Summing over Geometries in String Theory

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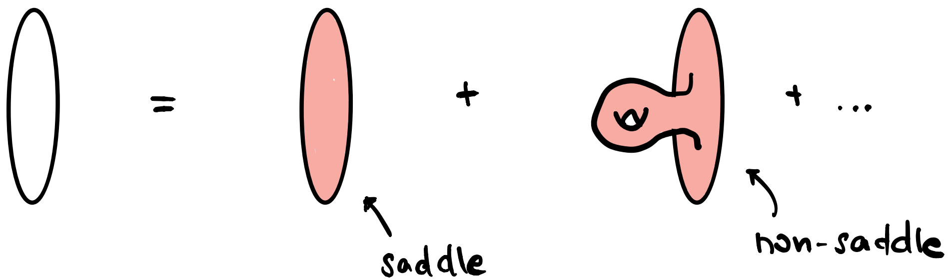
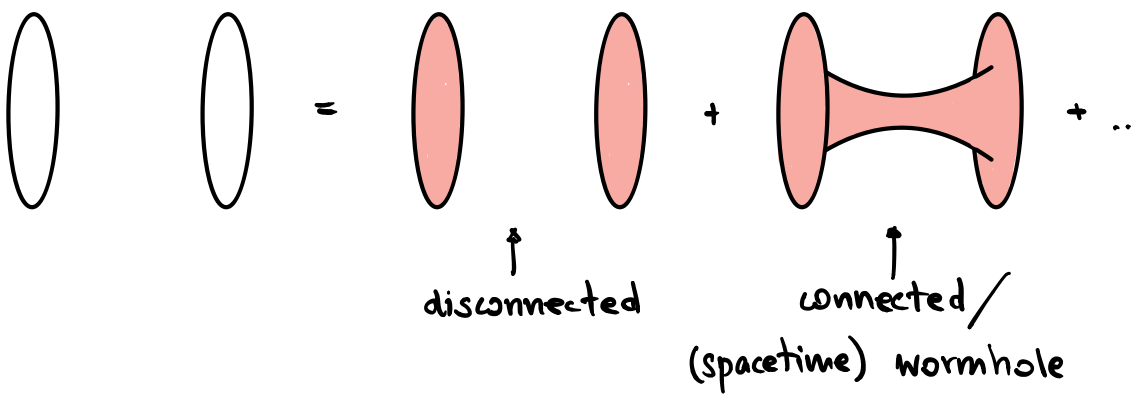
LE: [2008.07533], [21xx.xxxxx]

The gravitational path integral

- The Euclidean gravitational path integral includes a priori all possible geometries with given boundary conditions

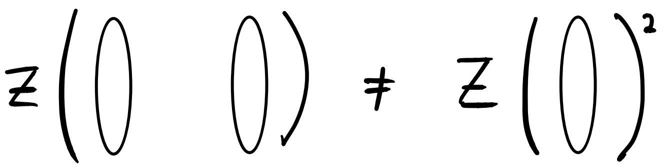
e.g. JT gravity (2d bulk / 1d boundary)





The gravitational path integral

- Including all possible geometries is very important to explain a number of phenomena:
  - Page curve for the entropy of Hawking radiation [Penington, Shenker, Stanford, Yang '19]
  - Ramp in the spectral form factor $|Z(\beta + it)|^2$. [Cotler, Gur-Ari, Hanada, Polchinski, Saad, Shenker, Stanford, Streicher, Tezuka '16]
  - Ensemble AdS/CFT correspondence



due to connected contributions

$\Rightarrow$ JT gravity is not dual to a single quantum mechanical system, but to an ensemble [Saad, Shenker, Stanford '19]

String theory

- String theory should work differently.
- We have AdS/CFT correspondences without ensemble average:

$$AdS_5 \times S^5 / \mathcal{N}=4 \text{ SYM}, \qquad AdS_3 \times S^3 \times T^4 / \text{Sym}^N(T^4)$$

[Maldacena '97]

- How does string theory achieve a sum over geometries without ensemble average?
- Usually we only consider string theory on a fixed background but here we should sum also over backgrounds

$$Z_{\text{string}}(\text{bdry conditions}) \overset{?}{=} \underbrace{\sum_{\text{bulks } \mathcal{M}}}_{\text{we know only how to include saddle geometries}} \exp\left(\sum_{g=0}^{\infty} g_s^{2g-2} \int D[\text{fields}]\; e^{-S_{\mathcal{M}}[\text{fields}]} \right)$$

$+$ nonperturbative in g_s

The tensionless string on AdS_3

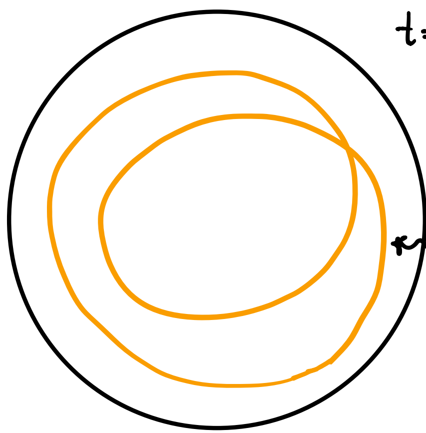
- We will discuss this with a controllable model [LE, Gaberdiel, Gopakumar '18]

$$\boxed{\begin{array}{l}\text{String theory on } AdS_3 \times S^3 \times T^4 \text{ with one unit of NSNS flux} \\ = \text{Symmetric product } \mathrm{Sym}^N(T^4) \qquad (AdS_3 \to \mathcal{M}_3)\end{array}}$$

N: number of fundamental strings in the background

- The string is tensionless: String excitations are very light, there massless higher spin fields in the spectrum
- Generic string state:

$t=0$ surface of AdS_3



string winds around the asymptotic boundary of AdS_3

The tensionless string on AdS_3

- Even the graviton is such a winding state

  $\Rightarrow$ No local notion of spacetime geometry ?

- We can still talk about background geometry

  background geometry $\Rightarrow$ worldsheet theory

  this remains well-defined

- Sum over geometries becomes sum over worldsheet theories

Bulk geometries

- We can consider this string theory on any geometry of the form

$$\mathcal{M}_3 \times S^3 \times T^4$$

$\uparrow$

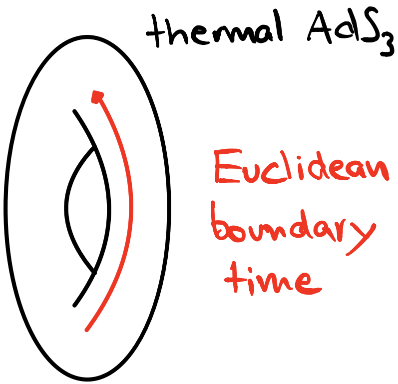
hyperbolic 3-manifold (locally Euclidean AdS_3)

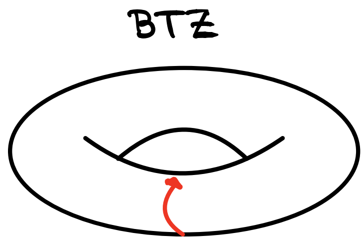
- Every $\mathcal{M}_3$ can be written as

$$\mathcal{M}_3 = \mathbb{H}^3/\Gamma \quad , \quad \Gamma \subset SL(2,\mathbb{C}) \text{ discrete subgroup}$$

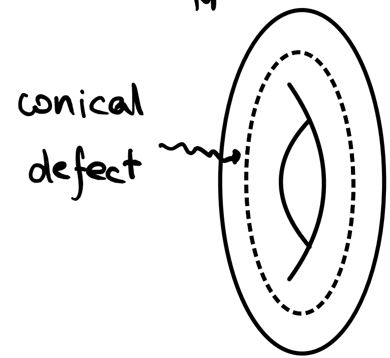
Examples

- $\Gamma \cong \mathbb{Z}$: $\mathcal{M}_3$ = solid torus = thermal AdS_3 / BTZ black hole

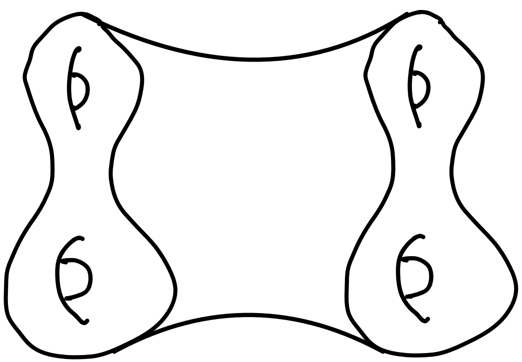




$\Gamma = \mathbb{Z} \times \mathbb{Z}_M$:



- $\Gamma \cong$ Fuchsian group $= \left\langle A_1,\dots,A_g,B_1,\dots,B_g \in SL(2,\mathbb{R}) \;\middle|\; \prod_{i=1}^{g} [A_i,B_i] = 1 \right\rangle$



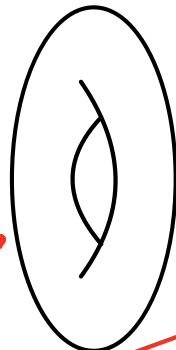
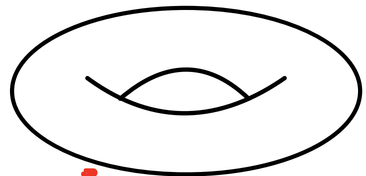
wormhole with two genus g boundaries

We necessarily have $g \geq 2$.

Thermal partition function

- The simplest object to compute is the boundary torus partition function $Z_{\mathrm{Sym}(T^4)}$.

- In 3d gravity:

$Z_{\mathrm{Sym}(T^4)} =$  $+$  $+$ $SL(2,\mathbb{Z})$ family of Euclidean black holes [Maloney, Witten '07]

$+$ conical defect geometries? [Benjamin, Collier, Maloney '20]

$+$ non saddle geometries? [Maxfield, Turiaci '20; Cotler, Jensen '20]

We can compute the perturbative string partition function on these and compare with the boundary CFT

The grand canonical ensemble

- Recall that

$$N = \# \text{ of fundamental strings in } AdS_3$$

- String perturbation theory does not fix N. [Kim, Porrati '15]
- Instead we fix a chemical potential and let N vary
- In the boundary CFT we compute the grand canonical partition function

$$\mathfrak{Z}_{\text{Sym}} = \sum_{N=0}^{\infty} p^N Z_{\text{Sym}^N(T^4)}.$$

$$= \exp\left(\sum_{L=1}^{\infty} \frac{p^L}{L} \sum_{a|L} \sum_{b \in \mathbb{Z}/\mathbb{Z}a} Z^{T^4}\begin{bmatrix} b/2 \\ a/2 \end{bmatrix}\left(\frac{L\,\tau_{\text{bdry}} + ab}{a^2}\right)\right).$$

$\uparrow$ sum over degree L coverings of the torus

Bulk computation

- We can compute the string partition function on thermal AdS_3 (& other geometries) to all orders in g_s.
- It is one-loop exact. [LE '20, wip]
- The worldsheet theory is a $\mathbb{Z}$-orbifold of the global AdS_3 worldsheet theory:

$$\text{global } AdS_3 \text{ worldsheet theory} = PSU(1,1|2)_1 \text{ WZW model}$$
$$\times \text{ top. twisted } T^4$$
$$\times \text{ ghosts}$$

this has a free field realization

$\Longrightarrow$ The worldsheet theory becomes solvable [Berkovits, Vafa, Witten '99]

Worldsheet partition function

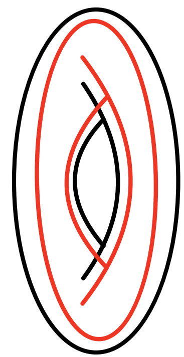
- The worldsheet torus partition function can be computed :

[LE '20]

$$Z^{\text{thermal AdS}_3}_{\text{worldsheet}} = \frac{1}{2} \operatorname{Im} \tau_{\text{bdry}} \sum_{a,b,c,d \in \mathbb{Z}} \delta^{(2)}\left(\tau_{\text{bdry}}(c\tau+d) - a\tau - b\right)$$

$$\times \; p^{\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \left| \exp\left(\frac{\pi i \tau_{\text{bdry}}}{2} \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)\right|^2 Z^{T^4}\begin{bmatrix} b/2 \\ a/2 \end{bmatrix}(\tau).$$

- $\sum_{a,b,c,d \in \mathbb{Z}}$ sums over topologically distinct embeddings of the worldsheet
- $p^{\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}}$ is the effect of including the chemical potential on the worldsheet



worldsheet torus inside target space solid torus

Localization

- The worldsheet partition function localizes in $\mathcal{M}_1$ on all holomorphic covering surfaces of the boundary torus

$$\tau_{\mathrm{bdry}} = \frac{a\tau + b}{c\tau + d}$$

- $\gamma : T^2_{\mathrm{ws}} \longrightarrow T^2_{\mathrm{bdry}}$ is a degree $-\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ covering map.

- This property can be proven for arbitrary genus & bulks $\mathcal{M}_3$.

[LE, Gaberdiel, Gopakumar '19; LE '20; Dei, Gaberdiel, Gopakumar, Knighton '20; LE '20; Knighton '20; LE, wip]

- Every covering map of a torus is a torus

$\Longrightarrow$ the string partition function is 1-loop exact

- Localization implies $\displaystyle\int_{\mathcal{M}_g} \longrightarrow \sum_{\substack{\text{covering} \\ \text{spaces}}}$

String partition function

- We can compute the string partition function on thermal AdS_3 by

$$\int_{\mathcal{M}_1} Z_{\text{worldsheet}}$$

- Easy to compute!

- We also have to account for the effect of the sphere partition function. This depends on the boundary conformal anomaly. We don't know how to compute it from first principles and take the gravity answer for it

$$Z_{\text{sphere}} = \frac{\pi c_{\text{bdry}}}{6} \operatorname{Im} T_{\text{bdry}} = \pi N \operatorname{Im} T_{\text{bdry}} .$$

String partition function

- This gives

$$\mathcal{Z}_{\text{thermal AdS}_3} = \exp\left(\sum_{a,d=1}^{\infty} \frac{p^{ad}}{ad} \sum_{b \in \mathbb{Z}/\mathbb{Z}a} Z^{T^4}\begin{bmatrix} b/2 \\ a/2 \end{bmatrix}\left(\frac{d\tau_{\text{bdry}}+b}{a} \right)\right)$$

$$= \mathcal{Z}_{\text{Sym}(T^4)} \ !$$

- Repeating the computation on other backgrounds,

$$\boxed{\begin{aligned} \mathcal{Z}_{\text{Sym}} &= \mathcal{Z}_{\text{thermal AdS}} = \mathcal{Z}_{\text{BTZ}} \\ &= \mathcal{Z}_{SL(2,\mathbb{Z})\text{ family of BH}} = \mathcal{Z}_{\text{conical defect}} \end{aligned}}$$

[LE '20]

No sum over geometries

- Instead of

$$\mathcal{Z}_{\mathrm{Sym}} = \sum_{\text{bulks } \mathcal{M}} Z_{\mathrm{string}}(\mathcal{M}),$$

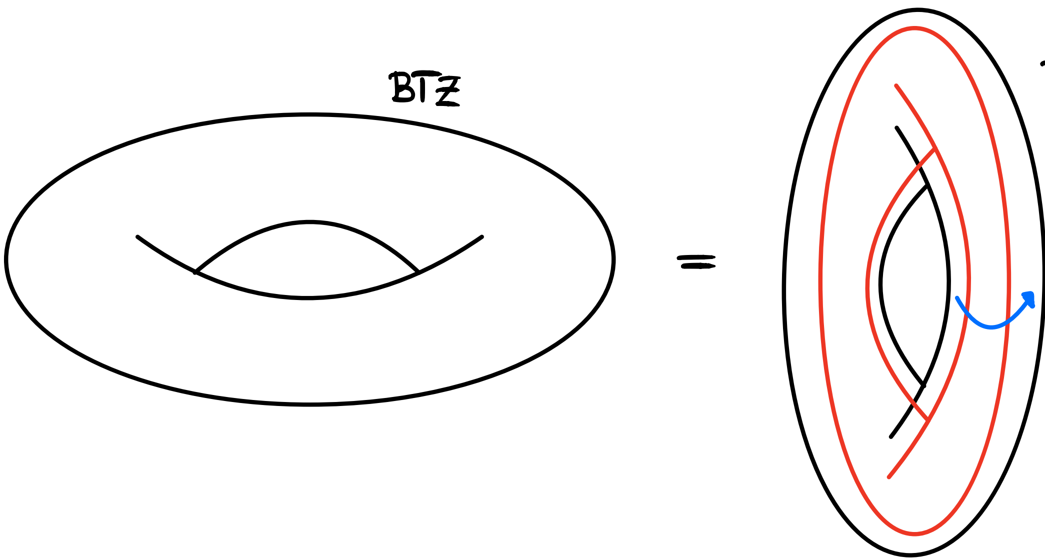
we have

$$\mathcal{Z}_{\mathrm{Sym}} = Z_{\mathrm{string}}(\text{any bulk geometry } \mathcal{M})$$

- Summing over geometries would overcount states.
- Every state in one bulk geometry can be identified with a state in another bulk geometry

Identification of states

- Let us identify the ground state of the BTZ black hole with an excited string state on thermal AdS_3.

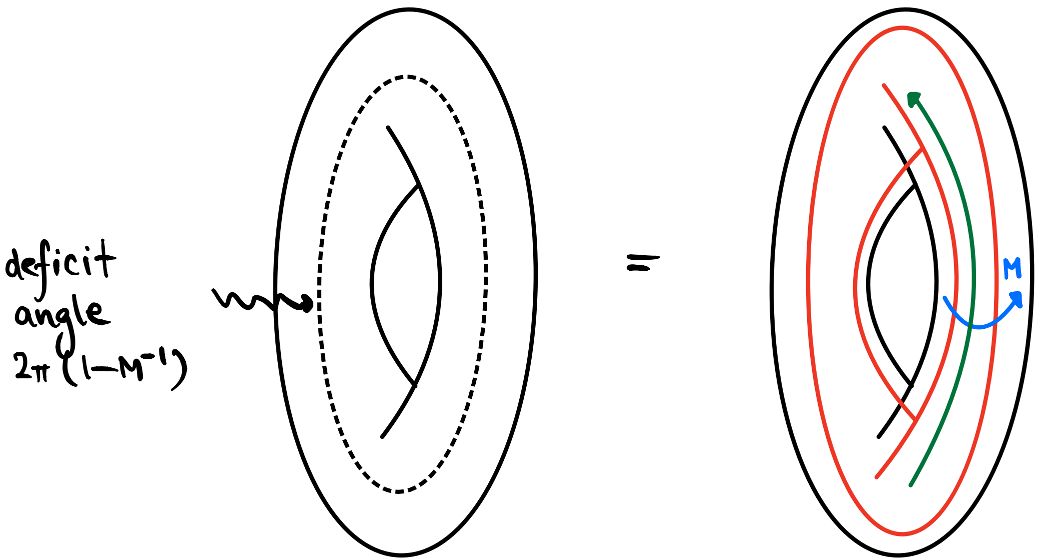


worldsheet winds arbitrarily many times around the boundary

$$\tau = -\frac{N}{\tau_{bdry}}$$

Identification of states

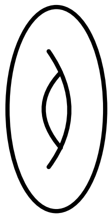
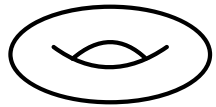
- The conical defect as an excited state in thermal AdS_3:



worldsheet winds M times around spatial cycle & arbitrarily many times around temporal cycle

Hawking Page transition

- Usually we learn in gravity

$\mathcal{Z} =$  $+$  $+ \dots$

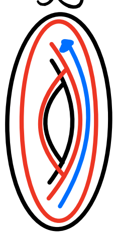
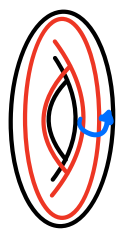
dominates at low temperatures

dominates at high temperatures

$\Rightarrow$ Phase transition!

[Hawking, Page '82]

- Here the HP transition comes from different dominating worldsheet configurations in the same background, e.g. thermal AdS_3:

$\mathcal{Z} =$  $+$  $+ \dots$

low temperature

high temperature

Analytic continuation of $\mathcal{Z}_{\mathrm{Sym}(T^4)}$

- Recall:

$$\mathcal{Z}_{\mathrm{Sym}(T^4)} = \sum_{N=0}^{\infty} p^N Z_{\mathrm{Sym}^N(T^4)}$$

- Low temperatures: $Z_{\mathrm{Sym}^N(T^4)} \sim |q|^{-\frac{N}{2}}$

  $\Rightarrow$ converges for $|p| < |q|^{1/2}$.

- But we can analytically continue in p & define $\mathcal{Z}_{\mathrm{Sym}(T^4)}$ for any complex $p = e^{2\pi i \sigma}$

Poles of $\mathcal{Z}_{\mathrm{Sym}(T^4)}$

- $\mathcal{Z}_{\mathrm{Sym}(T^4)}$ has a simple pole near

$$\sigma = \frac{i}{2} \operatorname{Im} \tau_{\mathrm{bdry}}$$

- In fact it has a pole for every semiclassical bulk geometry:

$$\sigma = \frac{i}{2M^2} \frac{\operatorname{Im} \tau_{\mathrm{bdry}}}{|c\tau_{\mathrm{bdry}} + d|^2}, \qquad \begin{array}{l} (c,d) = 1, \\ c+d \text{ odd}, \\ M \text{ odd} \end{array}$$

$\Rightarrow$ Also includes conical defects

$\Rightarrow$ No sum over geometries, but semiclassical geometries still appear

More general geometries

- We expect a pole corresponding to any bulk geometry.

- Fix $\partial\mathcal{M} = \Sigma$. For a bulk geometry $\mathcal{M}$ with $\partial\mathcal{M} = \Sigma$,

$$S_{bulk}[\mathcal{M}] = N\, S^{\circ}_{bulk}[\mathcal{M}]$$

because $N \propto \frac{1}{G_N}$ independent of N

$$Z \sim \sum_N p^N e^{-N S^{\circ}_{bulk}[\mathcal{M}]} \sim \frac{1}{1 - p\, e^{-S^{\circ}_{bulk}[\mathcal{M}]}}$$

$\Rightarrow$ Leads to a simple pole in the grand canonical partition function

- Even true for non-dominating bulk manifolds?

Condensation

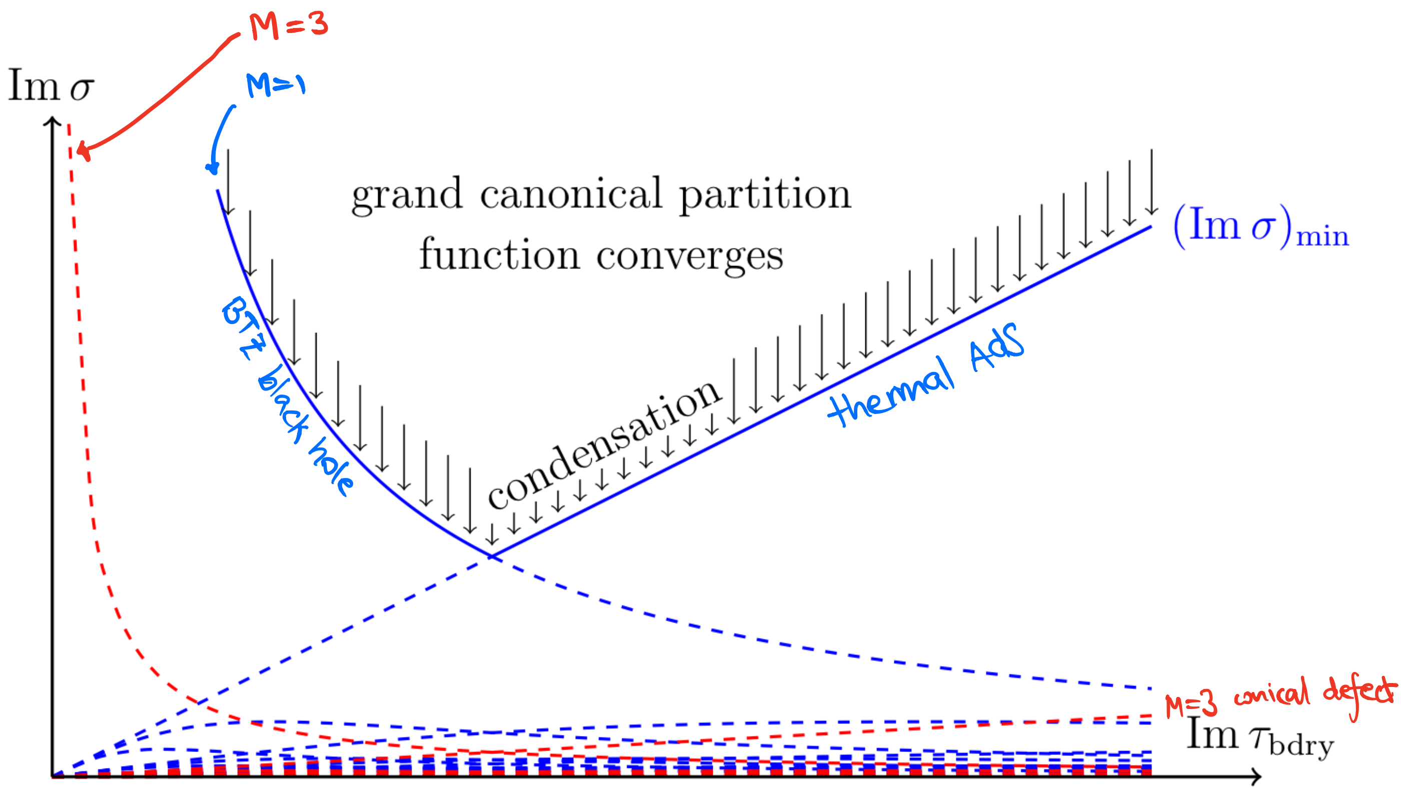
- Assume $\sigma \sim \sigma_p$, $\mathcal{Z}_{\mathrm{Sym}(T^4)}$ has a pole at σ_p.

  $\Rightarrow$ $\mathcal{Z}_{\mathrm{Sym}(T^4)}$ is dominated by contributions from very large degree

  $\Rightarrow$ semiclassical bulk geometry can emerge.

- Semiclassical geometry becomes a condensate of worldsheets

Phase diagram



A Fareytail?

[Dijkgraaf, Maldacena, Moore, Verlinde '00]

- Since all bulk geometries lead to simple poles in $\mathcal{Z}_{\mathrm{Sym}^N(T^4)}$, we can write for the thermal partition function

$$\mathcal{Z}_{\mathrm{Sym}(T^4)} = \sum_{\substack{\text{bulk manifolds} \\ \mathcal{M}}} \frac{Z^{\infty}(\mathcal{M})}{1 - p\, e^{-S^0_{\mathrm{bulk}}[\mathcal{M}]}} + \begin{array}{l}\text{contributions from} \\ \text{excited condensing} \\ \text{worldsheets}\end{array}$$

$Z^{\infty}(\mathcal{M})$: one-loop partition function on a fixed geometry $\mathcal{M}$.

- This was made precise for the symmetric orbifold of the Monster CFT [de Lange, Maloney, Verlinde '18]

Turning on interactions

- The symmetric orbifold is essentially a free field: the covering surfaces/worldsheets are not interacting.

- To make contact with semiclassical gravity more directly, we need to turn on an interaction.

- This is done by deforming the theory away from the symmetric orbifold point.

- Simpler operation that couples the covering maps: disorder average over the moduli space of T^4's (Narain moduli space)

Averaging

- The averaged partition function of T^4 can be interpreted holographically:

[Afkhami-Jeddi, Cohn, Hartman, Tajdini '20; Maloney, Witten '20]

$$\langle Z^{T^4} \rangle = \sum_{\gamma \in \Gamma_\infty \backslash PSL(2,\mathbb{Z})} \frac{1}{|\eta(\gamma \cdot \tau)^4|^2}$$

sum over bulk manifolds

one-loop determinant of CS-theory $\mathbb{R}^8$ on thermal AdS_3

$\Rightarrow$ $U(1)$-gravity.

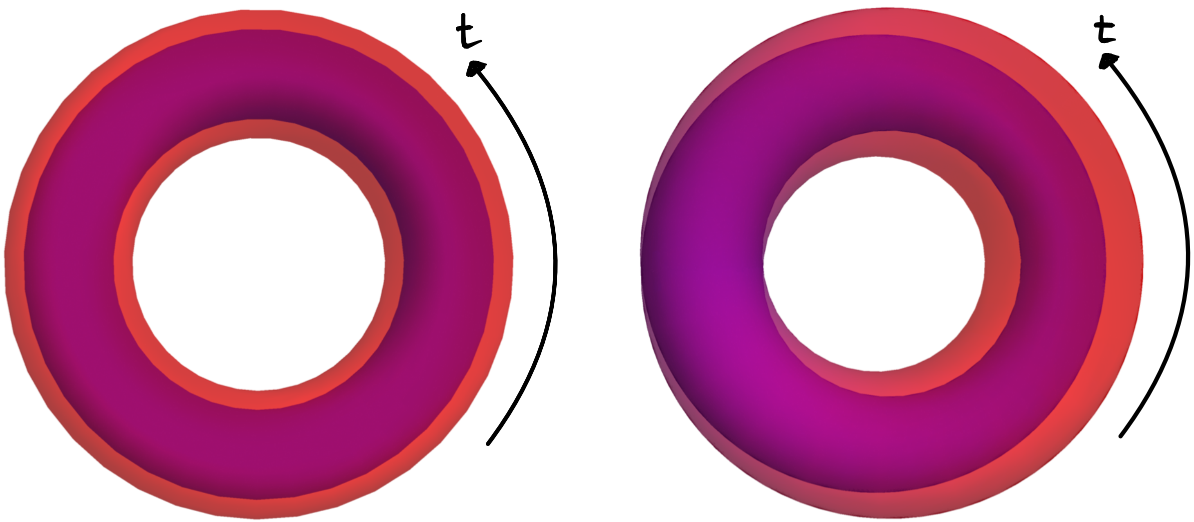
- For the symmetric orbifold, this implies that

$$\langle \mathfrak{Z}_{\mathrm{Sym}(T^4)} \rangle$$

can be computed by summing over all bulk geometries filling in the covering spaces/worldsheets

'Micro'-geometries

- This expresses $\mathcal{Z}_{\mathrm{Sym}(T^4)}$ as a sum over multiply sheeted geometries
- e.g. degree 2 covering:

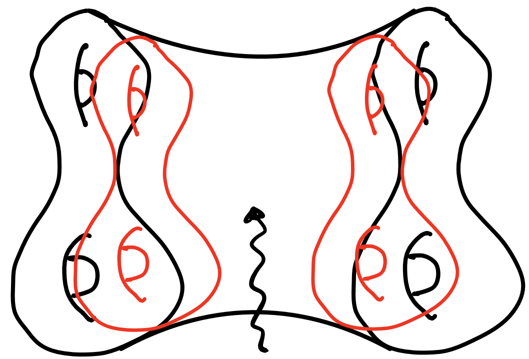


- Both these 'micro'-geometries condense to thermal AdS_3
- Semiclassical geometry can emerge when all sheets align

Factorization problem

- Consider again unaveraged theory.
- The theory can be defined on the genus g wormhole.
- How is it consistent to have a single boundary theory?
- We can compute

$$Z^{\text{string}}_{\text{wormhole}}$$



- Receives contributions from covering maps mapping to only one boundary (due to localization)
- With this one can argue

$$Z_{\text{wormhole}} = Z_L \times Z_R \quad \Rightarrow \text{Factorization restored}$$

Summary

- Worldsheet theory for the tensionless string localizes in $\mathcal{M}_g$ to covering maps of the boundary
- The partition function of the tensionless string on $\mathcal{M}_3 \times S^3 \times T^4$ depends only on $\partial\mathcal{M}_3$.
  - $\Rightarrow$ No sum over geometries
  - $\Rightarrow$ Resolution of the factorization problem
- The natural ensemble is the grand canonical ensemble
- The partition function has poles associated to bulk geometries & worldsheets can condense to these geometries
- Disorder average expresses the partition function as sum over 'micro'-geometries.