Reggeon Field Theory and Self-Duality

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A. Kovner, E. Levin, M. Li and M. Lublinsky,
"The JIMWLK evolution and the s-channel unitarity," arXiv:2006.15126, in JHEP

"Reggeon Field Theory and Self Duality: Making Ends Meet," arXiv:2007.12132, in JHEP

A. Kovner, E. Levin and M.L,
"QCD unitarity constraints on Reggeon Field Theory," arXiv:1605.03251, in JHEP

Motivation

- **QCD$(s \to \infty) \longrightarrow$ RFT**
  - Identify RFT Hilbert space and its basic degrees of freedom
  - Define algorithm for calculating the scattering amplitudes in terms of RFT degrees of freedom.
  - Realise symmetries and QCD unitarity constraints on the RFT "Fock states"
  - Construct a, hopefully unique, RFT Hamiltonian consistent with the symmetries and constraints.
  - Check for the known limits: dense-dilute (JIMWLK)

- **Get intuition from zero-dimensional toy models**

- **The main ingredients:**

  RFT Degrees of freedom: Pomerons/dipoles in toy models and, perhaps, large N_c QCD;

  Wilson lines in QCD

  s-channel unitarity = probabilistic interpretation of scattering amplitudes.

  t-channel unitarity = projectile-target symmetry (self-duality)

Summary of the results

- **In the world of toys**
  - **The Pomeron Hamiltonians containing only triple Pomeron vertex [BK($1 \to 2$) or Braun($1 \to 2$ and $2 \to 1$)] all violate s-channel unitarity.**
  - **It is possible to construct a Pomeron Hamiltonian, which respects both t- and s-channel unitarity conditions and reduces to BK in the dense-dilute limit.**

    Mueller & Salam (1996),

    Kovner and ML (2005),

    J. P. Blaizot, E. Iancu and D. N. Triantafyllopoulos (2006)

    Levin, ML, Kovner (2016)
  - **The Mueller-Salam Hamiltonian is not unique and presumably not adequate in the dense limit;** Levin, ML, Kovner, in preparation.

- **In QCD**
  - **JIMWLK Hamiltonian violates the s-channel unitarity**

    Levin, Li, ML, Kovner (2020)
  - **It is possible to construct a (family of) self-dual RFTs, which all reduce to JIMWLK in the dense-dilute limit. Yet, it is not known if any of the RFTs respect the s-channel unitarity.**

    Levin, Li, ML, Kovner (2020)

Zero dimensional toy world

A. Kovner, E. Levin and M.L, JHEP 08, 031 (2016)

1. Degrees of freedom: P **and** $\bar{P}$ **- Pomeron and its conjugate.**
$d = 1 - P$ **- target dipole;** $\bar{d} = 1 - \bar{P}$ **- projectile dipole.**

The commutation relations of P **and** $\bar{P}$ **are based on their perturbative identification:**

$$[\mathbf{P}, \bar{\mathbf{P}}] = -\gamma; \qquad \gamma \sim \mathbf{O}(\alpha_{\mathbf{s}}^{\mathbf{2}})$$

where γ **is the zero dimensional proxy for the dipole-dipole scattering amplitude.**

2. S**-matrix for scattering of** $\bar{n}$ **dipoles of the projectile on** m **dipoles of the target**

$$\langle \mathbf{m} | \bar{\mathbf{n}} \rangle = \langle \mathbf{0} | (\mathbf{1} - \mathbf{P})^{\mathbf{m}} (\mathbf{1} - \bar{\mathbf{P}})^{\bar{\mathbf{n}}} | \mathbf{0} \rangle$$

where the left and right Pomeron "Fock space vacua" are defined by

$$\langle \mathbf{0} | \bar{\mathbf{P}} = \mathbf{P} | \mathbf{0} \rangle = \mathbf{0}; \qquad \langle \mathbf{0} | \bar{\mathbf{d}} = \langle \mathbf{0} |; \qquad \mathbf{d} | \mathbf{0} \rangle = | \mathbf{0} \rangle$$

States in RFT

$$\langle \mathbf{m}| = \sum_{\mathbf{i}} \mathbf{a_i} \langle \mathbf{i}|; \qquad \mathbf{1} \geq \mathbf{a_i} \geq \mathbf{0}; \quad \sum_{\mathbf{i}} \mathbf{a_i} = \mathbf{1}$$

$$|\bar{\mathbf{n}}\rangle = \sum_{\mathbf{i}} \bar{\mathbf{a}}_{\mathbf{i}} |\mathbf{i}\rangle; \qquad \mathbf{1} \geq \bar{\mathbf{a}}_{\mathbf{i}} \geq \mathbf{0}; \quad \sum_{\mathbf{i}} \bar{\mathbf{a}}_{\mathbf{i}} = \mathbf{1}$$

Here a_i and $\bar{a}_i$ have the meaning of probabilities.

3. Rapidity evolution of the S matrix

$$\langle \mathbf{m}|\bar{\mathbf{n}}\rangle_{\mathbf{Y}} = \langle \mathbf{0}|(\mathbf{1}-\mathbf{P})^{\mathbf{m}} \mathbf{e}^{\mathbf{H}(\mathbf{P},\bar{\mathbf{P}})\mathbf{Y}} (\mathbf{1}-\bar{\mathbf{P}})^{\bar{\mathbf{n}}}|\mathbf{0}\rangle$$

The unitarity condition

$$\langle \mathbf{m}|\mathbf{e}^{\mathbf{HY}} = \sum_{\mathbf{i}} \mathbf{a_i}(\mathbf{Y}) \langle \mathbf{i}|; \qquad \mathbf{1} \geq \mathbf{a_i} \geq \mathbf{0}; \quad \sum_{\mathbf{i}} \mathbf{a_i} = \mathbf{1}$$

$$\mathbf{e}^{\mathbf{HY}}|\bar{\mathbf{n}}\rangle = \sum_{\mathbf{i}} \bar{\mathbf{a}}_{\mathbf{i}}(\mathbf{Y}) |\mathbf{i}\rangle; \qquad \mathbf{1} \geq \bar{\mathbf{a}}_{\mathbf{i}} \geq \mathbf{0}; \quad \sum_{\mathbf{i}} \bar{\mathbf{a}}_{\mathbf{i}} = \mathbf{1}$$

Fan diagram (BK) Hamiltonian

$$\mathbf{H_{BK}} = -\frac{\mathbf{1}}{\gamma}\left[\mathbf{P}\bar{\mathbf{P}} - \mathbf{P}\bar{\mathbf{P}}^{\mathbf{2}}\right]$$

Action on the dilute projectile (unitarity is preserved)

$$\mathbf{e^{\Delta H_{BK}}|\bar{n}\rangle \approx (1-\Delta\bar{n})|\bar{n}\rangle + \Delta\bar{n}|\bar{n}+1\rangle}$$

Action on the dense target (unitarity is violated)

$$\mathbf{\langle m|e^{\Delta H_{BK}} = (1+\Delta m)\langle m| - \Delta m[1+\gamma(m-1)]\langle m-1| + \Delta\gamma m(m-1)\langle m-2|}$$

Braun's Hamiltonian (projectile-target symmetric or self-dual under $P \leftrightarrow \bar{P}$)

$$\mathbf{H_{Braun} = -\frac{1}{\gamma}\left[P\bar{P} - P\bar{P}^2 - \bar{P}P^2\right]}$$

Unitarity is violated both in the evolution of the projectile and target

The source of the problem is in the algebra

$$[\mathbf{P}, \bar{\mathbf{P}}] = -\gamma$$

This commutation relation implies each dipole (either projectile or target) can scatter only once. We should allow, say, each projectile dipole to scatter multiply on target dipoles.

$$(\mathbf{1}-\mathbf{P})(\mathbf{1}-\bar{\mathbf{P}}) = [\mathbf{1}-\gamma]\,(\mathbf{1}-\bar{\mathbf{P}})(\mathbf{1}-\mathbf{P}) \qquad \longrightarrow \qquad \langle \mathbf{m}|\bar{\mathbf{n}}\rangle = (\mathbf{1}-\gamma)^{\mathbf{m}\bar{\mathbf{n}}}$$

Unitarized Toy Model: $\mathbf{H}_{\mathrm{UTM}} = -\frac{1}{\gamma}\bar{\mathbf{P}}\mathbf{P}$

$$\mathbf{e}^{\Delta \mathbf{H}_{\mathrm{UTM}}}|\bar{\mathbf{n}}\rangle \approx (\mathbf{1} - \frac{\Delta}{\gamma}(\mathbf{1}-(\mathbf{1}-\gamma)^{\bar{\mathbf{n}}}))|\bar{\mathbf{n}}\rangle + \frac{\Delta}{\gamma}(\mathbf{1}-(\mathbf{1}-\gamma)^{\bar{\mathbf{n}}})|\bar{\mathbf{n}}+\mathbf{1}\rangle$$

- This is the model by Mueller and Salam
- Both self-dual and unitary
- Reduces to BK in either dilute projectile or target limit

Yet, the model is not unique. Furthermore, at each step of the evolution it produces one dipole only. This is somewhat unphysical for dense systems of partons.

$$\mathbf{H} = \mathbf{e}^{-\frac{1}{\gamma}\bar{\mathbf{P}}\mathbf{P}} - \mathbf{1}$$

Back to QCD

A. Kovner, E. Levin, M. Li, and M.L, JHEP (2020)

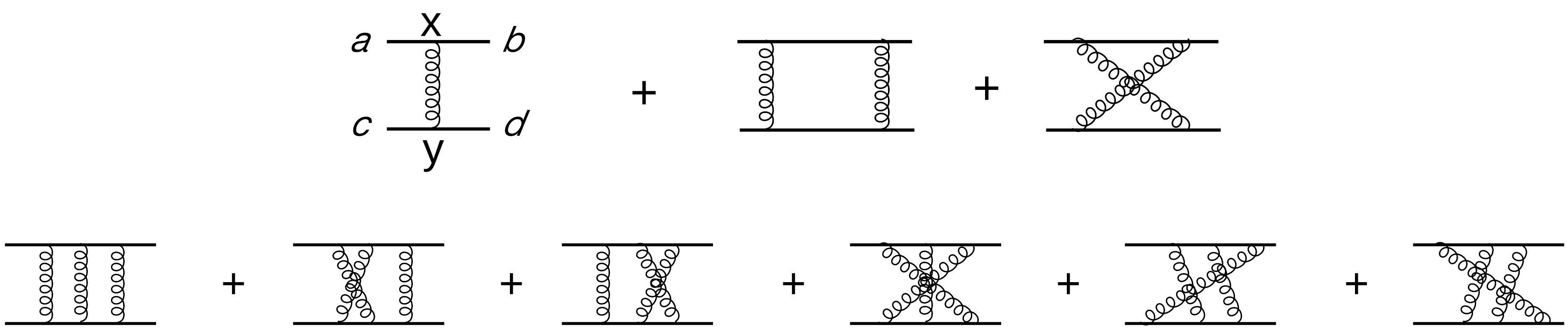
1. RFT degrees of freedom: Wilson line operator and its dual

$$\bar{\mathbf{U}}(\mathbf{x}) = \mathbf{e}^{\mathbf{T^a}\frac{\delta}{\delta\rho^{\mathbf{a}}(\mathbf{x})}} \; ; \qquad \mathbf{U}(\mathbf{x}) = \mathbf{e}^{\mathbf{igT^a}\int_{\mathbf{y}}\phi(\mathbf{x}-\mathbf{y})\rho^{\mathbf{a}}(\mathbf{y})}$$

Here ρ is the color charge density and $\phi(\mathbf{x}-\mathbf{y}) = \frac{\mathbf{g}}{\nabla^2} = \frac{\mathbf{g}}{2\pi}\ln|\mathbf{x}-\mathbf{y}|$

The algebra encodes a gluon-gluon scattering and is very complicated

$$[\bar{\mathbf{U}}(\mathbf{x}), \mathbf{U}(\mathbf{y})] = \mathbf{1} + \mathbf{g}\phi + \ldots.$$



2. Left and Right Fock vacuum states

$$\langle \mathbf{L}|\bar{\mathbf{U}}_{\mathbf{ab}} = \delta_{\mathbf{ab}}\langle \mathbf{L}|; \qquad \mathbf{U}_{\mathbf{ab}}|\mathbf{R}\rangle = \delta_{\mathbf{ab}}|\mathbf{R}\rangle$$

S-matrix

$$\mathbf{S_{if}} = \langle \mathbf{L}|\mathbf{U}^{\mathbf{a_1 b_1}}(\mathbf{x_1})\dots\mathbf{U}^{\mathbf{a_N b_N}}(\mathbf{x_N})\bar{\mathbf{U}}^{\mathbf{c_1 d_1}}(\mathbf{y_1})\dots\bar{\mathbf{U}}^{\mathbf{c_M d_M}}(\mathbf{y_M})|\mathbf{R}\rangle$$

RFT States:

$$|\mathbf{P}\rangle = \sum_{\mathbf{m},\{\bar{\mathbf{c}},\bar{\mathbf{d}};\bar{\mathbf{y}}\}} \mathbf{G}^{\mathbf{m}}_{\mathbf{M}}(\mathbf{Y},\{\mathbf{c},\mathbf{d},\mathbf{y};\ \bar{\mathbf{c}},\bar{\mathbf{d}},\bar{\mathbf{y}}\})\prod_{\mathbf{i}=1}^{\mathbf{m}}[\bar{\mathbf{U}}^{\bar{\mathbf{c}}_\mathbf{i}\bar{\mathbf{d}}_\mathbf{i}}(\bar{\mathbf{y}}_\mathbf{i})]|\mathbf{R}\rangle$$

$$|\mathbf{T}\rangle = \sum_{\mathbf{n},\{\bar{\mathbf{a}},\bar{\mathbf{b}};\bar{\mathbf{x}}\}} \mathbf{F}^{\mathbf{n}}_{\mathbf{N}}(\mathbf{Y},\{\mathbf{a},\mathbf{b},\mathbf{x};\ \bar{\mathbf{a}},\bar{\mathbf{b}},\bar{\mathbf{x}}\})\langle \mathbf{L}|\prod_{\mathbf{i}=1}^{\mathbf{n}}[\mathbf{U}^{\bar{\mathbf{a}}_\mathbf{i}\bar{\mathbf{b}}_\mathbf{i}}(\bar{\mathbf{x}}_\mathbf{i})]$$

Projectile and Target unitarity condition:

$$1 > \mathbf{G}^{\mathbf{m}}_{\mathbf{M}}(\mathbf{Y},\{\mathbf{c},\mathbf{c},\mathbf{y};\ \bar{\mathbf{c}},\bar{\mathbf{c}},\bar{\mathbf{y}}\}) > 0; \qquad \sum_{\mathbf{m}}\sum_{\bar{\mathbf{c}}}\int\{\mathbf{d}\bar{\mathbf{y}}\}\mathbf{G}^{\mathbf{m}}_{\mathbf{M}}(\mathbf{Y},\{\mathbf{c},\mathbf{c},\mathbf{y};\ \bar{\mathbf{c}},\bar{\mathbf{c}},\bar{\mathbf{y}}\}) = 1$$

$$1 > \mathbf{F}^{\mathbf{n}}_{\mathbf{N}}(\mathbf{Y},\{\mathbf{a},\mathbf{a},\mathbf{x};\ \bar{\mathbf{a}},\bar{\mathbf{a}},\bar{\mathbf{x}}\}) > 0; \qquad \sum_{\mathbf{n}}\sum_{\bar{\mathbf{a}}}\int\{\mathbf{d}\bar{\mathbf{x}}\}\mathbf{F}^{\mathbf{n}}_{\mathbf{N}}(\mathbf{Y},\{\mathbf{a},\mathbf{a},\mathbf{x};\ \bar{\mathbf{a}},\bar{\mathbf{a}},\bar{\mathbf{x}}\}) = 1$$

3. Energy evolution with RFT Hamiltonian $H_{RFT}[U,\bar{U}]$.

$$\mathbf{S_{if}(Y) = \langle L|U^{a_1b_1}(x_1)\ldots U^{a_Nb_N}(x_N)e^{YH_{RFT}[U,\bar{U}]}\bar{U}^{c_1d_1}(y_1)\ldots\bar{U}^{c_Md_M}(y_M)|R\rangle}$$

Projectile evolution

$$\mathbf{\bar{U}^{c_1d_1}(y_1)\ldots\bar{U}^{c_Md_M}(y_M)|R\rangle \to e^{YH_{RFT}[U,\bar{U}]}\bar{U}^{c_1d_1}(y_1)\ldots\bar{U}^{c_Md_M}(y_M)|R\rangle}$$

$$\mathbf{= \sum_{m,\{\bar{c},\bar{d};\bar{y}\}} G^m_M(Y,\{c,d,y;\ \bar{c},\bar{d},\bar{y}\})\prod_{i=1}^{m}[\bar{U}^{\bar{c}_i\bar{d}_i}(\bar{y}_i)]|R\rangle}$$

Projectile unitarity condition is satisfied by H_{JIMWLK}

Target evolution

$$\mathbf{\langle L|U^{a_1b_1}(x_1)\ldots U^{a_Nb_N}(x_N) \to \langle L|U^{a_1b_1}(x_1)\ldots U^{a_Nb_N}(x_N)e^{YH_{RFT}}}$$

$$\mathbf{= \sum_{n,\{\bar{a},\bar{b};\bar{x}\}} F^n_N(Y,\{a,b,x;\ \bar{a},\bar{b},\bar{x}\})\langle L|\prod_{i=1}^{n}[U^{\bar{a}_i\bar{b}_i}(\bar{x}_i)]}$$

Target unitarity condition is violated by H_{JIMWLK}!

JIMWLK Hamiltonian

$$\mathbf{H_{JIMWLK}} = \frac{\alpha_s}{2\pi^2}\int_{x,y,z} \frac{(\mathbf{x}-\mathbf{z})\cdot(\mathbf{y}-\mathbf{z})}{(\mathbf{x}-\mathbf{z})^2(\mathbf{y}-\mathbf{z})^2}\left[2\mathcal{J}_L^a(\mathbf{x})\mathcal{J}_R^b(\mathbf{y})\bar{\mathbf{U}}^{ab}(\mathbf{z}) - \mathcal{J}_L^a(\mathbf{x})\mathcal{J}_L^a(\mathbf{y}) - \mathcal{J}_R^a(\mathbf{x})\mathcal{J}_R^a(\mathbf{y})\right] .$$

$$[\mathcal{J}_L^a(\mathbf{x}), \bar{\mathbf{U}}^{mn}(\mathbf{y})] = -(\mathbf{T}^a\bar{\mathbf{U}}(\mathbf{y}))^{mn}\delta(\mathbf{x}-\mathbf{y}) ,$$
$$[\mathcal{J}_R^a(\mathbf{x}), \bar{\mathbf{U}}^{mn}(\mathbf{y})] = -(\bar{\mathbf{U}}(\mathbf{y})\mathbf{T}^a)^{mn}\delta(\mathbf{x}-\mathbf{y}) .$$

$\mathbf{SU_L(N)\times SU_R(N)}$:

$$[\mathcal{J}_L^a(\mathbf{x}), \mathcal{J}_L^b(\mathbf{y})] = \mathbf{i}f^{abc}\mathcal{J}_L^c(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}) ,$$

$$[\mathcal{J}_R^a(\mathbf{x}), \mathcal{J}_R^b(\mathbf{y})] = -\mathbf{i}f^{abc}\mathcal{J}_R^c(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})$$

$$[\mathcal{J}_L^a(\mathbf{x}), \mathcal{J}_R^a(\mathbf{y})] = 0$$

H_{JIMWLK} is obviously not symmetric (self-dual) under $U \leftrightarrow \bar{U}$ (interchange of projectile and target)

Towards Self-dual H_{RFT}

$$[\mathbf{U^{mn}}(\mathbf{y}), \mathcal{I}^{\mathbf{a}}_{\mathbf{L}}(\mathbf{x})] = -(\mathbf{T^a U}(\mathbf{y}))^{\mathbf{mn}} \delta(\mathbf{x}-\mathbf{y}) \,,$$

$$[\mathbf{U^{mn}}(\mathbf{y}), \mathcal{I}^{\mathbf{a}}_{\mathbf{R}}(\mathbf{x})] = -(\mathbf{U}(\mathbf{y})\mathbf{T^a})^{\mathbf{mn}} \delta(\mathbf{x}-\mathbf{y}) \,.$$

Another $\mathbf{SU_L(N)} \times \mathbf{SU_R(N)}$:

$$[\mathcal{I}^{\mathbf{a}}_{\mathbf{L}}(\mathbf{x}), \mathcal{I}^{\mathbf{b}}_{\mathbf{L}}(\mathbf{y})] = -\mathbf{i f^{abc}} \mathcal{I}^{\mathbf{c}}_{\mathbf{L}}(\mathbf{x}) \delta(\mathbf{x}-\mathbf{y}) \,,$$
$$[\mathcal{I}^{\mathbf{a}}_{\mathbf{R}}(\mathbf{x}), \mathcal{I}^{\mathbf{b}}_{\mathbf{R}}(\mathbf{y})] = \mathbf{i f^{abc}} \mathcal{I}^{\mathbf{c}}_{\mathbf{R}}(\mathbf{x}) \delta(\mathbf{x}-\mathbf{y})$$

$$[\mathcal{I}^{\mathbf{a}}_{\mathbf{L}}(\mathbf{x}), \mathcal{I}^{\mathbf{a}}_{\mathbf{R}}(\mathbf{y})] = \mathbf{0}$$

Because $[\bar{U}, U] \neq 0$, the algebra $[\mathcal{J}_{L,R}, \mathcal{I}_{L,R}] \neq 0$ and is very complicated. So far, we know how to deal with these algebras perturbatively only.

Wilson line like operators in the *fundamental representation*

$$\mathbf{V_L(x) = \mathrm{Exp}\left\{i\int_y g\phi(x-y)t^e\mathcal{J}^e_L(y)\right\}}$$

$$\mathbf{V_R(x) = \mathrm{Exp}\left\{-i\int_y g\phi(x-y)t^e\mathcal{J}^e_R(y)\right\}}$$

$$\mathbf{\bar{V}_L(x) = \mathrm{Exp}\left\{i\int_y g\phi(x-y)t^e\mathcal{I}^e_L(y)\right\}}$$

$$\mathbf{\bar{V}_R(x) = \mathrm{Exp}\left\{-i\int_y g\phi(x-y)t^e\mathcal{I}^e_R(y)\right\}}.$$

A Self-dual Hamiltonian

$$\mathbf{H_{RFT} = \frac{1}{\pi g^2}\int d^2x\,\partial^2[\bar{V}_L^{\beta\gamma}(x)\bar{V}_R^{\delta\alpha}(x)]V_L^{\alpha\beta}(x)V_R^{\gamma\delta}(x)}$$

Reduces to JIMWLK in the dense-dilute limit

Looks very similar to the "diamond action" of
Y. Hatta, E. Iancu, L. McLerran, A. Stasto, D. Triantafyllopoulos, NPA 764, 423 (2006)
but in fact different because it is built of Wilson lines obeying different algebra.