The Chirality-Flow Formalism for the SM

MCNET MEETING 20 APRIL 2021 - ANDREW LIFSON

BASED ON HEP-PH:2003.05877 (EPJC) AND HEP-PH:2011.10075

IN COLLABORATION WITH JOAKIM ALNEFJORD, CHRISTIAN REUSCHLE, AND MALIN SJÖDAHL

Outline of the Talk

Aim: To fully simplify calculations of spin structure in Feynman diagrams

- Will first go through spinor helicity methods
- Then show our new chirality-flow method
  - Will first show massless QED and examples of how to implement it
  - Then massless QCD and example
  - Then massive QED and example
- Finally show preliminary results on 3-point amplitudes
- More information available at hep-ph:2003.05877 (published in EPJC) and hep-ph:2011.10075 (accepted for publication by EPJC)

Reminder: Lorentz Group Representations

Lorentz group elements: $e^{i(\theta_i J_i + \eta_i K_i)}$ $J_i \equiv$ rotations, $K_i \equiv$ boosts

- Lorentz group generators $\simeq$ 2 copies of su(2) generators
  - $so(3,1)_{\mathbb{C}} \cong su(2) \oplus su(2)$

Group algebra defined by commutator relations

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \quad [J_i, K_j] = i\epsilon_{ijk} K_k, \quad [K_i, K_j] = -i\epsilon_{ijk} J_k$$

$$N_i^{\pm} = \frac{1}{2}(J_i \pm iK_i) \; , \quad [N_i^-, N_j^+] = 0 \; ,$$

$$[N_i^-, N_j^-] = i\epsilon_{ijk} N_k^- \; , \qquad [N_i^+, N_j^+] = i\epsilon_{ijk} N_k^+$$

- Representations
  - $(0,0)$ scalar particles
  - $(\frac{1}{2},0)$ left-chiral and $(0,\frac{1}{2})$ right-chiral Weyl (2-component) spinors.
  - $(\frac{1}{2},0) \oplus (0,\frac{1}{2})$, Dirac (4-component) spinors.
  - $(\frac{1}{2},\frac{1}{2})$ vectors, e.g. gauge bosons

Spinor-Helicity: its Building Blocks

Lorentz algebra $so(3,1) \cong su(2) \oplus su(2)$
Consider massless particles: chirality $\sim$ helicity

Spinors (use chiral basis):

$$u^+(p) = v^-(p) = \begin{pmatrix} 0 \\ |p\rangle \end{pmatrix} \qquad u^-(p) = v^+(p) = \begin{pmatrix} |p] \\ 0 \end{pmatrix}$$

$$\bar{u}^+(p) = \bar{v}^-(p) = \begin{pmatrix} [p| & 0 \end{pmatrix} \qquad \bar{u}^-(p) = \bar{v}^+(p) = \begin{pmatrix} 0 & \langle p| \end{pmatrix}$$

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad P_{L/R} = \frac{1 \mp \gamma^5}{2}$$

- Amplitude written in terms of Lorentz-invariant spinor inner products

$$\langle ij\rangle = -\langle ji\rangle \equiv \langle i||j\rangle \text{ and } [ij] = -[ji] \equiv [i||j]$$

$$\langle i||j] \equiv \bar{u}^-(p_i) P_R P_L u^-(p_j) = 0$$

- These are well known complex numbers, $\langle ij\rangle \sim [ij] \sim \sqrt{2p_i \cdot p_j}$

Spinor-Helicity: Vectors and Removing μ Indices

Lorentz algebra $so(3,1) \cong su(2) \oplus su(2)$
Consider massless particles: chirality $\sim$ helicity

Dirac matrices in chiral basis

$$\gamma^\mu = \begin{pmatrix} 0 & \sqrt{2}\tau^\mu \\ \sqrt{2}\bar{\tau}^\mu & 0 \end{pmatrix} \qquad \sqrt{2}\tau^\mu = (1, \vec{\sigma}), \quad \sqrt{2}\bar{\tau}^\mu = (1, -\vec{\sigma}),$$

Remove $\tau/\bar{\tau}$ matrices in amplitude with

$$\langle i|\bar{\tau}^\mu|j][k|\tau_\mu|l\rangle = \langle il\rangle[kj] \,, \qquad \langle i|\bar{\tau}^\mu|j] = [j|\tau^\mu|i\rangle$$

Express $\epsilon^\mu_\pm$, p^μ in terms of spinors (r arbitrary, $r \cdot p \neq 0$, $r^2 = 0$)

$$\epsilon^\mu_+(p,r) = \frac{\langle r|\bar{\tau}^\mu|p]}{\langle rp\rangle} \,, \qquad \epsilon^\mu_-(p,r) = \frac{[r|\tau^\mu|p\rangle}{[pr]}$$

$$\sqrt{2}p^\mu\tau_\mu \equiv \not{p} = |p]\langle p| \,, \qquad \sqrt{2}p^\mu\bar{\tau}_\mu \equiv \bar{\not{p}} = |p\rangle[p|$$

Define Problem

Kinematic part of amplitude slowed by spin and vector structures

- Slowed by:
  - Textbook calculations: Traces of gamma matrices, spin sums, squaring matrix rather than complex number
  - Spinor-helicity: charge conjugations and Fierz identities, antisymmetries
- Can we effectively remove these spinor-helicity issues?
- Bonus question: Can we make it intuitive to obtain the inner products?
- In SU(N) use graphical reps for calculations
  - E.g. using the colour-flow method
  - (Also birdtracks etc.)

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- In SU(N) use graphical reps for calculations
  - E.g. using the colour-flow method
  - (Also birdtracks etc.)
- Spinor-helicity $\equiv \color{blue}{su(2)} \oplus \color{red}{su(2)}$
  - Can we do the same?

Creating Chirality Flow: Building Blocks

A flow is a directed line from one object to another
$su(2)$ objects have dotted indices and $su(2)$ objects undotted indices

- Compare to colour flow:
  - Colour $\equiv$ single $SU(N)$: generators $t^a \to \delta$'s
  - Spinor-hel $\equiv su(2) \oplus su(2)$: $\tau^\mu, \bar{\tau}^\mu, g_{\mu\nu}, |i\rangle_\alpha, |j]^{\dot{\alpha}}, \epsilon^\mu_\pm, \to \langle ij\rangle, [kl]$
- First step: Ansatz for spinor inner products (only possible Lorentz invariant)

$$\langle i|^\alpha |j\rangle_\alpha \equiv \langle ij\rangle = -\langle ji\rangle =$$ 

$$[i|_{\dot{\beta}} |j]^{\dot{\beta}} \equiv [ij] = -[ji] =$$ 

- Spinors and Kronecker deltas follow

$$\langle i|^\alpha =$$ 

, $$|j\rangle_\alpha =$$ 

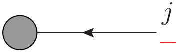
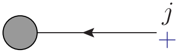
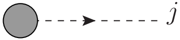
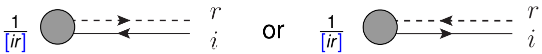
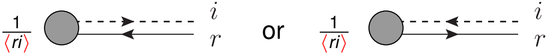
$$[i|_{\dot{\beta}} =$$ 

, $$|j]^{\dot{\beta}} =$$ 

$$\delta_\alpha^{\ \beta} \equiv \mathbb{1}_\alpha^{\ \beta} =$$ 

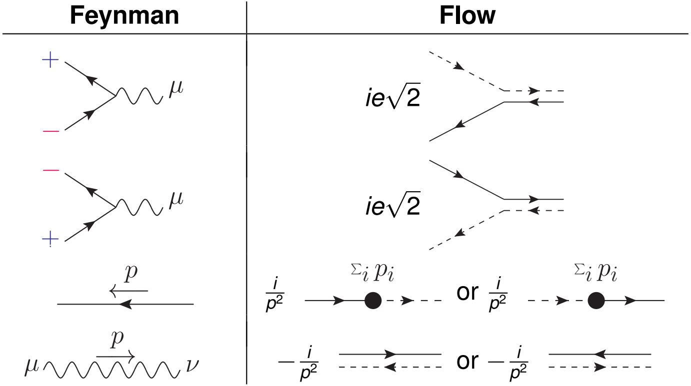
, $$\delta^{\dot{\beta}}_{\ \dot{\alpha}} \equiv \mathbb{1}^{\dot{\beta}}_{\ \dot{\alpha}} =$$ 

The QED Flow Rules: External Particles

Species	Feynman	Flow
$\bar{u}^{-}(p_i)$		
$v^{-}(p_j)$		
$v^{+}(p_j)$		
$\bar{u}^{+}(p_i)$		
$\epsilon^{\mu}_{-}(p_i, r)$		
$\epsilon^{\mu}_{+}(p_i, r)$		

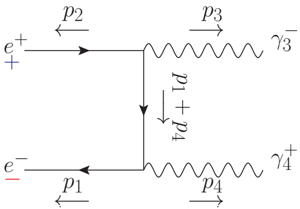
Lorentz algebra $so(3,1) \cong \underbrace{su(2)}_{\text{dotted}} \oplus \underbrace{su(2)}_{\text{undotted}}$

The QED Flow Rules: Vertices and Propagators



Lorentz algebra $so(3,1) \cong \underbrace{su(2)}_{\text{dotted}} \oplus \underbrace{su(2)}_{\text{undotted}}$

An Illuminating Example: $e^+e^- \to \gamma\gamma$



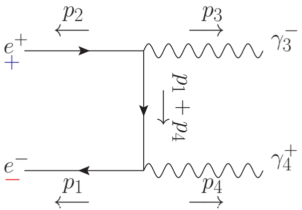
Spinor helicity:

$$\sim \langle p_1|\bar{\tau}^\mu \underbrace{\left(|p_1]\langle p_1| + |p_4]\langle p_4|\right)}_{\not{p}_1+\not{p}_4} \bar{\tau}^\nu |p_2] \underbrace{\frac{\langle r_3|\bar{\tau}_\nu|p_3]}{\langle r_3 3\rangle}}_{\epsilon_3^-} \underbrace{\frac{[r_4|\tau_\mu|p_4\rangle}{[4 r_4]}}_{\epsilon_4^+}$$

$$= \frac{\left(\langle p_1|\bar{\tau}^\mu|p_1] + \langle p_1|\bar{\tau}^\mu|p_4]\right)[r_4|\tau_\mu|p_4\rangle \left(\langle p_1|\bar{\tau}^\nu|p_2] + \langle p_4|\bar{\tau}^\nu|p_2]\right)[p_3|\tau_\nu|r_3\rangle}{\langle r_3 3\rangle [4 r_4]}$$

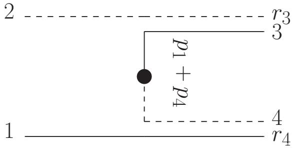
$$= \underbrace{\frac{\langle 1 r_4\rangle\left([41]\langle 13\rangle + [44]\langle 43\rangle\right)[r_3 2]}{\langle r_3 3\rangle[4 r_4]}}_{\text{Fierz identities like } \langle i|\bar{\tau}^\mu|j][k|\tau_\mu|l\rangle = \langle il\rangle[kj]} = \underbrace{\frac{\langle 1 r_4\rangle[41]\langle 13\rangle[r_3 2]}{\langle r_3 3\rangle[4 r_4]}}_{[ii]=0}$$

An Illuminating Example: $e^+e^- \to \gamma\gamma$

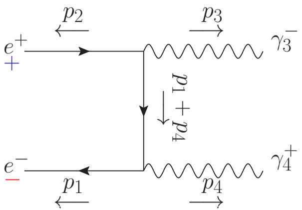


Chirality flow:

$$\sim \frac{1}{\langle r_3 3\rangle [4 r_4]}$$

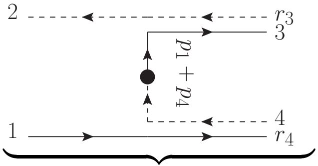


An Illuminating Example: $e^+e^- \to \gamma\gamma$



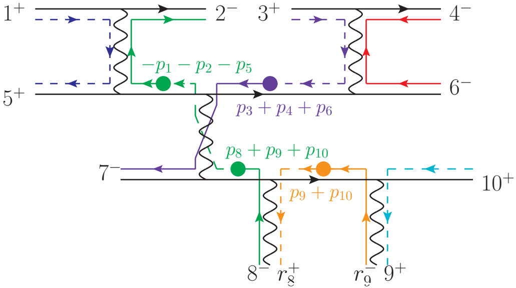
Chirality flow:

$$\sim \frac{1}{\langle r_3 3\rangle [4 r_4]}$$



$$\langle 1 r_4\rangle [41]\langle 13\rangle [r_3 2]$$

A complicated QED Example



Compare to:

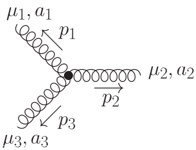
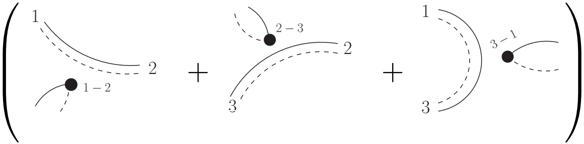
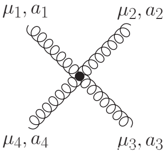
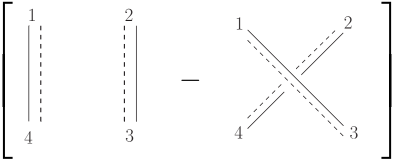
- Standard QFT:
  $2 \times \mathrm{Tr}(\gamma^{\mu_1} \cdots \gamma^{\mu_{12}})$,
  $2 \times \mathrm{Tr}(\gamma^{\mu_1} \cdots \gamma^{\mu_4})$,
  $2 \times$ photon spin sum
- Standard spinor-helicity:
  5 charge conjugation/Fierz
  $+$ rearranging

$$= \underbrace{(\sqrt{2}ei)^8}_{\text{vertices}} \underbrace{\frac{(-i)^3}{s_{1\,2}\; s_{3\,4}\; s_{8\,9\,10}}}_{\text{photon propagators}} \underbrace{\frac{(i)^4}{s_{1\,2\,5}\; s_{3\,4\,6}\; s_{8\,9\,10}\; s_{9\,10}}}_{\text{fermion propagators}} \underbrace{\frac{1}{[8r_8]\langle r_9 9\rangle}}_{\text{polarization vectors}} [15]\langle 64\rangle[10\;9]$$

$$\times \left(\langle r_9 9\rangle[9 r_8] + \langle r_9 10\rangle[10 r_8]\right)\left(\underbrace{[33]}_{0}\langle 37\rangle + [34]\langle 47\rangle + [36]\langle 67\rangle\right)$$

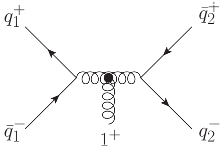
$$\times \left(-\langle 89\rangle[91]\langle 12\rangle - \langle 89\rangle[95]\langle 52\rangle - \langle 8\,10\rangle[10\;1]\langle 12\rangle - \langle 8\,10\rangle[10\;5]\langle 52\rangle\right)$$

The Non-abelian Massless QCD Flow Vertices

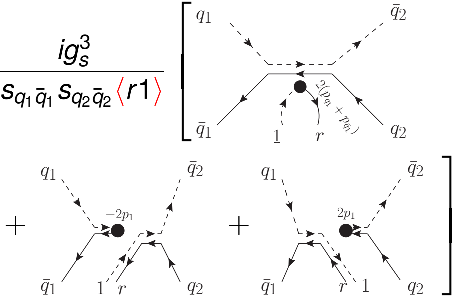
Feynman	Flow
	$-\frac{g_s f^{abc}}{2}$ 
	$ig_s^2 \sum_{Z(2,3,4)} f^{a_1 a_2 b} f^{b a_4 a_3}$ 

Arrow directions only consistently set within full diagram
Double line $\equiv g_{\mu\nu}$, momentum dot $\equiv p_\mu$

QCD Example: $q_1 \bar{q}_1 \to q_2 \bar{q}_2 g$



$$= \frac{i g_s^3}{2 s_{q_1\bar{q}_1} s_{q_2\bar{q}_2} \langle r 1\rangle}$$



$$\left[\cdots\right] \equiv \Big\{ 2[q_1 \bar{q}_2]\langle q_2 \bar{q}_1\rangle \Big([1 q_1]\langle q_1 r\rangle + [1 \bar{q}_1]\langle 1 r\rangle \Big)$$
$$- 2[q_1 1]\langle 1 \bar{q}_1\rangle\langle q_2 r\rangle[1 \bar{q}_2] + 2[q_1 1]\langle r \bar{q}_1\rangle\langle q_2 1\rangle[1 q_2] \Big\}$$

Massive Spinor Helicity Basics

Decompose massive momentum p as sum of massless ones

$$p^\mu = p^{\flat,\mu} + \alpha q^\mu \,, \quad (p^\flat)^2 = q^2 = 0 \,, \quad \alpha = \frac{p^2}{2p\cdot q}$$

- Massive spinors and polarisation vectors written in terms of massless Weyl spinors of momentum $p^\flat$, q
- We recycle results from massless chirality flow
  - E.g. $\not{p} \equiv \sqrt{2} p^\mu \tau_\mu = |p^\flat]\langle p^\flat| + \alpha |q]\langle q|$
- q is arbitrary but physical, as defines spin direction s^μ

$$s^\mu = \frac{1}{m}(p^{\flat,\mu} - \alpha q^\mu) = \frac{1}{m}(p^\mu - 2\alpha q^\mu)$$

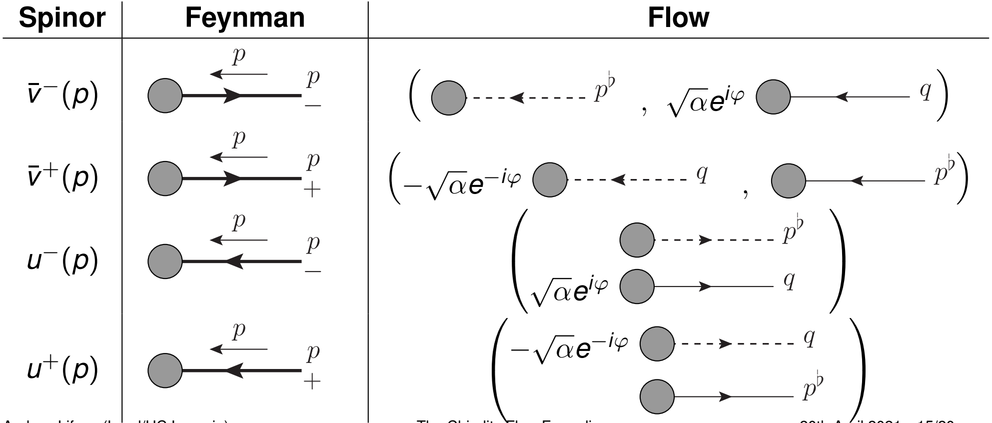
- Spin-summed amplitude independent of q choice

See e.g. hep-ph:0510157 for more details

Incoming Massive Spinors in Chirality Flow

$$p^\mu = p^{\flat,\mu} + \alpha q^\mu \,, \quad (p^\flat)^2 = q^2 = 0 \,, \quad e^{i\varphi}\sqrt{\alpha} = \frac{m}{\langle p^\flat q \rangle} \,, \qquad e^{-i\varphi}\sqrt{\alpha} = \frac{m}{[q p^\flat]}$$

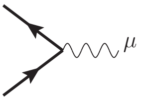
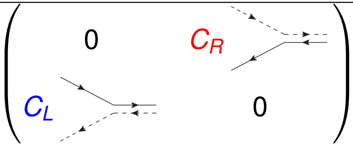
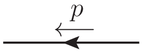
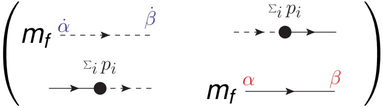
$$\text{Spin operator } -\frac{\Sigma^\mu s_\mu}{2} = \frac{\gamma^5 s^\mu \gamma_\mu}{2}, \qquad s^\mu = \frac{1}{m}(p^{\flat,\mu} - \alpha q^\mu)$$



Remaining Massive QED Flow Rules

$$p^\mu = p^{\flat,\mu} + \alpha q^\mu \,, \quad (p^\flat)^2 = q^2 = 0 \,, \quad \alpha = \frac{p^2}{2p\cdot q}$$

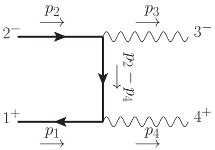
Left and right chiral couplings may differ

Feynman	Dirac	Flow
	$ie(P_L C_L + P_R C_R)\gamma^\mu$	$ie\sqrt{2}$ 
	$i\frac{\not{p}+m}{p^2-m^2}$	$\frac{i}{p^2-m_f^2}$ 

Remaining massive chirality-flow rules in backup slides

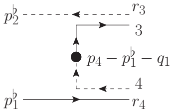
A Massive *Illuminating* Example

Consider the same diagram of $e_1^+ e_2^- \to \gamma_3^+ \gamma_4^-$ as before but include mass m_e

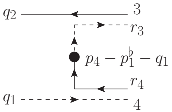


- Obtain 3 new terms
- Simplify with choices of q_1, q_2, r_3, r_4
- $e^{i\varphi_i}\sqrt{\alpha_i} = \frac{m_i}{\langle p_i^\flat q_i \rangle}$, $\quad e^{-i\varphi_i}\sqrt{\alpha_i} = \frac{m_i}{[q_i p_i^\flat]}$

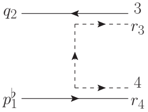
$$= \frac{-2ie^2}{(s_{23} - m_f^2)\langle r_3 3\rangle [4 r_4]} \Bigg\{$$



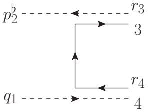
$$- \sqrt{\alpha_1 \alpha_2} e^{i(\varphi_2 - \varphi_1)}$$



$$+ m_e \Bigg(\sqrt{\alpha_2} e^{i\varphi_2}$$



$$- \sqrt{\alpha_1} e^{-i\varphi_2}$$



$$\Bigg) \Bigg\}$$

Little Group Scaling

Little group definition

Subset of Lorentz transformation which leaves momentum p invariant

Scaling for spinors (flow lines) is:

$$\not{p} = |p]\langle p| \xrightarrow{\text{litle group}} \not{p} = (t^{-1}|p])(t\langle p|)$$

Amplitude scales as:

$$A(\ldots,\{t_i|i\rangle, t_i^{-1}|i], h_i\}\ldots) = t_i^{-2h_i} A(\ldots,\{|i\rangle, |i], h_i\}\ldots)$$

Can show that 3-pt amplitudes contain only $\langle ij\rangle$ or $[ij]$ but not both

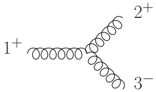
Combining above information gives

$$A_3(1^{h_1}, 2^{h_2}, 3^{h_3}) = c'\langle 12\rangle^{+(h_3-h_1-h_2)}\langle 13\rangle^{+(h_2-h_1-h_3)}\langle 23\rangle^{+(h_1-h_2-h_3)}, \quad \text{or}$$
$$A_3(1^{h_1}, 2^{h_2}, 3^{h_3}) = c[12]^{-(h_3-h_1-h_2)}[13]^{-(h_2-h_1-h_3)}[23]^{-(h_1-h_2-h_3)},$$

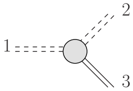
Little Group Scaling and Chirality Flow

$$A_3(1^{h_1}, 2^{h_2}, 3^{h_3}) = c[12]^{-(h_3-h_1-h_2)}[13]^{-(h_2-h_1-h_3)}[23]^{-(h_1-h_2-h_3)},$$
$$A(\ldots, \{t_i|i\rangle, t_i^{-1}|i], h_i\} \ldots) = t_i^{-2h_i} A(\ldots, \{|i\rangle, |i], h_i\} \ldots)$$

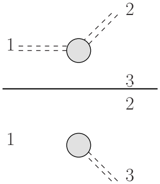
Possible new way to derive 3-pt amplitude



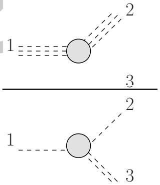
$= A_3(1^+, 2^+, 3^-) \sim c$



$\xrightarrow{\text{only } \langle ij\rangle \text{ or } [ij]}$ c



$\xrightarrow{\times 1}$ c

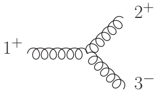


$\xrightarrow{\text{connect lines}}$ $c\dfrac{[12]^3}{[13][23]}$

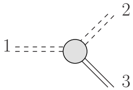
Little Group Scaling and Chirality Flow

$$A_3(1^{h_1}, 2^{h_2}, 3^{h_3}) = c[12]^{-(h_3-h_1-h_2)}[13]^{-(h_2-h_1-h_3)}[23]^{-(h_1-h_2-h_3)} \,,$$
$$A(\ldots, \{t_i|i\rangle, t_i^{-1}|i], h_i\}\ldots) = t_i^{-2h_i} A(\ldots, \{|i\rangle, |i], h_i\}\ldots)$$

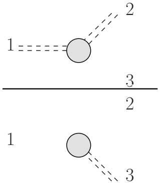
Possible new way to derive 3-pt amplitude



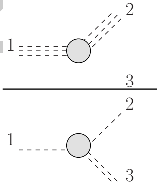
$= A_3(1^+, 2^+, 3^-) \sim c$



$\xrightarrow{\text{only } \langle ij\rangle \text{ or } [ij]}$ c



$\xrightarrow{\times 1}$ c



$\xrightarrow{\text{connect lines}}$ $c\dfrac{[12]^3}{[13][23]}$ ✓

Conclusions and Outlook

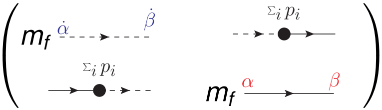
- Chirality flow offers the shortest possible journey from Feynman diagram to complex number
  - Further simplifies the spinor helicity formalism
  - Calculations often performed in a single step, particularly for massless diagrams
- Full standard model at tree level understood
  - Useful at tree level for *any* model with only Dirac fermions and matries (Pauli matrices), Minkowski metric, momenta, spin 0 and 1 bosons in Feynman rules
- Loops next on the agenda
- 3 point amplitudes and recursions look interesting!
- Useful for generators based on Feynman diagrams
- Useful for quick pen and paper calculations and checks

Fermion Lines with Multiple Emissions

$$p^\mu = p^{\flat,\mu} + \alpha q^\mu \,, \quad (p^\flat)^2 = q^2 = 0 \,, \quad \alpha = \frac{p^2}{2p\cdot q}$$

Fermion propagator

$$\frac{i}{p^2 - m_f^2}\begin{pmatrix} m_f \delta^{\dot\alpha}{}_{\dot\beta} & \sqrt{2}p^{\dot\alpha\beta} \\ \sqrt{2}\bar{p}_{\alpha\dot\beta} & m_f \delta_\alpha{}^\beta \end{pmatrix} = \frac{i}{p^2 - m_f^2}$$



- Propagators and vertices don't always contribute factor $\tau/\bar\tau$
- Have to update arrow swap procedure to include even number of $\tau/\bar\tau$

$$\langle i|\bar\tau^{\mu_1}\tau^{\mu_2}\ldots\bar\tau^{\mu_{2n+1}}|j] = [j|\tau^{\mu_{2n+1}}\bar\tau^{\mu_{2n}}\ldots\tau^{\mu_1}|i\rangle$$

$$\langle i|\bar\tau^{\mu_1}\tau^{\mu_2}\ldots\tau^{\mu_{2n}}|j\rangle = -\langle j|\bar\tau^{\mu_{2n}}\bar\tau^{\mu_{2n-1}}\ldots\tau^{\mu_1}|i\rangle$$

$$[i|\tau^{\mu_1}\bar\tau^{\mu_2}\ldots\bar\tau^{\mu_{2n}}|j] = -[j|\tau^{\mu_{2n}}\bar\tau^{\mu_{2n-1}}\ldots\bar\tau^{\mu_1}|i]$$

Arrow flips may induce minus signs! Care must be taken

Massive Polarisation Vectors

Backup Slides

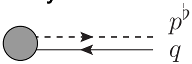
$$p^\mu = p^{\flat,\mu} + \alpha q^\mu \;, \quad (p^\flat)^2 = q^2 = 0 \;, \quad \alpha = \frac{p^2}{2p\cdot q}$$

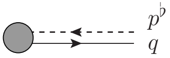
- External gauge bosons

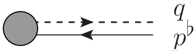
$$\epsilon_+^\mu(p) = \frac{[p^\flat|_{\dot\alpha}\tau^{\mu,\dot\alpha\beta}|q\rangle_\beta}{\langle q p^\flat\rangle} \;, \qquad \epsilon_-^\mu(p) = \frac{\langle p^\flat|^\alpha \bar\tau^\mu_{\alpha\dot\beta}|q]^{\dot\beta}}{[p^\flat q]}$$

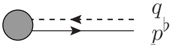
$$\epsilon_0^\mu(p) = s^\mu = \frac{1}{m}(p^{\flat,\mu} - \alpha q^\mu)$$

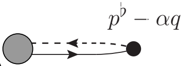
- Translate to chirality flow

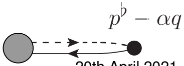
$\epsilon_+^\mu(p) \to \frac{1}{\langle ri\rangle}$ 

, or $\epsilon_+^\mu(p) \to \frac{1}{\langle ri\rangle}$ 

$\epsilon_-^\mu(p) \to \frac{1}{[p^\flat q]}$ 

, or $\epsilon_-^\mu(p) \to \frac{1}{[p^\flat q]}$ 

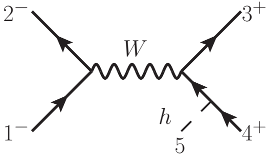
$\epsilon_0^\mu(p) \to \frac{1}{m\sqrt{2}}$ 

, or $\epsilon_0^\mu(p) \to \frac{1}{m\sqrt{2}}$ 

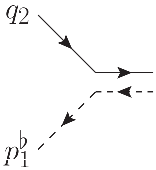
A Second Massive Example: $f_1 \bar{f}_2 \to W \to f_3 \bar{f}_4 h_5$

Backup Slides

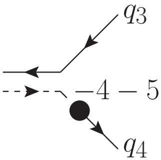
- W bosons simplifies ($C_R = 0$)
- Simplify with choices of $q_1, \cdots q_5$
- $e^{i\varphi_i}\sqrt{\alpha_i} = \frac{m_i}{\langle p_i^\flat q_i \rangle}$, $\quad e^{-i\varphi_i}\sqrt{\alpha_i} = \frac{m_i}{[q_i p_i^\flat]}$
- Scalar has no flow line



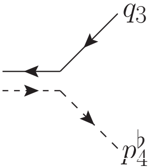
Step 1: Draw fermion lines: $\sim C_{L,12}\sqrt{\alpha_2}e^{i\varphi_2}$



$\times\, C_{L,34}\sqrt{\alpha_3}(-e^{i\varphi_3})$ [$\sqrt{\alpha_4}(-e^{i\varphi_4})$



$+\, m_4$



]

A Second Massive Example: $f_1 \bar{f}_2 \to W \to f_3 \bar{f}_4 h_5$

- W bosons simplifies ($C_R = 0$)
- Simplify with choices of $q_1, \cdots q_5$
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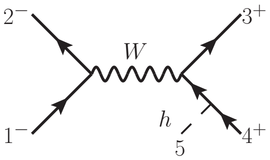
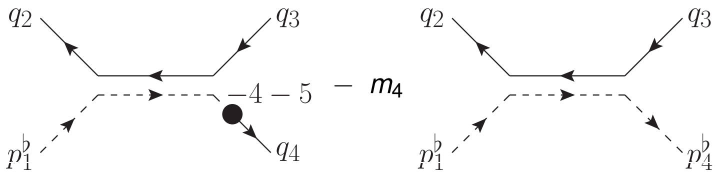


Step 2: Flip arrows and connect: $C_{L,12} C_{L,34} \sqrt{\alpha_2 \alpha_3} e^{i(\varphi_2+\varphi_3)}$

$\times$ [$\sqrt{\alpha_4} e^{i\varphi_4}$ ]

The Helicity Basis in Massive Spinor Helicity

Decompose massive momentum p as sum of massless ones

$p^\mu = p^{\flat,\mu} + \alpha q^\mu \,, \quad (p^\flat)^2 = q^2 = 0 \,, \quad \alpha = \frac{p^2}{2p\cdot q}$

Spin measured along $s^\mu = \frac{1}{m}(p^{\flat,\mu} - \alpha q^\mu) = \frac{1}{m}(p^\mu - 2\alpha q^\mu)$

- Consider eigenvectors/values of $\not{p}, \bar{\not{p}}$

$$\not{p}|p_{f/b}] = \lambda_{f/b}|p_{f/b}] \qquad \bar{\not{p}}|p_{f/b}\rangle = \lambda_{f/b}|p_{f/b}\rangle$$

$$\lambda_{f/b} = p^0 \pm |\vec{p}| \qquad p^\mu_{f/b} = \frac{\lambda_{f/b}}{2}(1, \pm\hat{p})$$

See e.g. hep-ph:9805445, hep-ph:2011.10075 for more details

The Helicity Basis in Massive Spinor Helicity

Decompose massive momentum p as sum of massless ones

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$$\not{p}|p_{f/b}] = \lambda_{f/b}|p_{f/b}] \qquad \bar{\not{p}}|p_{f/b}\rangle = \lambda_{f/b}|p_{f/b}\rangle$$

$$\lambda_{f/b} = p^0 \pm |\vec{p}| \qquad p^\mu_{f/b} = \frac{\lambda_{f/b}}{2}(1, \pm\hat{p})$$

Conclusion: in helicity basis!

$$p^\mu = p_f^\mu + p_b^\mu , \quad p_f^2 = p_b^2 = 0 , \qquad p^\flat \to p_f , \quad \alpha \to 1 , \quad q \to p_b$$

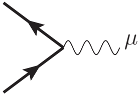
Spin measured along $s^\mu = \frac{1}{m}(p_f^\mu - p_b^\mu) = \frac{1}{m}(|\vec{p}|, p^0\hat{p}) \equiv$ direction of motion!

See e.g. hep-ph:9805445, hep-ph:2011.10075 for more details

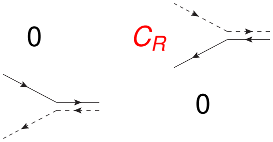
Fermion Vertices

$$p^\mu = p^{\flat,\mu} + \alpha q^\mu\,, \quad (p^\flat)^2 = q^2 = 0\,, \quad \alpha = \frac{p^2}{2p\cdot q}$$

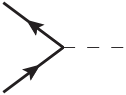
Fermion-vector vertex



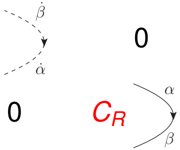
$$= ie(P_L C_L + P_R C_R)\gamma^\mu = ie\sqrt{2}\begin{pmatrix} 0 & C_R \\ C_L & 0 \end{pmatrix}$$



Fermion-scalar vertex



$$= ie(P_L C_L + P_R C_R) = ie\begin{pmatrix} C_L & 0 \\ 0 & C_R \end{pmatrix}$$



Left and right chiral couplings may differ

Massless and Massive Particles

Little group

Little group is set of Lorentz transformation leaving p^μ invariant
Little group is responsible for spin degrees of freedom

Massless spin J

- $U(1)$ little group
- Two helicity states $h = \pm J$
- Spinors: one chirality per helicity:
  - $u^+ = u_R$
  - $u^- = u_L$
- Two-component Weyl spinors
- Gauge bosons/gauge redundancy

Massive spin J

- $SU(2)$ little group
- States have fixed-spin along axis s: $J_s = J, J-1, \cdots, -J$
- Spinors: both chiralities per spin-j_s:
  - $u^+ \sim u_L + u_R$
  - $u^- \sim u_L + u_R$
- Four-component Dirac spinors
- Vector bosons

Momentum: The Last Piece of the Flow Puzzle

- Gordon identity: $p^\mu = \frac{1}{\sqrt{2}} \langle p|^{\alpha} \bar{\tau}^{\mu}_{\alpha\dot{\beta}} |p]^{\dot{\beta}} = \frac{1}{\sqrt{2}} [p|_{\dot{\alpha}} \tau^{\mu\dot{\alpha}\beta} |p\rangle_{\beta}$

p^μ is a pseudo vertex $\Rightarrow$ can be written as a flow

What does p^μ get contracted with?

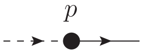
Momentum: The Last Piece of the Flow Puzzle

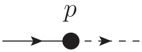
Backup Slides

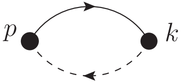
- Gordon identity: $p^\mu = \frac{1}{\sqrt{2}} \langle p|^{\alpha} \bar{\tau}^{\mu}_{\alpha\dot{\beta}} |p]^{\dot{\beta}} = \frac{1}{\sqrt{2}} [p|_{\dot{\alpha}} \tau^{\mu\dot{\alpha}\beta} |p\rangle_{\beta}$

p^μ is a pseudo vertex $\Rightarrow$ can be written as a flow

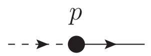
What does p^μ get contracted with?

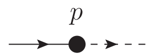
- $\tau_\mu \to \not{p}/\sqrt{2} = \frac{1}{\sqrt{2}}$ 

, $\bar{\tau}_\mu \to \bar{\not{p}}/\sqrt{2} = \frac{1}{\sqrt{2}}$ 

- $k_\mu \to p \cdot k = \underbrace{\frac{\mathrm{Tr}(\not{p}\bar{\not{k}})}{2}}_{\mathrm{Tr}(\tau^\mu \bar{\tau}^\nu) = g^{\mu\nu}} = \frac{1}{2}$ 

Chirality-flow rule for p^μ

$p^\mu \to \frac{1}{\sqrt{2}}$ 

, or $p^\mu \to \frac{1}{\sqrt{2}}$ 

Fermion Propagators in Chirality Flow

$su(2)$ objects have dotted indices and $su(2)$ objects undotted indices

- We split $\not{p}_{4d} \equiv p_\mu \gamma^\mu$ split into two terms

$$\not{p} \equiv \sqrt{2} p^\mu \tau_\mu^{\dot{\alpha}\beta} =$$

$$\bar{\not{p}} \equiv \sqrt{2} p_\mu \bar{\tau}^\mu_{\alpha\dot{\beta}} =$$

- Momentum dot defined to represent slashed momenta
- In a propagator, we have $p^\mu = \sum p_i^\mu$, $p_i^2 = 0$

$$\not{p} =$$

$$= \sum_i |i]^{\dot{\alpha}} \langle i|^{\beta} \quad \text{for} \quad p_i^2 = 0$$

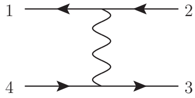
$$\bar{\not{p}} =$$

$$= \sum_i |i\rangle_\alpha |i]_{\dot{\beta}} \quad \text{for} \quad p_i^2 = 0$$

QED Chirality Flow: Vertices & Internal Photons

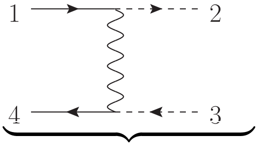
Backup Slides

Calculate helicity amplitude for photon exchange using chirality flow



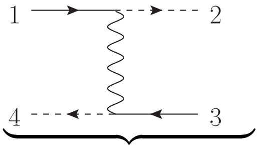
contains two types of terms:

- Arrows aligned: e.g. $\mathcal{M}(1^-, 2^+, 3^+, 4^-) \sim$



$$(\langle 1|^{\alpha}\bar{\tau}^{\mu}_{\alpha\dot{\beta}}|2]^{\dot{\beta}})([3|_{\dot{\gamma}}\tau_{\mu}^{\dot{\gamma}\eta}|4\rangle_{\eta})$$

- Arrows opposed: e.g. $\mathcal{M}(1^-, 2^+, 3^-, 4^+) \sim$



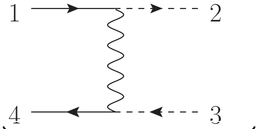
$$(\langle 1|^{\alpha}\bar{\tau}^{\mu}_{\alpha\dot{\beta}}|2]^{\dot{\beta}})(\langle 3|^{\gamma}\bar{\tau}_{\mu,\gamma\dot{\eta}}|4]^{\dot{\eta}})$$

QED Chirality Flow: Vertices & Internal Photons

Arrows aligned?

Use Fierz identity $\bar{\tau}^{\mu}_{\alpha\dot{\beta}}\tau_{\mu}^{\dot{\gamma}\eta} = \delta_{\alpha}^{\ \eta}\delta^{\dot{\gamma}}_{\ \dot{\beta}}$ to remove vector index

$\mathcal{M}(1^-, 2^+, 3^+, 4^-) \sim$



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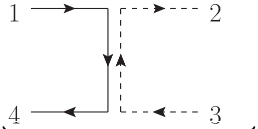


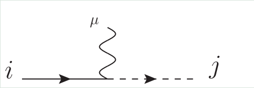
$(\langle 1|^{\alpha}\bar{\tau}^{\mu}_{\alpha\dot{\beta}}|2]^{\dot{\beta}})([3|_{\dot{\gamma}}\tau_{\mu}^{\dot{\gamma}\eta}|4\rangle_{\eta})$

$\langle 14\rangle[32]$

QED Chirality Flow: Vertices & Internal Photons

Arrows opposed? Flip them

First use charge conjugation



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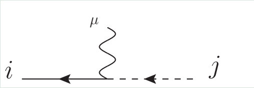
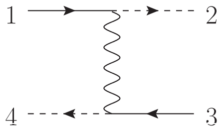


$$\langle i|^{\alpha}\bar{\tau}^{\mu}_{\alpha\dot{\beta}}|j]^{\dot{\beta}} = [j|_{\dot{\alpha}}\tau^{\mu,\dot{\alpha}\beta}|i\rangle_{\beta}$$

Then use Fierz identity $\bar{\tau}^{\mu}_{\alpha\dot{\beta}}\tau_{\mu}^{\dot{\gamma}\eta} = \delta_{\alpha}^{\eta}\delta^{\dot{\gamma}}_{\dot{\beta}}$ to remove vector index

$\mathcal{M}(1^-, 2^+, 3^-, 4^+)$

$\sim$



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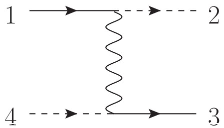

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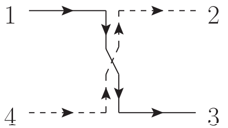


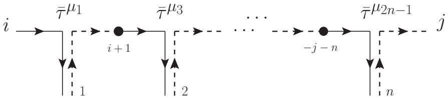
$$\left(\langle 1|^{\alpha}\bar{\tau}^{\mu}_{\alpha\dot{\beta}}|2]^{\dot{\beta}}\right)\left(\langle 3|^{\gamma}\bar{\tau}_{\mu,\gamma\dot{\eta}}|4]^{\dot{\eta}}\right) = \left(\langle 1|^{\alpha}\bar{\tau}^{\mu}_{\alpha\dot{\beta}}|2]^{\dot{\beta}}\right)\left(\langle 4|_{\dot{\eta}}\tau_{\mu}^{\dot{\eta}\gamma}|3]_{\gamma}\right) = \langle 13\rangle[42]$$

Fermion Lines with Multiple Photons

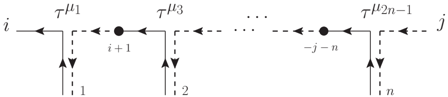
What if fermions emit more than one photon? Is flow picture valid?

- Yes (at least at tree level)
- Conjugation of a current holds for full fermion line

$$\langle i | \bar{\tau}^{\mu_1} \tau^{\mu_2} \dots \bar{\tau}^{\mu_{2n+1}} | j] =$$



$$=$$

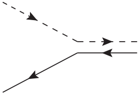


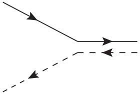
$$= [j | \tau^{\mu_{2n+1}} \bar{\tau}^{\mu_{2n}} \dots \tau^{\mu_1} | i \rangle$$

i.e. arrow swap (and Fierz) works for any fermion line!

Quick Summary: QED Vertices & Internal Photons

Charge conjugation $\Rightarrow$ arrow flips $\Rightarrow$ replace vector with double line

$\Rightarrow \tau^{\mu,\dot{\alpha}\beta} =$  ,

$\bar{\tau}^{\mu}_{\alpha\dot{\beta}} =$ 

$\Rightarrow g_{\mu\nu} =$  ,

or

$g_{\mu\nu} =$ 

Fierz identity already built into flow formalism

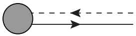
In Feynman diagram choose arrow direction which gives aligned arrows

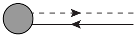
Creating Chirality Flow: External Gauge Bosons

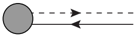
$$\epsilon_+^\mu(p,r) = \frac{[p|_{\dot\alpha}\tau^{\mu,\dot\alpha\beta}|r\rangle_\beta}{\langle rp\rangle}, \qquad \epsilon_-^\mu(p,r) = \frac{\langle p|^{\alpha}\bar\tau^{\mu}_{\alpha\dot\beta}|r]^{\dot\beta}}{[pr]}$$

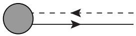
Pseudo vertex

External gauge bosons are just $f\bar{f}$ pairs with a denominator!

$\epsilon_+^\mu(p,r) \rightarrow \frac{1}{\langle ri\rangle}$ 

, or $\epsilon_+^\mu(p,r) \rightarrow \frac{1}{\langle ri\rangle}$ 

$\epsilon_-^\mu(p,r) \rightarrow \frac{1}{[ir]}$ 

, or $\epsilon_-^\mu(p,r) \rightarrow \frac{1}{[ir]}$ 

In Feynman diagram choose arrow direction which gives aligned arrows

Spinor-Helicity: Gauge Bosons in Terms of Spinors

Lorentz algebra $so(3,1) \cong su(2) \oplus su(2)$
Consider massless particles: chirality $\sim$ helicity

Outgoing polarisation vectors:

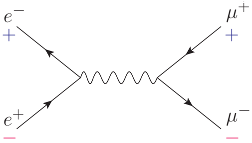
$$\epsilon_+^\mu(p,r) = \frac{\langle r|\bar\tau^\mu|p]}{\langle rp\rangle}\,, \qquad \epsilon_-^\mu(p,r) = \frac{[r|\tau^\mu|p\rangle}{[pr]}$$

$$p\cdot\epsilon_+(p,r) = \underbrace{\frac{\langle r|p^\mu\bar\tau_\mu|p]}{\langle rp\rangle} = 0}_{\text{Weyl eq. } p^\mu\bar\tau_\mu|p]=0} \qquad p\cdot\epsilon_-^\mu(p,r) = \underbrace{\frac{[r|p^\mu\tau_\mu|p\rangle}{[pr]} = 0}_{\text{Weyl eq. } p^\mu\tau_\mu|p\rangle=0}$$

$$\epsilon_+(p,r)\cdot(\epsilon_-)^*(p,r) = \underbrace{\frac{\langle r|\bar\tau^\mu|p]}{\langle rp\rangle}\frac{[r|\tau_\mu|p\rangle}{[pr]}}_{\epsilon_\pm=(\epsilon_\mp)^*} = \frac{\langle rp\rangle[rp]}{\langle rp\rangle[pr]} = \underbrace{-1}_{[pr]=-[rp]}$$

Simplest QED Example

- Regular spinor-helicity ≡ easy



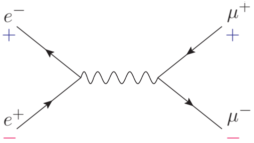
$$= \frac{2ie^2}{s_{e^+e^-}} (\tilde{\lambda}_{e^-,\dot{\alpha}} \tau_{\mu}^{\dot{\alpha}\beta} \lambda_{e^+,\beta})(\lambda_{\mu^-}^{\alpha} \bar{\tau}^{\mu}_{\alpha\dot{\beta}} \tilde{\lambda}_{\mu^+}^{\dot{\beta}})$$

$$= \frac{2ie^2}{s_{e^+e^-}} \tilde{\lambda}_{e^-,\dot{\alpha}} \tilde{\lambda}_{\mu^+}^{\dot{\alpha}} \lambda_{\mu^-}^{\beta} \lambda_{e^+,\beta} \equiv \frac{2ie^2}{s_{e^+e^-}} [e^- \mu^+] \langle \mu^- e^+ \rangle$$

Simplest QED Example

Backup Slides

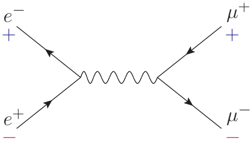
- Regular spinor-helicity ≡ easy



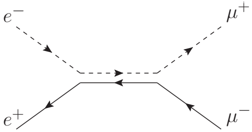
$$= \frac{2ie^2}{s_{e^+e^-}}(\tilde{\lambda}_{e^-,\dot{\alpha}}\tau_{\mu}^{\dot{\alpha}\beta}\lambda_{e^+,\beta})(\lambda^{\alpha}_{\mu^-}\bar{\tau}^{\mu}_{\alpha\dot{\beta}}\tilde{\lambda}^{\dot{\beta}}_{\mu^+})$$

$$= \frac{2ie^2}{s_{e^+e^-}}\tilde{\lambda}_{e^-,\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}}_{\mu^+}\lambda^{\beta}_{\mu^-}\lambda_{e^+,\beta} \equiv \frac{2ie^2}{s_{e^+e^-}}[e^-\mu^+]\langle\mu^- e^+\rangle$$

- Helicity flow ≡ super easy and intuitive

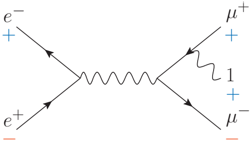


$$= \frac{2ie^2}{s_{e^+e^-}}$$



Next Simplest QED Example

- Regular spinor-helicity $\equiv$ easy



$$= \frac{-i2\sqrt{2}e^3}{s_{e^+e^-}s_{\mu^+1}}\left(\tilde{\lambda}_{e^-,\dot{\alpha}}\tau_\mu^{\dot{\alpha}\beta}\lambda_{e^+,\beta}\right)\left(\lambda^{\alpha}_{\mu^-}\bar{\tau}^{\mu}_{\alpha\dot{\beta}}(\not{p}_1+\not{p}_{\mu^+})^{\dot{\beta}\delta}\not{\epsilon}^{+}_{\delta\dot{\gamma}}(1,r)\tilde{\lambda}^{\dot{\gamma}}_{\mu^+}\right)$$

$$= \frac{-i2\sqrt{2}e^3}{s_{e^+e^-}s_{\mu^+1}\langle r1\rangle}\left(\tilde{\lambda}_{e^-,\dot{\alpha}}\tau_\mu^{\dot{\alpha}\beta}\lambda_{e^+,\beta}\right)\tilde{\lambda}_{1,\dot{\delta}}\tilde{\lambda}^{\dot{\delta}}_{\mu^+}$$

$$\times\left(\lambda^{\alpha}_{\mu^-}\bar{\tau}^{\mu}_{\alpha\dot{\beta}}\tilde{\lambda}_1^{\dot{\beta}}\lambda_1^{\delta}\lambda_{r,\delta}+\lambda^{\alpha}_{\mu^-}\bar{\tau}^{\mu}_{\alpha\dot{\beta}}\tilde{\lambda}^{\dot{\beta}}_{\mu^+}\lambda^{\delta}_{\mu^+}\lambda_{r,\delta}\right)$$

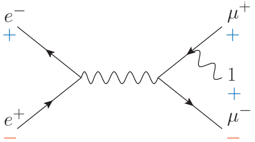
$$\sim \lambda^{\beta}_{\mu^-}\lambda_{e^+,\beta}\left(\tilde{\lambda}_{e^-,\dot{\alpha}}\tilde{\lambda}_1^{\dot{\alpha}}\lambda_1^{\delta}\lambda_{r,\delta}+\tilde{\lambda}_{e^-,\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}}_{\mu^+}\lambda^{\delta}_{\mu^+}\lambda_{r,\delta}\right)\tilde{\lambda}_{1,\dot{\delta}}\tilde{\lambda}^{\dot{\delta}}_{\mu^+}$$

Correct Answer

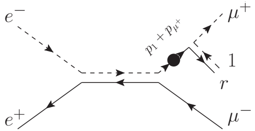
$$\frac{-i2\sqrt{2}e^3}{s_{e^+e^-}s_{\mu^+1}\langle r1\rangle}\left([e^-1]\langle 1r\rangle+[e^-\mu^+]\langle\mu^+r\rangle\right)[1\mu^+]\langle\mu^-e^+\rangle$$

Next Simplest QED Example

- Helicity flow $\equiv$ super easy and intuitive



$$= \frac{-i2\sqrt{2}e^3}{s_{e^+e^-}\, s_{\mu^+1} \langle r1 \rangle}$$

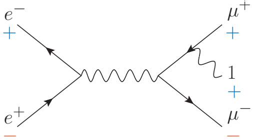


Correct Answer

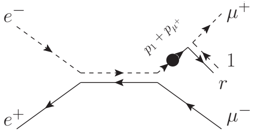
$$\frac{-i2\sqrt{2}e^3}{s_{e^+e^-}\, s_{\mu^+1} \langle r1 \rangle} \Big([e^- 1] \langle 1 r \rangle + [e^- \mu^+] \langle \mu^+ r \rangle \Big) [1 \mu^+] \langle \mu^- e^+ \rangle$$

Next Simplest QED Example

- Helicity flow $\equiv$ super easy and intuitive



$$= \frac{-i2\sqrt{2}e^3}{s_{e^+e^-}s_{\mu^+1}\langle r1\rangle}$$



- Immediately read off inner products

Correct Answer

$$\frac{-i2\sqrt{2}e^3}{s_{e^+e^-}s_{\mu^+1}\langle r1\rangle}\Big([e^-1]\langle 1r\rangle + [e^-\mu^+]\langle \mu^+r\rangle\Big)[1\mu^+]\langle \mu^-e^+\rangle$$

How to Calculate a (Massless) Scattering Amplitude

- QCD often factorise colour, use helicity basis for kinematics

$$\mathcal{M}_h \left(1^{h_1}, \ldots, n^{h_n}\right) = \sum_i C_i A_i \left(p_1^{h_1}, \ldots, p_n^{h_n}\right)$$

- $C_i \equiv$ colour factor
  - QED: $C_i = 1$
- $A_i \equiv$ kinematic amplitude
  - Cross incoming particles to outgoing
  - Each particle j is given a specific helicity h_j
  - Since massless, helicity $\sim$ chirality

The Spinor-Helicity Method: a Better Way

Assign each particle has a specific helicity
Lorentz algebra $so(3,1) \cong su(2) \oplus su(2)$

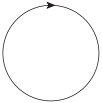
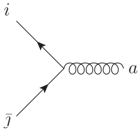
- Ideal for (approximately) massless particles (e.g. most particles at LHC)
  - Helicity is the spin quantum number massless particles
- Spinors of given helicity have given chirality/Lorentz representation
  - For incoming (anti)spinors chirality $((\frac{1}{2},0)$ or $(0,\frac{1}{2})) \sim$ helicity $(-\frac{1}{2}$ or $+\frac{1}{2})$
  - For outgoing (anti)spinors chirality $((\frac{1}{2},0)$ or $(0,\frac{1}{2})) \sim$ helicity $(-\frac{1}{2}$ or $+\frac{1}{2})$
- Amplitude itself is a number rather than a matrix
  - Easy to square
- Different helicity amplitudes are orthogonal
  - Only sum over helicities after squaring

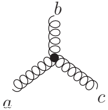
Colour Flow: a Quick Introduction

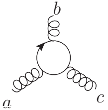
Standard method in $SU(N)$-colour calculations:

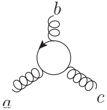
Write all objects in terms of $\delta_{i\bar{\jmath}} \equiv$ flows of colour (for simplicity $\mathrm{T_R} = 1$)

$\delta_{i\bar{\jmath}} =$ 

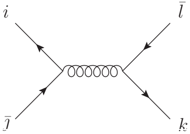
, $\sum_i \delta_{ii} = N =$ 
, $t^a_{i\bar{\jmath}} =$ 

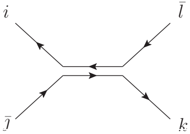
$if^{abc} =$ 

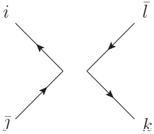
$=$ 

$-$ 

$= \mathrm{Tr}(t^a[t^b, t^c])$



$=$ 

$-\frac{1}{N}$ 

$t^a_{i\bar{\jmath}} t^a_{k\bar{l}}$ $\quad \delta_{i\bar{l}}\delta_{k\bar{\jmath}}$ $\quad \delta_{i\bar{\jmath}}\delta_{k\bar{l}}$

Spinor-Helicity: Gauge Bosons in Terms of Spinors

Lorentz algebra $so(3,1) \cong su(2) \oplus su(2)$
Consider massless particles: chirality $\sim$ helicity

Outgoing polarisation vectors:

$$\epsilon_+^\mu(p,r) = \frac{\langle r|\bar{\tau}^\mu|p]}{\langle rp\rangle}\,, \qquad \epsilon_-^\mu(p,r) = \frac{[r|\tau^\mu|p\rangle}{[pr]}$$

- r is a (massless) arbitrary reference momentum ($p \cdot r \neq 0$)
- Different r choices correspond to different gauges

$$\epsilon_+^\mu(p,r') - \epsilon_+^\mu(p,r) = -p^\mu \frac{\langle r' r\rangle}{\langle r' p\rangle\langle rp\rangle}$$

- Gauge invariant quantities must be r-invariant
  - Choose r as conveniently as possible (remember $\langle ij\rangle = -\langle ji\rangle$ s.t. $\langle ii\rangle = 0$)
  - Variance under $r \to r'$ good check of gauge invariance of (partial) amplitude

Spinor-Helicity: Vectors and Removing μ Indices

Lorentz algebra $so(3,1) \cong su(2) \oplus su(2)$
Consider massless particles: chirality $\sim$ helicity

Dirac matrices in chiral basis

$$\gamma^\mu = \begin{pmatrix} 0 & \sqrt{2}\tau^\mu \\ \sqrt{2}\bar{\tau}^\mu & 0 \end{pmatrix} \qquad \sqrt{2}\tau^\mu = (1, \vec{\sigma}), \quad \sqrt{2}\bar{\tau}^\mu = (1, -\vec{\sigma}),$$

Remove $\tau/\bar{\tau}$ matrices in amplitude with

$$\underbrace{\langle i|\bar{\tau}^\mu|j][k|\tau_\mu|l\rangle = \langle il\rangle[kj]}_{\text{Fierz identity}}, \qquad \underbrace{\langle i|\bar{\tau}^\mu|j] = [j|\tau^\mu|i\rangle}_{\text{Charge Conjugation}}$$

Express (massless) p^μ in terms of spinors

$$p^\mu = \frac{[p|\tau^\mu|p\rangle}{\sqrt{2}} = \frac{\langle p|\bar{\tau}^\mu|p]}{\sqrt{2}}, \quad \sqrt{2}p^\mu\tau_\mu \equiv \not{p} = |p]\langle p|, \quad \sqrt{2}p^\mu\bar{\tau}_\mu \equiv \bar{\not{p}} = |p\rangle[p|$$