four. We have very good evidence that such a theory cannot exist[22].

In the other approach, we imagine that the elementary fermions couple linearly to fermionic operators of the strong sector [23]. Schematically, the lagrangian is

$$\mathcal{L} \sim q\mathcal{O}_{q^c} + u^c\mathcal{O}_u + \mathcal{O}_{q^c}\mathcal{O}_q + \mathcal{O}_{u^c}\mathcal{O}_u + \mathcal{O}_{q^c}\mathcal{O}_H\mathcal{O}_u \quad (7.7)$$

(where I have left out the Λ s) and the light fermion masses arise by mixing with heavy fermionic resonances of the strong sector, which feel the electroweak symmetry breaking. The beauty of this mechanism is that fermion masses can now be generated by relevant operators (*cf.* the operator that generates masses in (7.5), which is at best marginal, since $d > 1$); this means that one can, in principle, send Λ to infinity and the problems with flavour physics can be completely decoupled. There is even a further bonus, in that the light fermions of the first and second generations, which are the ones that flavour physics experiments have most stringently probed, are the ones that are least mixed with the strong sector and the flavour-changing physics that lies therein. In this model, the observed SM fermions are mixtures of elementary and composite fermions, with the lightest fermions being mostly elementary, and the top quark mostly composite. The scenario therefore goes by the name of *partial compositeness*.

It turns out (see, *e. g.*, [24]) that the fermions can give negative corrections to the mass-squared in the Higgs potential, and thus result in EWSB. Since the top quark Yukawa is somewhat bigger than the gauge couplings, this is (at least naively) the most likely outcome.

We now have something approaching a realistic model of EWSB via strong dynamics. Having built it up, we should now do our best to knock it down.

A first problem is that no one actually knows how to get a pattern of $SO(5) \rightarrow SO(4)$ global symmetry breaking out of an explicit strongly-coupled gauge theory coupled to fermions.²⁰

A second problem is the S -parameter. We have argued that the necessary suppression can be obtained if v turns out to be somewhat smaller than f , the scale of strong dynamics. Well, v is obtained by minimizing the Higgs potential $V(h)$, which contains contributions of very roughly equal size, but opposite in sign, from the top quark and gauge bosons. Thus it is possible to imagine that there is a slight cancellation due to an accident of the particular strong dynamics, such that the v that emerges is small enough. A measure of the required tuning is $\frac{v^2}{f^2}$, and the observed S -parameter requires tuning at the level of ten *per cent* or so.

The third problem concerns flavour physics. To argue, as we have done above, that the flavour problem can be decoupled, is not the same as arguing that it *is* solved. To do that, one needs to find an explicit model which possesses all the required operators, with

²⁰The breaking $SO(6) \rightarrow SO(5)$ [17] is easier to achieve, since $SO(6) \simeq SU(4)$, and unitary groups are easier to obtain.

the right dimensions. Needless to say, our ignorance of strongly-coupled dynamics means we have no idea whether such a model exists. Certainly, in all cases that have been studied (either models with large rank of the gauge group, or lattice studies), there *is* a problem with flavour constraints.

Note that in both cases, we have an ‘advantage’ with respect to technicolour models, in that we can always suppose that the peculiar dynamics of the model makes v somewhat smaller than f , whereas in technicolour we are stuck with $v \sim f$. Still we should remember that it is precisely to avoid having to do this kind of tuning that we built such models in the first place!

Despite these problems, composite Higgs models seem just as good (or just as bad) as solutions to the hierarchy problem as supersymmetric models, and so they deserve thorough investigation at the LHC. This itself is not so easy to do. Naïvely, the obvious place to look for deviations is in the Higgs sector itself, for example in the couplings of the Higgs boson to other particles. However, we know that (since such models reproduce the SM in the limit $v^2/f^2 \rightarrow 0$) the deviations must be proportional to v^2/f^2 and hence at most 10 % or so. Such deviations are hard to see at the LHC, and even at a future e^+e^- collider. Perhaps a better way is to look for the composite partners of the top quark, which must be not too heavy in order to reproduce the observed Higgs mass. Many suggestions for how to do so have been put forward and the experiments are beginning to implement them.

8 The axion

The axion offers an elegant solution to a mysterious problem of the SM called the ‘strong CP problem’. It is, moreover, an excellent candidate for the dark matter that makes up 20 % of the energy density of the Universe today. It behoves us to discuss it.

We begin by rectifying some earlier sleight of hand. In introducing the composite Higgs, we claimed that strong-coupling in QCD breaks the $SU(2)_L \times SU(2)_R$ (approximate) chiral symmetries to the vectorial $SU(2)_V$, resulting in 3 (nearly) massless pions. But the chiral symmetry is really $U(2)_L \times U(2)_R$ which gets broken to $SU(2)_V \times U(1)_B$, so why isn’t there a fourth pion, corresponding to the axial $U(1)$? This old mystery (which went by the name of the $U(1)_A$ problem) is explained by the fact that the $U(1)_A$ is not a symmetry, because of a quantum *anomaly*. (If you don’t know about these already, you’d better look in a QFT textbook now.) For our purposes, it is enough to note that rotating $(q, q^c) \rightarrow e^{i\theta}(q, q^c)$, does not send the QCD lagrangian to itself, but rather generates the term

$$\frac{\theta g_s^2}{16\pi^2} F_{\mu\nu}^a \tilde{F}_a^{\mu\nu}, \quad (8.1)$$

where $\tilde{F}_a^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\sigma\rho} F_{a\sigma\rho}$. Now, we omitted this term from our original gauge lagrangian (3.2) because it may be written as a total derivative and therefore does not contribute at any order in perturbation theory. But it does not vanish non-perturbatively (in particular, it gets contributions from instanton configurations). Thus, there is no symmetry to break, and no fourth Goldstone boson. The $U(1)_A$ problem is solved.

In its place, we find a different problem. The term (8.2) violates CP . If all quarks are massive, then we cannot completely rotate it away using the above chiral rotation, and

indeed we find that the combination $\bar{\theta} = \theta + \arg \det M$ is physical, where M is the quark mass matrix. Physical quantities like the vacuum energy and the electric dipole moment (EDM) of the neutron will depend on $\bar{\theta}$. These can be computed using the EFT for the pions (called *chiral perturbation theory*; for a review, see [25]); the vacuum energy density, for example, is given by

$$E(\theta) = -m_\pi^2 f_\pi^2 \frac{m_u m_d}{(m_u + m_d)^2} \cos^2 \bar{\theta}. \quad (8.2)$$

Measurement (or rather lack of) a neutron EDM leads to a bound of $\bar{\theta} \lesssim 10^{-9}$. Since $\bar{\theta}$ is just an angular parameter (and a combination of strong and weak sector physics at that, with the latter already known to contain a distinct, $O(1)$, CP -violating phase), a naïve guess for its size would be $O(1)$. The strong CP problem is to understand why it is, in fact, so small.

The axion provides a compelling solution. To introduce it, note that if we could somehow turn $\bar{\theta}$ into a field, then the vacuum energy (8.2) would act as a potential for the field, which we should minimize to obtain $\bar{\theta} = 0$. We can achieve this miracle by making the chiral rotation of quarks a symmetry of the lagrangian, in the absence of the anomaly [26]. This requires extra fields. The original way of doing it (due, independently, to Weinberg [27] and Wilczek [28]), for example, was to imagine that there are 2 Higgs doublets, one of which is responsible for up quark masses and the other for down quark masses (like in SUSY). The corresponding Yukawa terms are

$$\lambda^u q H_u u^c + \lambda^d q H_d d^c + \text{h. c.} \quad (8.3)$$

and it is clear that the chiral rotations of the quarks can now be undone (modulo the anomaly) by equal and opposite transformations of H_u and H_d . At this point, we might be tempted to argue that, just as if we had a massless quark, we could remove $\bar{\theta}$ by an anomalous symmetry transformation, making it unphysical.

But we cannot do so, because the extra symmetry must be spontaneously broken in the vacuum. Indeed, the symmetry acts chirally on quarks, and we know that this must be broken in their vacuum by their masses. So (in the absence of the anomaly) there is a Goldstone boson $a(x)$, called the axion. The axion is a spacetime dependent field, and we cannot remove it by a constant rephasing. The net effect is that $\bar{\theta}$ is replaced everywhere by the dimensionless combination $a(x)/f_a$ (up to a constant, which we *can* re-phase away), where f_a is the symmetry breaking order parameter (just like f_π for pions or f for the composite Higgs). Note that, because of the anomaly, $a(x)$ is not really a Goldstone boson. It gets a potential $V(a) = -m_\pi^2 f_\pi^2 \frac{m_u m_d}{m_u + m_d} \cos^2 \frac{a}{f_a}$ from (8.2), which is minimised at $a = 0$, solving the strong CP problem. It also acquires couplings to other matter, which are proportional to $1/f_a$.

So all the phenomenology is fixed by f_a and we can translate experimental data into bounds on it. In the original Weinberg-Wilczek model, the symmetry is broken by the Higgs vevs, which are also responsible for EWSB, and so f_a is of order v . This model was very quickly ruled out, but it was soon realised that f_a can be given whatever value one

likes, by making the axion symmetry breaking scale independent of the weak scale. This is called the ‘invisible axion’ scenario.

What then are the observational constraints on f_a (see, *e. g.*, [29, 30] for more details)? If f_a is below about 10^9 GeV, the axion is sufficiently strongly coupled to other matter (whilst remaining sufficiently light) to be able to compete with neutrini in transporting energy out of cooling stars. In particular, there would have been observational consequences in *e. g.*, the 1987 supernova and red giants.

We can get an upper bound as follows. On cosmological scales, the axion field is Hubble damped until $t^{-1} \sim m_a$. There is no reason to suppose that the axion sits at its minimum during this period. Subsequently, for $t^{-1} \lesssim m_a$, the axion undergoes damped oscillations about its minimum with the amplitude being diluted by the expansion. It thus behaves like cold dark matter.²¹ As f_a is increased, the oscillations start later and are diluted less, and so we end up with too much dark matter today. If $f_a \gtrsim 10^{12}$ GeV, the Universe is overclosed, but if not, then the axion is an excellent dark matter candidate. Unfortunately, the axion is so weakly coupled to matter in the allowed window of f_a that it is very difficult to detect, but several ingenious experiments have either been proposed or are already under way, *e. g.* [31]. I encourage you to read more about them.

9 Afterword

I have only had time to discuss one or two examples, in the briefest terms, but I hope it is clear to you that there is plenty of evidence for physics BSM, and plenty of ideas for what constitutes it. This is good news. The bad news is that a lot of that evidence (the cosmological constant, neutrino masses, GUTs, inflation, the axion, quantum gravity) hints at scales that are a long, long way away from our current high-energy frontier (*c.* TeV). Of course, the electroweak hierarchy problem is still a very strong motivation for probing the TeV scale, but if nothing turns up in the next LHC run, then you (and your experimental colleagues) are going to need to get a lot more creative in the future. I wish you the best of luck!

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²¹If the symmetry breaking that yields the axion occurs after inflation, then there should also be defects in the axion field in the cosmos that contribute to dark matter, making the bounds more model dependent.

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