Deep Inelastic Scattering (DIS)

Ingo Schienbein
UGA/LPSC Grenoble

MCnet-CTEQ Summer School 2021

Virtual
5-16 September 2021

Disclaimer

This lecture has profited a lot from the following resources:

- The text book by **Halzen&Martin**, Quarks and Leptons
- The text book by **Ellis, Stirling & Webber**, QCD and Collider Physics
- The lecture on DIS at the CTEQ school in 2012 by **F. Olness**
- The lecture on DIS given by **F. Gelis** Saclay in 2006

Lecture 1

1. Kinematics of Deep Inelastic Scattering
2. Cross sections for inclusive DIS (photon exchange)
3. OPE
4. Longitudinal and Transverse Structure functions
5. CC and NC DIS
6. Bjorken scaling

Lecture II

7. The Parton Model

8. Which partons?

9. Structure functions in the parton model

10. The pQCD formalism

11. NLO corrections to DIS

12. Parton evolution

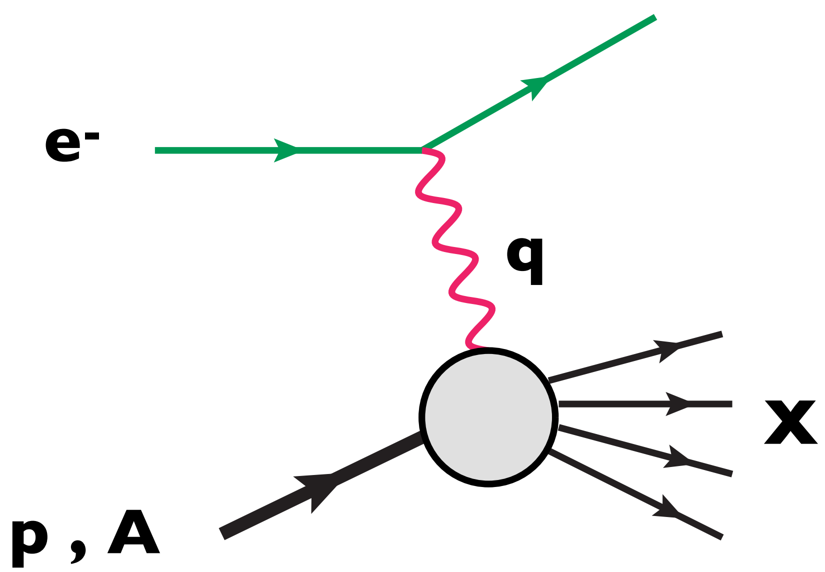
Not covered

- DIS with massive quarks
- Target mass corrections
- QCD studies with neutrinos
- Polarised DIS
- DIS off spin-1 targets

1. Kinematics of Deep Inelastic Scattering

What is inside nucleons?

- Basic idea: smash a well known probe on a nucleon or nucleus in order to try to figure out what is inside
- Photons are well suited for that purpose because their interactions are well understood
- Deep inelastic scattering: collision between an electron and a nucleon or nucleus by exchange of a virtual photon



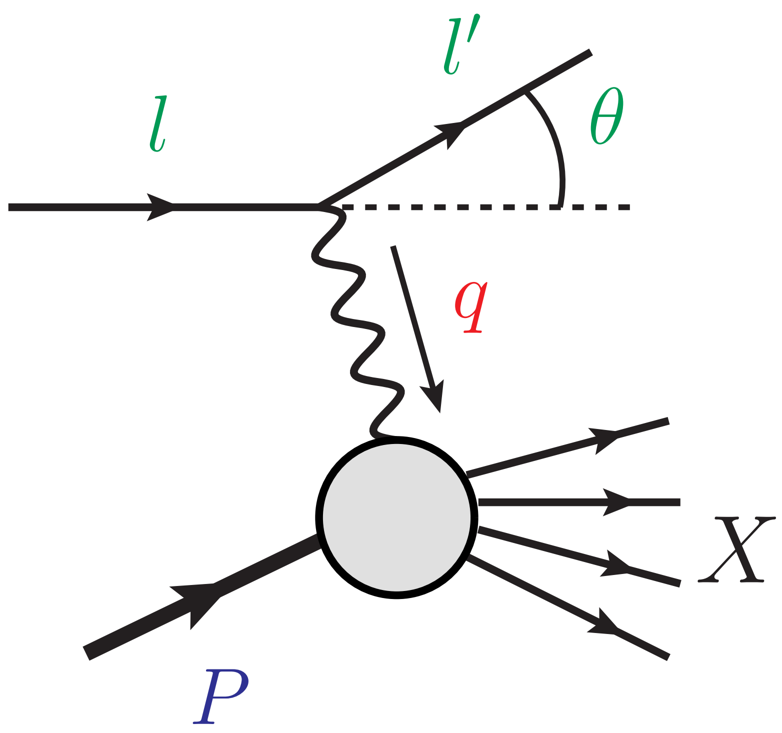
- Note: the virtual photon is spacelike: $q^2 < 0$
- **Deep**: $Q^2 = -q^2 >> M_N^2 \sim 1\ GeV^2$
- **Inelastic**: $W^2 \equiv M_X^2 > M_N^2$

- Variant: collision with a neutrino, by exchange of a Z^0 or $W^\pm$

Kinematic variables

- Let's consider **inclusive DIS** where a sum over all **hadronic final states X** is performed:

  $e^-(l)+N(p) \rightarrow e^-(l')+X(p_X)$



- On-shell conditions: **$p^2=M^2$, $l^2=l'^2=m^2$**
- Measure energy and polar angle of scattered electron (**E',θ**)
- Other invariants of the reaction:

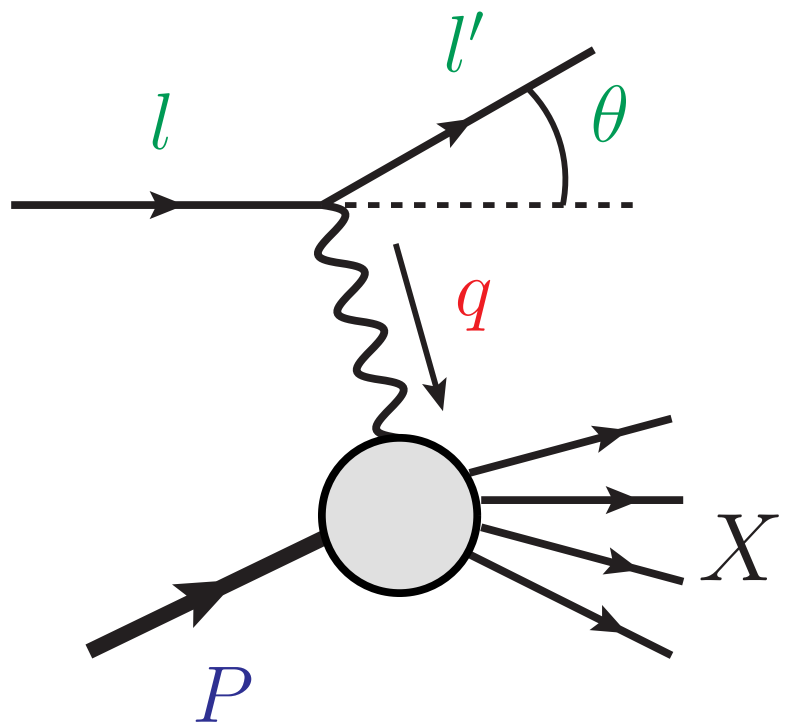
  - $Q^2 = -q^2 = -(l-l')^2 > 0$, the square of the momentum transfer,
  - $\nu = p \cdot q / M \overset{\text{lab}}{=} E_l - E_{l'}$,
  - $0 \leq x = Q^2/(2p \cdot q) = Q^2/(2M\nu) \leq 1$, the (dimensionless) Bjorken scaling variable,
  - $0 \leq y = p \cdot q / p \cdot l \overset{\text{lab}}{=} (E_l - E_{l'})/E_l \leq 1$, the inelasticity parameter,

* Here 'lab' designates the proton rest frame p=(M,0,0,0) which coincides with the lab frame for fixed target experiments

Kinematic variables

- There are two independent variables to describe the kinematics of inclusive DIS (up to trivial **φ** dependence):

  (E',,θ) or **(x,Q²)** or **(x,y)** or ...



- Relation between **Q²**, **x**, and **y**:

$$Q^2 = (2p\cdot l)\left(\frac{Q^2}{2p\cdot q}\right)\left(\frac{p\cdot q}{p\cdot l}\right) = Sxy = 2MExy$$

$$S = 2p\cdot l = (p+l)^2 - p^2 - l^2$$

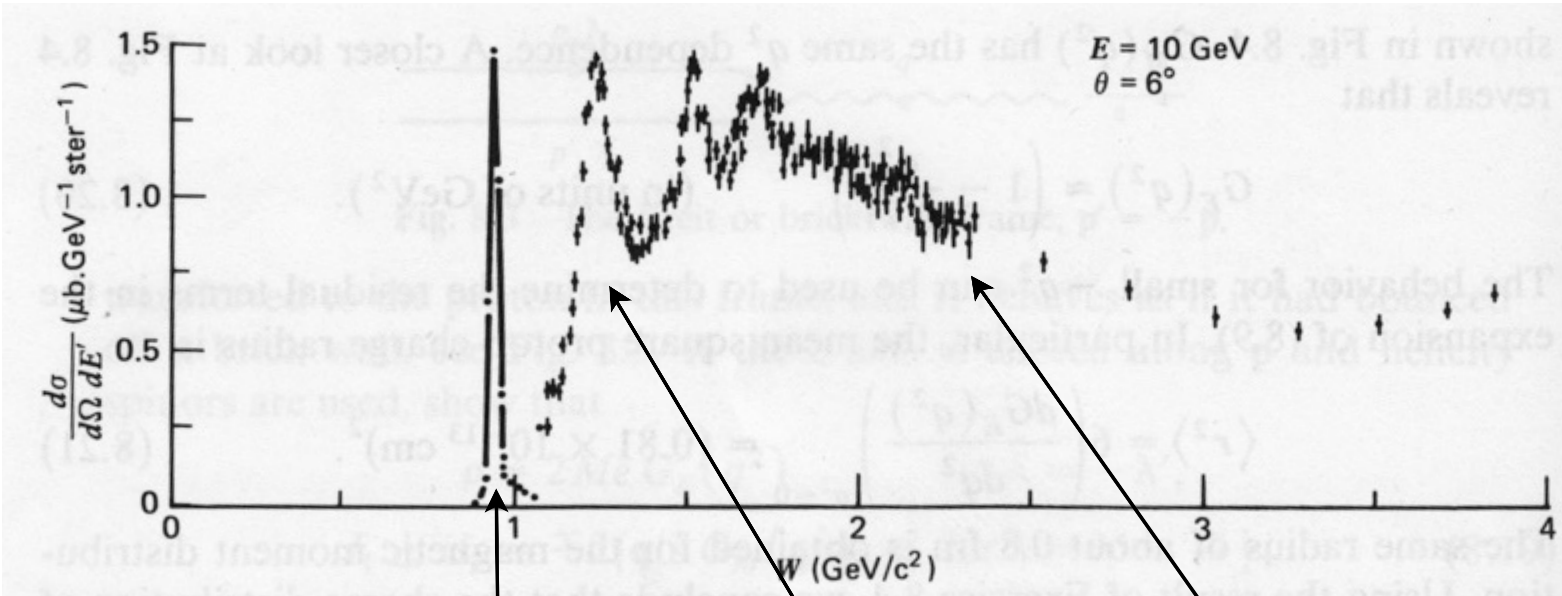
- Invariant mass **W** of the hadronic final state **X**:
  (also called missing mass since only outgoing electron measured)

$$W^2 \equiv M_X^2 = (p+q)^2 = M_N^2 + 2p\cdot q + q^2 = M_N^2 + \frac{Q^2}{x} - Q^2 = M_N^2 + \frac{Q^2}{x}(1-x)$$

elastic scattering: **W =M$_N$, x=1**

inelastic: **W ≥ M$_N$ + m$_\pi$, x<1**

The ep→eX cross section as function of W



Halzen&Martin, Quarks&Leptons, Fig. 8.6

Data from SLAC;
The elastic peak at **W=M** has been reduced by a factor 8.5

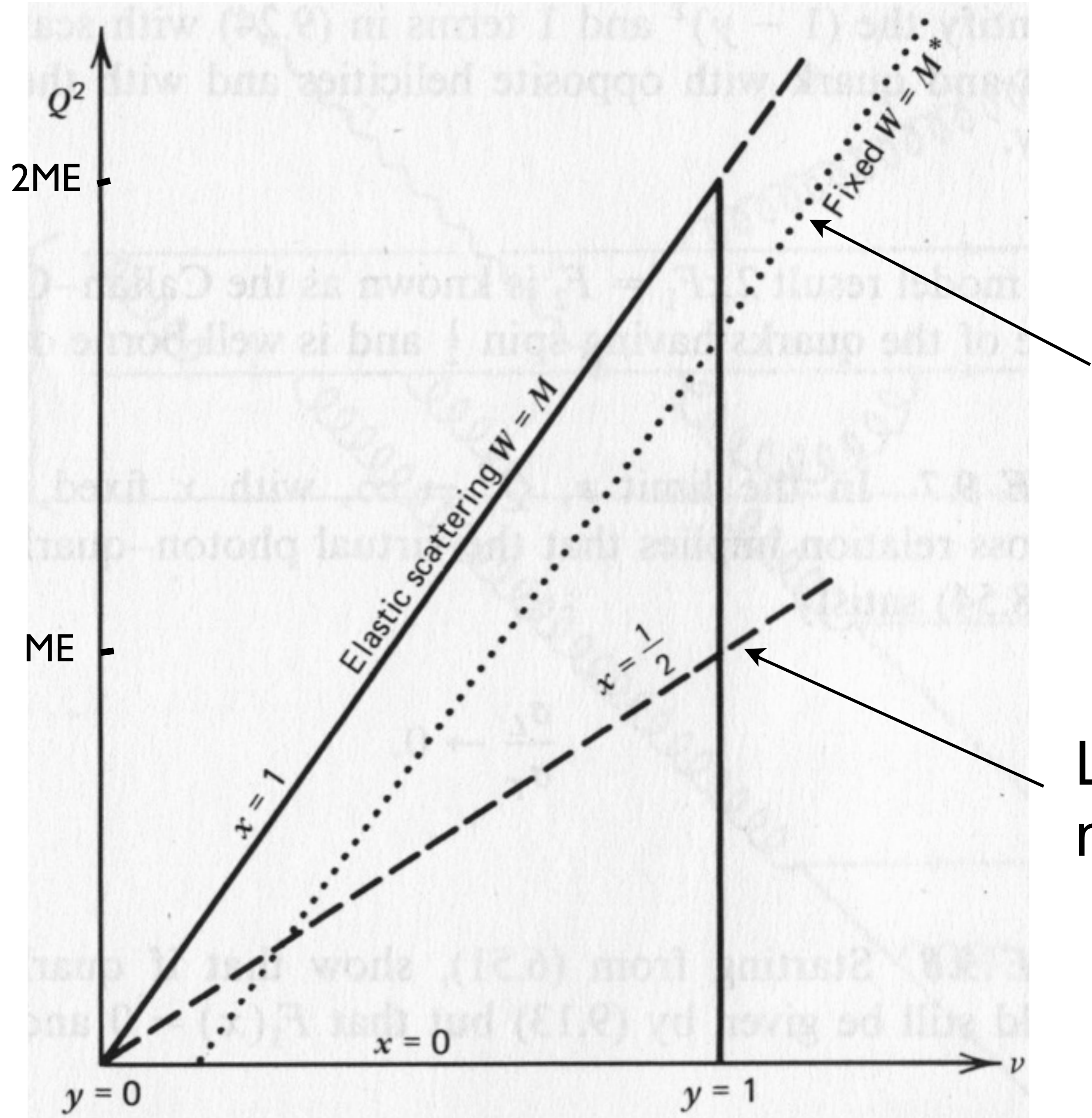
Elastic peak	Δ resonance $ep \rightarrow e\Delta^+ \rightarrow ep\pi^0$	Inelastic region

- Elastic peak: **W=M**, **x=1** (proton doesn't break up: **ep→ep**)
- Resonances: **W=M$_R$, ω=1/x=1+(M$_R^2$-M^2)/Q^2**
(Note that there is also a non-resonant background in the resonance region!)
- 'Continuum' or 'inelastic region': **W>~1.8 GeV**
complicated multiparticle final states resulting in a smooth distribution in **W**
(Note there are also charmonium and bottonium resonances at W~3 and 9 GeV)

Phase Space in (ν,Q^2) plane

Q^2=(2MEx)y

Halzen&Martin,
Quarks&Leptons, Fig. 9.3



Allowed kinematic region

Line of constant invariant mass

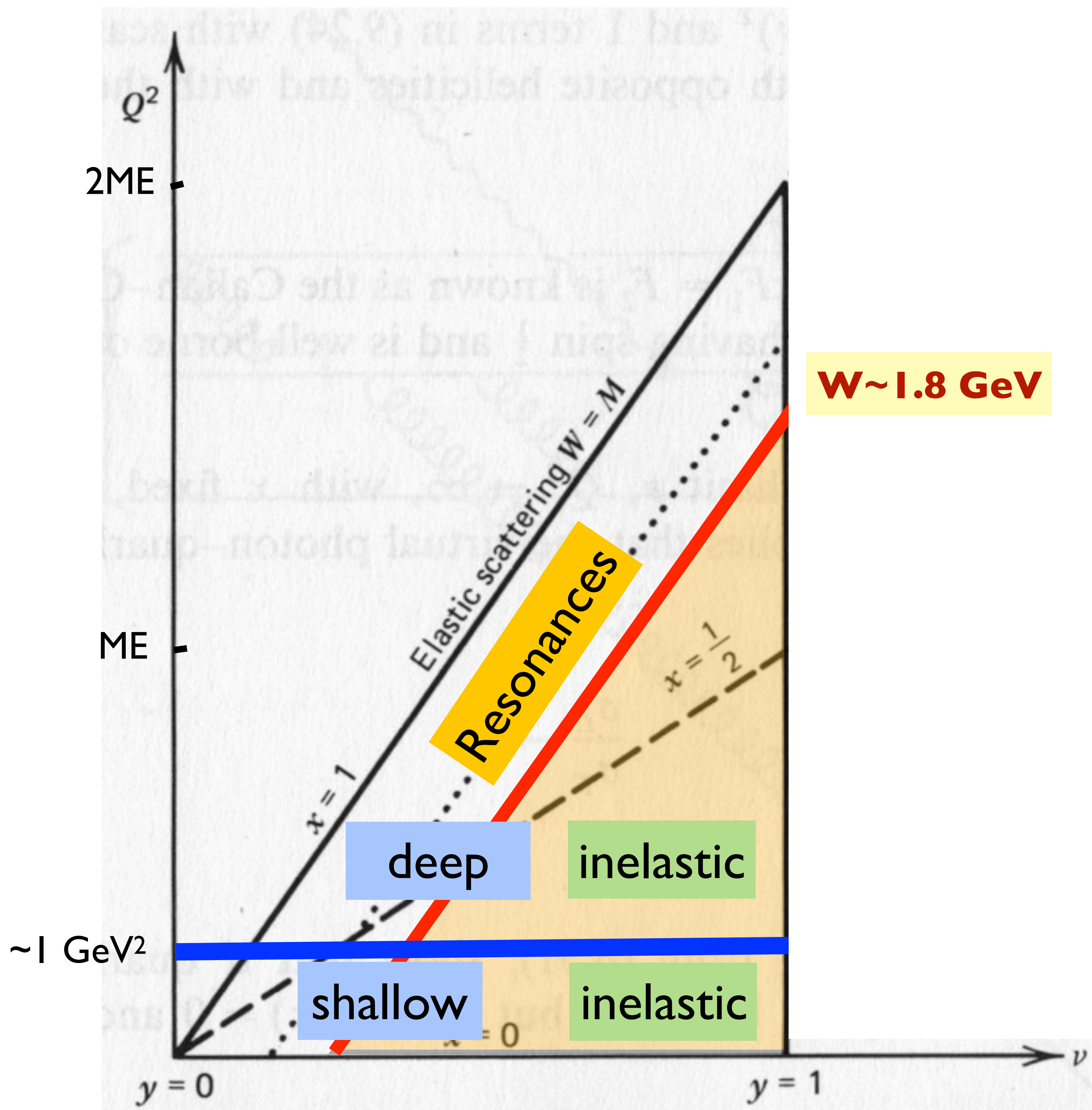
Line of constant momentum fraction

Fixed **$W=M_R$**: **$Q^2=2MExy$, $x=Q^2/(Q^2+M_R^2-M^2)$** → **$Q^2 = M^2-M_R^2+ 2MEy$**

Hence: fixed **W** curves are **parallel** to **W=M** curve!

Phase Space in (ν,Q^2) plane

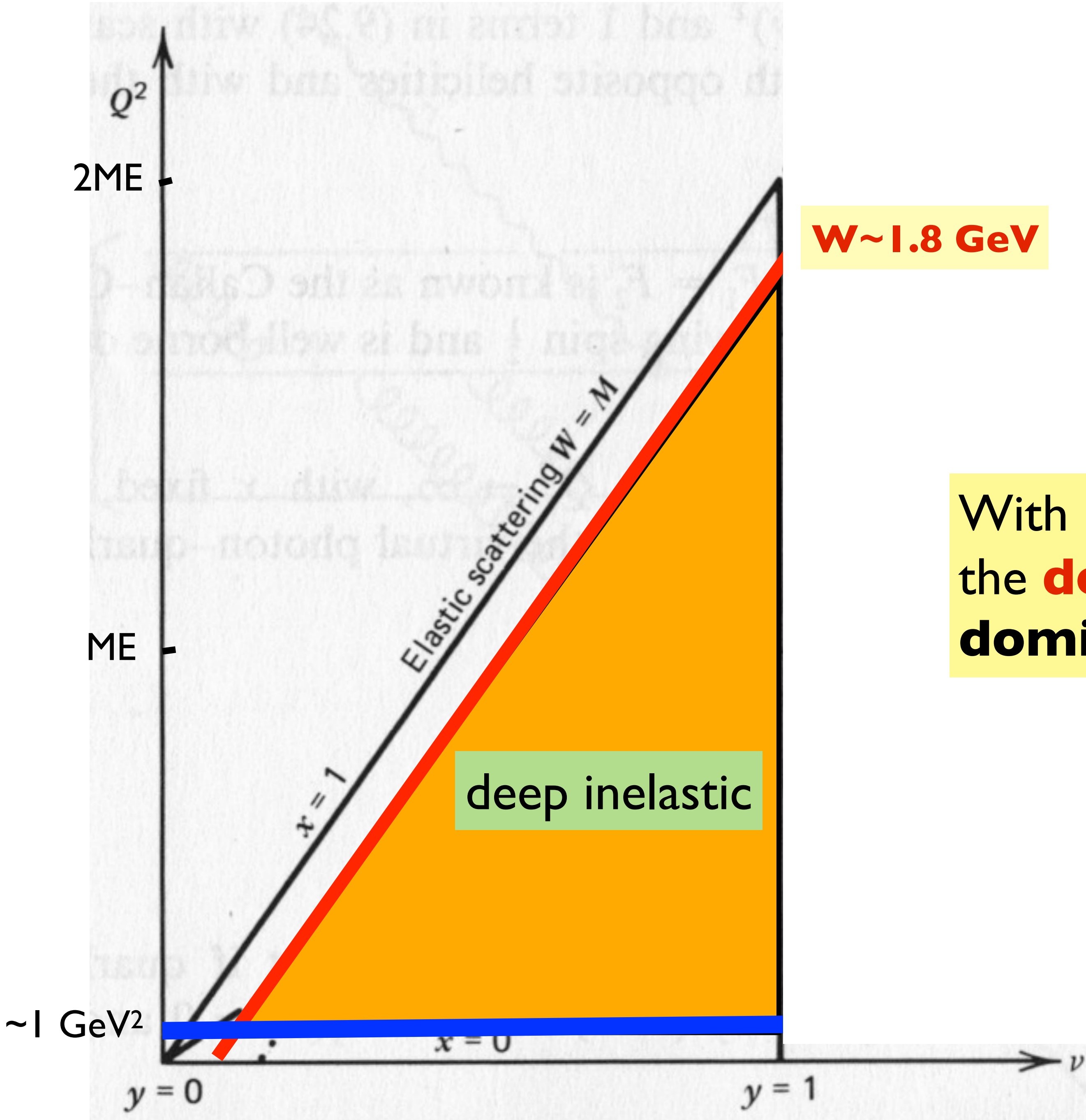
$$Q^2=(2MEx)y$$



- The phase space is separated into a resonance region (RES) and the inelastic region at **W~1.6 ... 1.8 GeV** (red line)
- The phase space is separated into a deep and a shallow region at **Q^2 ~1 GeV^2** (blue horizontal line)
- In global analyses of DIS data often the DIS cuts **Q^2>4 GeV^2, W>3.5 GeV** are employed
- The W-cut removes the large x region: **$W^2 = M^2 + Q^2/x\ (1-x) > 3.5$ GeV**
- The Q-cut removes the smallest x: **$Q^2 = S\ x\ y > 4$ GeV^2**

Phase Space in (ν,Q^2) plane

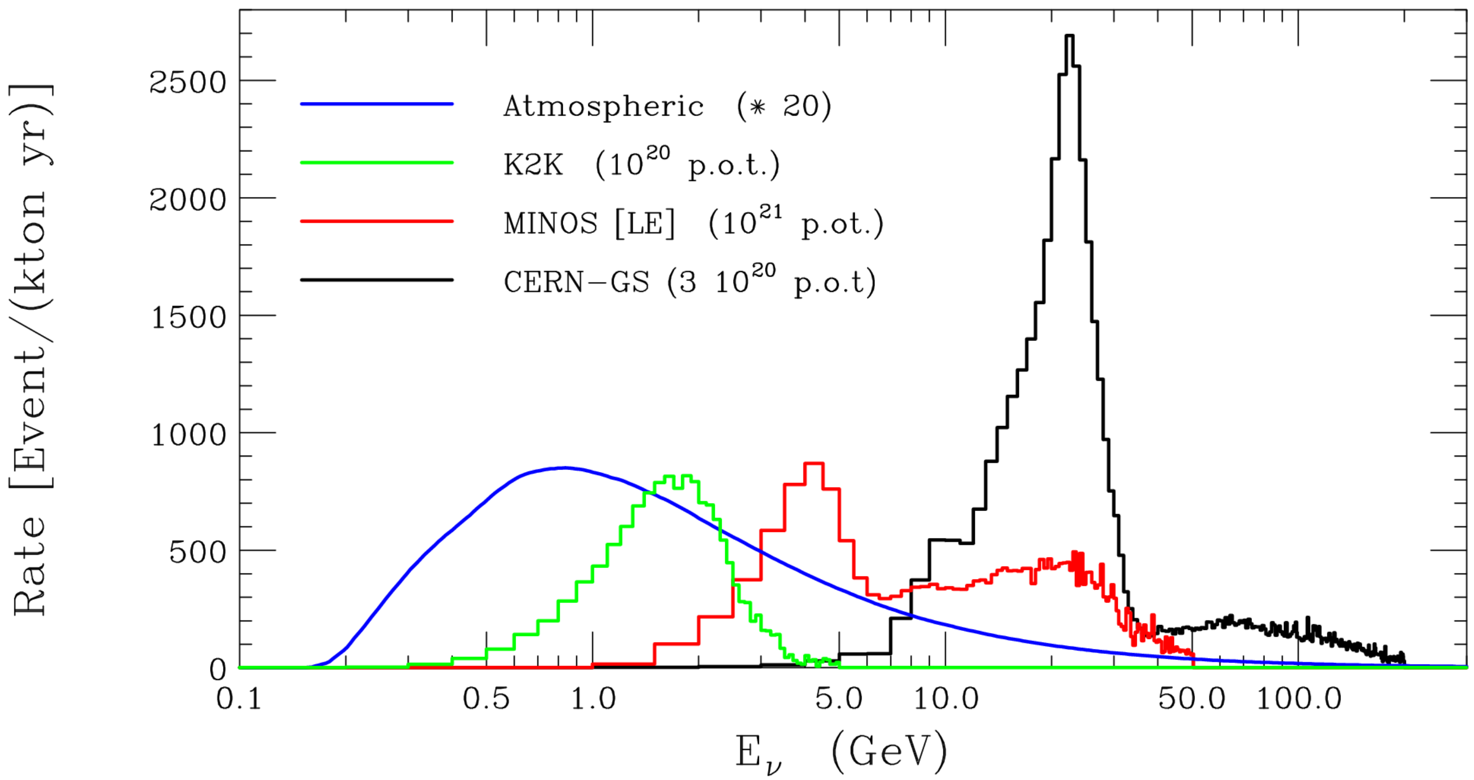
$$Q^2=(2MEx)y$$



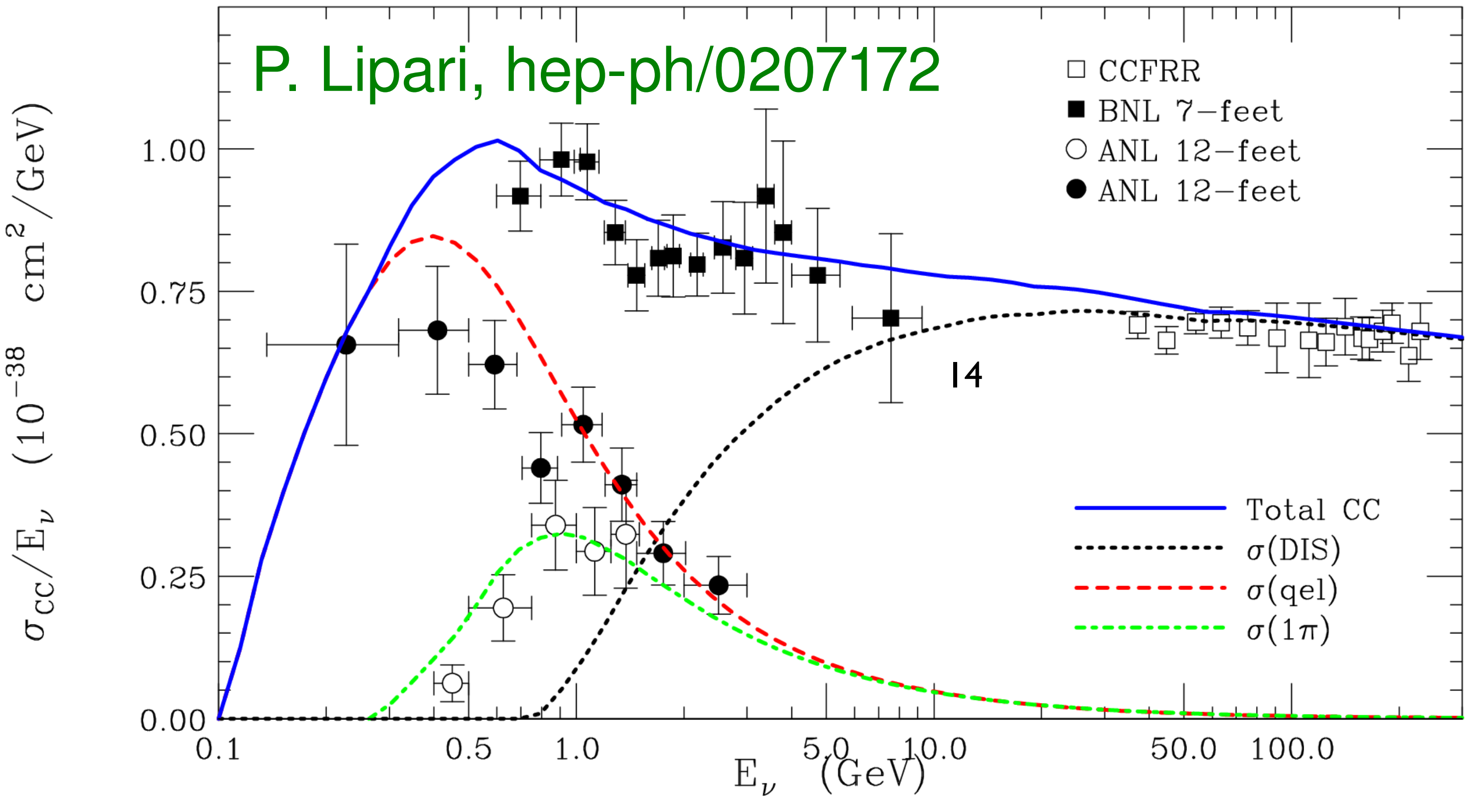
With increasing energy **E**
the **deep inelastic** region
dominates the phase space!

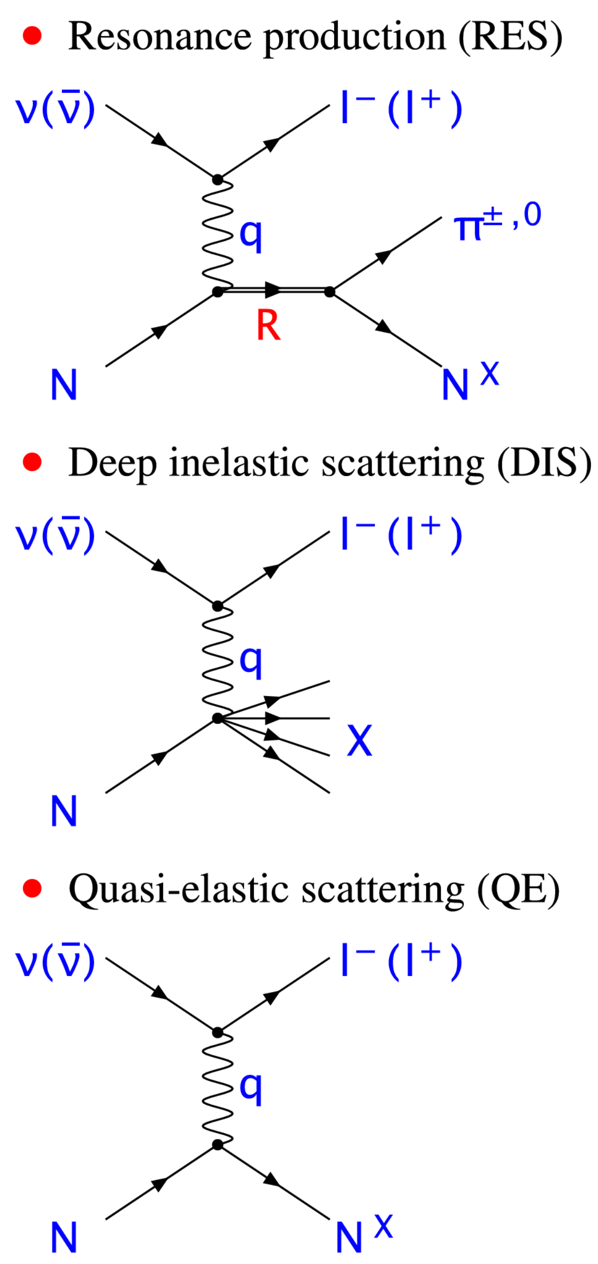
Neutrino cross sections at atmospheric ν energies

With increasing energy **E** the **deep inelastic** region **dominates** the phase space!



Paschos,JYY,PRD65(2002)033002





Homework Problems

1. Recap that the allowed kinematic region for $ep \rightarrow eX$ is $\mathbf{0 \leq x \leq 1}$ and $\mathbf{0 \leq y \leq 1}$.
   Construct the phase space in the (ν, Q^2)-plane yourself. [Ex. 8.11 in Halzen]

2. Show that $Q^2 = 2\,E\,E'\,(1 - \cos(\theta)) = 4\,E\,E'\,\sin^2(\theta/2)$ neglecting the lepton mass.
   Here, the z-axis coincides with the incoming lepton direction and θ is the polar angle of the outgoing lepton with respect to the z-axis

3. Show that in the target rest frame $x = [2\,E\,E'\,\sin^2(\theta/2)]/[M(E-E')]$
   still neglecting the lepton mass and the energies E, E' are now in the target rest frame

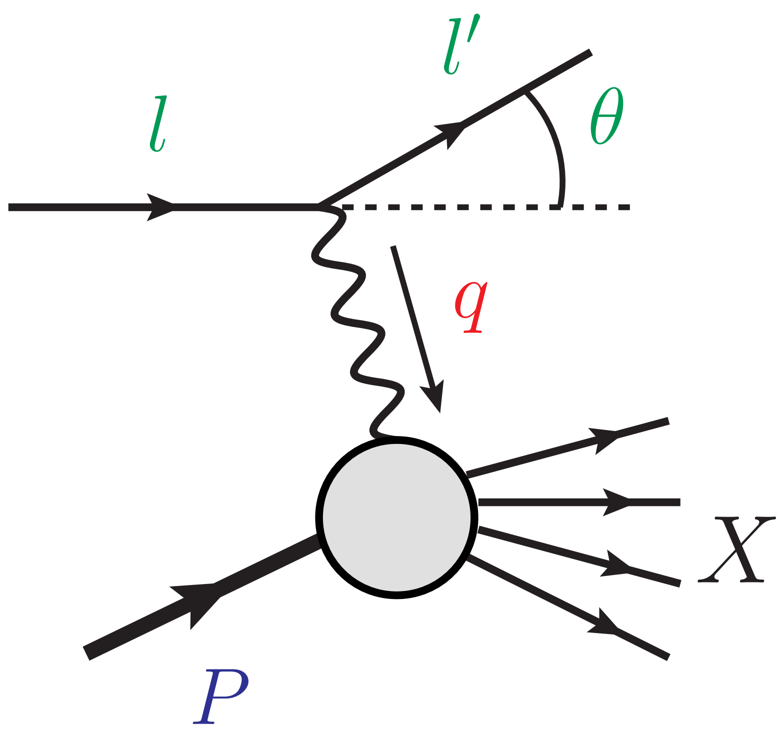
II. Cross section for inclusive DIS (photon exchange)

The cross section for inclusive ep→eX

- Let's consider **inclusive DIS** where a sum over all **hadronic final states X** is performed:

  e⁻(l)+N(p) → e⁻(l')+X(pX)

- The amplitude (**A**) is proportional to the interaction of a **leptonic current** (**j**) with a **hadronic current** (**J**):



$$A \sim \frac{1}{q^2} j^\mu J_\mu$$

- The leptonic current is well-known perturbatively in QED:

$$j^\mu = \langle l', s_{l'} | \hat{j}^\mu | l, s_l \rangle = \bar{u}(l', s_{l'}) \gamma^\mu u(l, s_l)$$

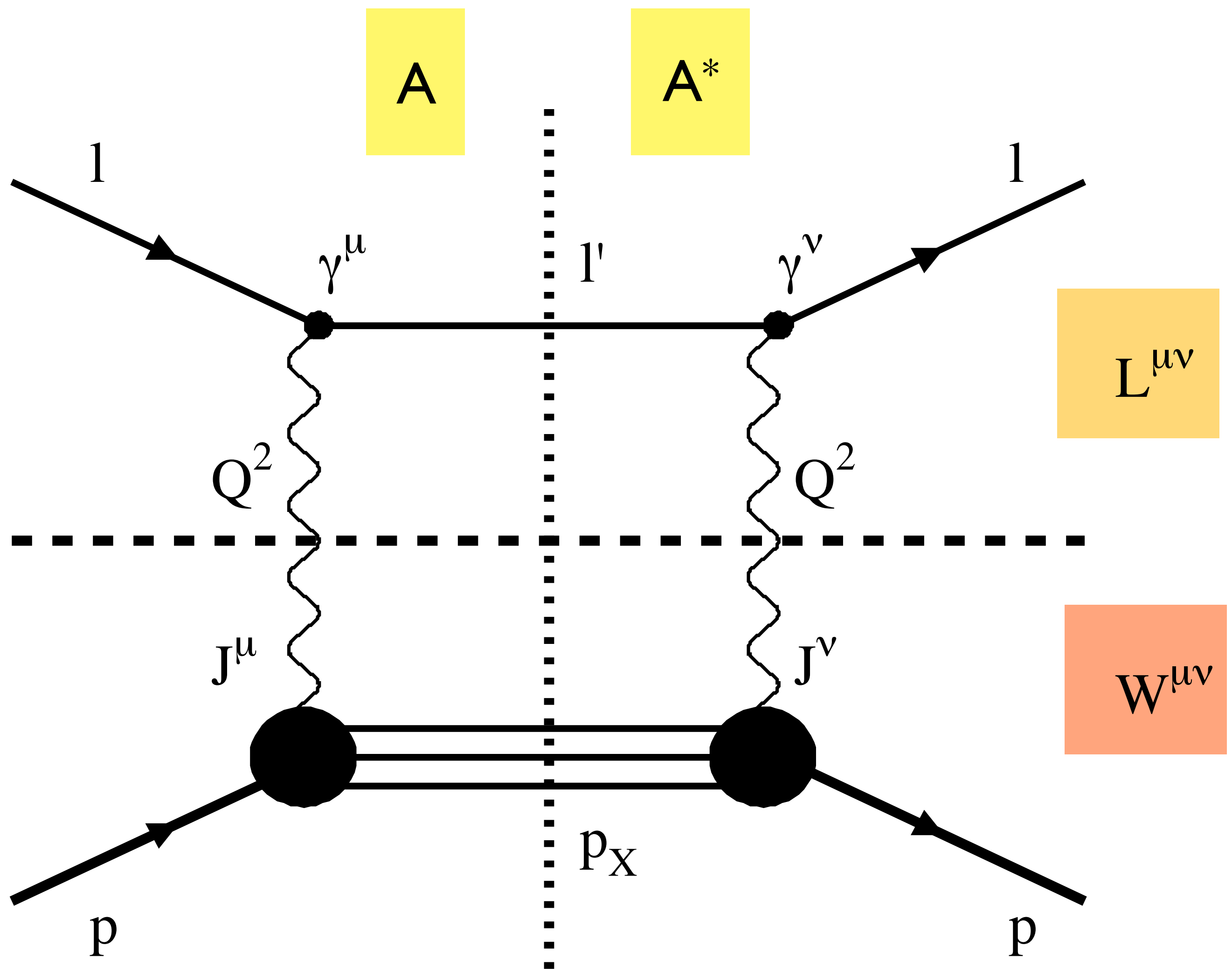
- The hadronic current is non-pert. and depends on the multi-particle final state over which we sum:

$$J^\mu = \langle X, \mathrm{spins} | \hat{J}^\mu | p, s_p \rangle$$

The cross section for inclusive ep→eX

The cross section which is proportional to the amplitude squared can be **factored** into a **leptonic and a hadronic piece**:

$$d\sigma \sim |A|^2 \sim L_{\mu\nu} W^{\mu\nu}$$



Leptonic tensor
calculable in pert. theory

Hadronic tensor
not calculabe in pert. theory

The cross section for inclusive ep→eX

$$d\sigma = \sum_X \frac{1}{F} \langle |A_X|^2 \rangle_{\rm spin} dQ_X \frac{d^3 l'}{(2\pi)^3 2E'} = \frac{1}{F} \left[\frac{e^4}{(q^2)^2} L_{\mu\nu} W^{\mu\nu} 4\pi \right] \frac{d^3 l'}{(2\pi)^3 2E'}$$

- With the Møller flux:

$$F = 4\sqrt{(l \cdot p)^2 - l^2 p^2} = 4\sqrt{(l \cdot p)^2 - m^2 M^2} \simeq 2S$$

- The phase space of the hadronic final state **X** with **N$_X$** particles:

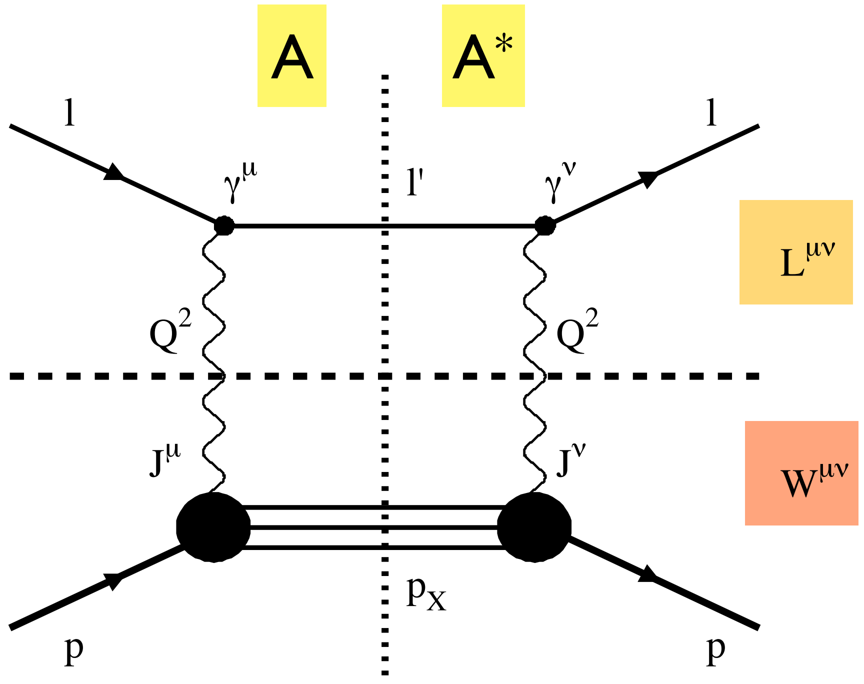
$$dQ_X = (2\pi)^4 \delta^{(4)}(p + q - p_X) \prod_{k=1}^{N_X} \frac{d^3 p_k}{(2\pi)^3 2E_k} = (2\pi)^4 \delta^{(4)}(p + q - p_X) d\Phi_X$$

- The amplitude with final state **X**:

$$A_X = \frac{e^2}{q^2} [\bar{u}(l')\gamma^\mu u(l)] \langle X | J_\mu(0) | N(p) \rangle \qquad A_X^* = \frac{e^2}{q^2} [\bar{u}(l)\gamma^\nu u(l')] \langle N(p) | J_\nu^\dagger(0) | X \rangle$$

The cross section for inclusive ep→eX

$$d\sigma = \sum_X \frac{1}{F} \langle |A_X|^2 \rangle_{\rm spin} dQ_X \frac{d^3 l'}{(2\pi)^3 2E'} = \frac{1}{F} \left[\frac{e^4}{(q^2)^2} L_{\mu\nu} W^{\mu\nu} 4\pi \right] \frac{d^3 l'}{(2\pi)^3 2E'}$$



In the amplitude squared appears the leptonic tensor:

$$\begin{aligned} L_{\mu\nu} &= \frac{1}{2} \sum_{s_l} \sum_{s_{l'}} \bar{u}(l')\gamma_\mu u(l) \bar{u}(l) \gamma_\nu u(l') \\ &= \frac{1}{2} \mathrm{Tr}[\gamma_\mu (\not{l} + m) \gamma_\nu (\not{l}' + m)] \\ &= 2[l_\mu l'_\nu + l_\nu l'_\mu - g_{\mu\nu}(l \cdot l' - m^2)] \end{aligned}$$

The hadronic tensor is defined as:

$$4\pi W_{\mu\nu} = \sum_{\text{states } X} \int d\Phi_X (2\pi)^4 \delta^{(4)}(p + q - p_X) \Big\langle \langle N(p)|J_\nu^\dagger(0)|X\rangle \langle X|J_\mu(0)|N(p)\rangle \Big\rangle_{\rm spin}$$

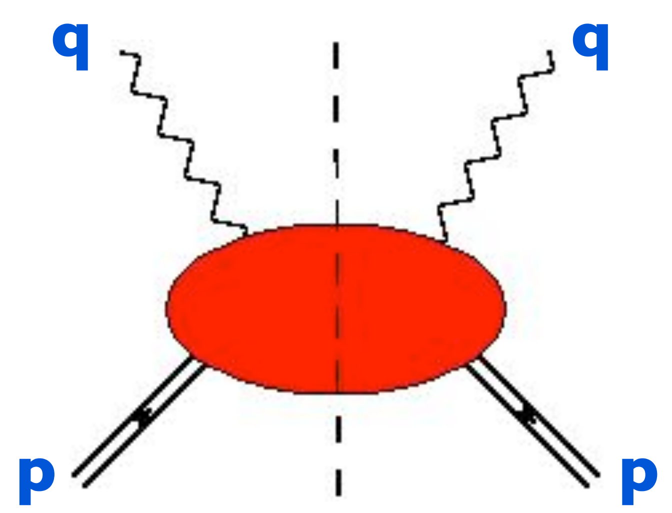
Note that the factor 4π is a **convention**. In this case the hadronic tensor is dimensionless (Exercise!).
Halzen&Martin, for example, use a factor $4\pi M$ and the hadronic tensor has dimension **mass^{-1}**.

The hadronic tensor and structure functions

- **$W_{\mu\nu}(p,q)$** <u>cannot</u> be calculated in perturbation theory.
  It parameterizes our ignorance of the nucleon.

- Goal: write down most general covariant expression for **$W_{\mu\nu}(p,q)$**

- Other symmetries (current conservation, parity, time-reversal inv.) have to be respected as well, depending on the interaction



- All possible tensors using the independent momenta **p, q** and the metric **g** are:

$$g_{\mu\nu}, \quad p_\mu p_\nu, \quad q_\mu q_\nu, \quad p_\mu q_\nu + p_\nu q_\mu,$$
$$\epsilon_{\mu\nu\rho\sigma} p^\rho q^\sigma, \quad p_\mu q_\nu - p_\nu q_\mu$$

- For a (spin-averaged) nucleon, the most general covariant expression for **$W_{\mu\nu}(p,q)$** is:

$$W^{\mu\nu}(p,q) = -g^{\mu\nu} W_1 + \frac{p^\mu p^\nu}{M^2} W_2 - i\epsilon^{\mu\nu\rho\sigma} \frac{p_\rho q_\sigma}{M^2} W_3 + \frac{q^\mu q^\nu}{M^2} W_4 + \frac{p^\mu q^\nu + p^\nu q^\mu}{M^2} W_5 + \frac{p^\mu q^\nu - p^\nu q^\mu}{M^2} W_6$$

- The structure functions **W_i** can depend only on the Lorentz-invariants **$p^2=M^2$**, **q^2**, and **p.q**

The hadronic tensor and structure functions

$$W^{\mu\nu}(p,q) = -g^{\mu\nu}W_1 + \frac{p^\mu p^\nu}{M^2}W_2 - i\epsilon^{\mu\nu\rho\sigma}\frac{p_\rho q_\sigma}{M^2}W_3 + \frac{q^\mu q^\nu}{M^2}\cancel{W_4} + \frac{p^\mu q^\nu + p^\nu q^\mu}{M^2}\cancel{W_5} + \frac{p^\mu q^\nu - p^\nu q^\mu}{M^2}\cancel{W_6}$$

$$d\sigma_{|W_4} \sim m_l^2 \qquad d\sigma_{|W_5} \sim m_l^2 \qquad d\sigma_{|W_6} = 0$$

- Instead of **p.q** use **ν** or **x** as argument: **$W_i = W_i(\nu,q^2)$ or $W_i=W_i(x,Q^2)$**
- **W_6** doesn't contribute to the cross section! No (**$l_\mu q_\nu$ - $l_\nu q_\mu$**) in the leptonic tensor
- **W_4** and **W_5** terms are proportional to the lepton masses squared in the cross section since **$q^\mu L_{\mu\nu} \sim m_l^2$**. Only place where they are relevant is charged current ν_τ-DIS.
- Parity and Time reversal symmetry implies **$W_{\mu\nu}=W_{\nu\mu}$**
- **$W_3=0$ and $W_6=0$** for parity conserving currents (like the e.m. current)

The hadronic tensor and structure functions

$$W^{\mu\nu}(p,q) = -g^{\mu\nu} W_1 + \frac{p^\mu p^\nu}{M^2} W_2 - i\epsilon^{\mu\nu\rho\sigma} \frac{p_\rho q_\sigma}{M^2} W_3 + \frac{q^\mu q^\nu}{M^2} \cancel{W_4} + \frac{p^\mu q^\nu + p^\nu q^\mu}{M^2} \cancel{W_5} + \frac{p^\mu q^\nu - p^\nu q^\mu}{M^2} \cancel{W_6}$$

$d\sigma_{|W_4} \sim m_l^2$ $\quad d\sigma_{|W_5} \sim m_l^2$ $\quad d\sigma_{|W_6} = 0$

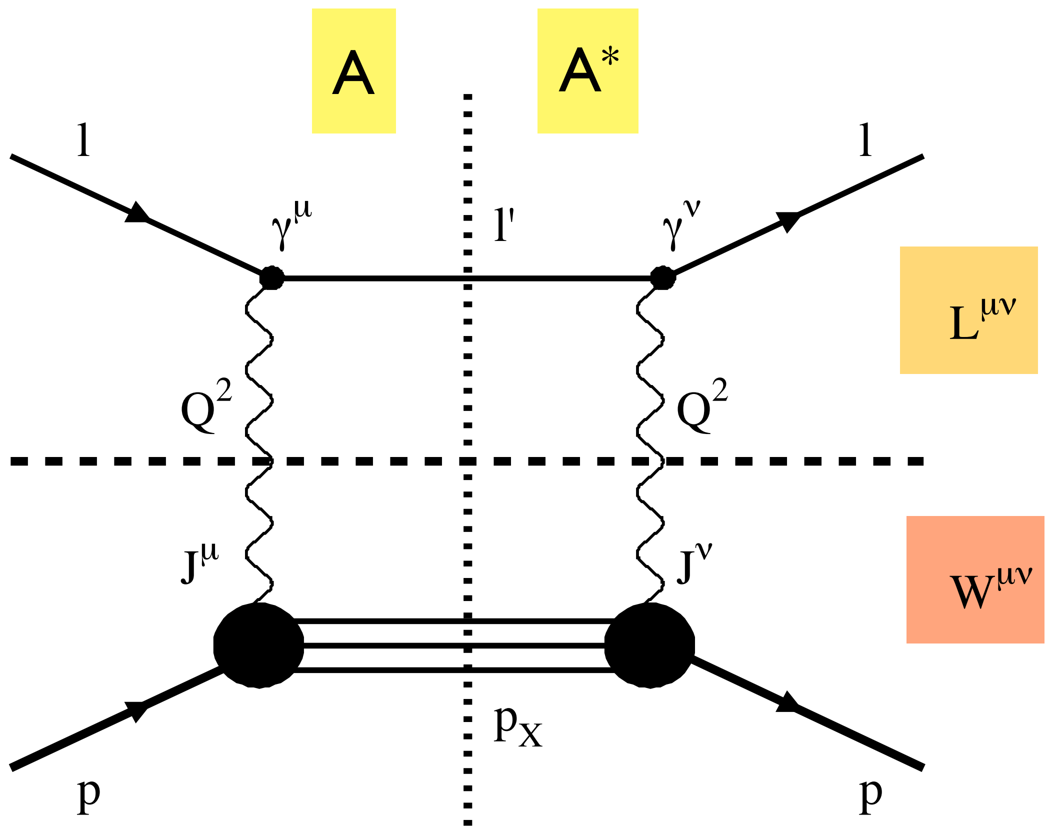
- Current conservation at the hadronic vertex requires **$q^\mu W_{\mu\nu} = q^\nu W_{\mu\nu} = 0$** implying

$$W_5 = -\frac{p \cdot q}{q^2} W_2\,, \quad W_4 = \left(\frac{p \cdot q}{q^2}\right)^2 W_2 + \frac{M^2}{q^2} W_1$$

- With current conservation+parity symmetry we are left with 2 independent sfs:

$$W^{\mu\nu} = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) W_1 + \frac{1}{M^2}\left(p^\mu - \frac{p \cdot q}{q^2} q^\mu\right)\left(p^\nu - \frac{p \cdot q}{q^2} q^\nu\right) W_2$$

The cross section for inclusive ep→eX



$$d\sigma = \frac{1}{F}\left[\frac{e^4}{(q^2)^2} L_{\mu\nu} W^{\mu\nu} 4\pi\right] \frac{d^3 l'}{(2\pi)^3 2E'}$$

$$L_{\mu\nu} = 2[l_\mu l'_\nu + l_\nu l'_\mu - g_{\mu\nu}(l \cdot l' - m^2)]$$

$$W^{\mu\nu} = -g_\perp^{\mu\nu} W_1 + \frac{1}{M^2} p_\perp^\mu p_\perp^\nu W_2$$

$$g_\perp^{\mu\nu} = g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\,,\; p_\perp^\mu = p^\mu - \frac{q \cdot p}{q^2} q^\mu$$

Show that (m=0): $$L_{\mu\nu} W^{\mu\nu} = 4(l \cdot l') W_1 + \frac{2}{M^2}\left[2(p \cdot l)(p \cdot l') - M^2 l \cdot l'\right] W_2$$

Giving in the nucleon rest frame: $$L_{\mu\nu} W^{\mu\nu} = 4EE'\left[2\sin^2(\theta/2) W_1 + \cos^2(\theta/2) W_2\right]$$

The DIS cross section in the nucleon rest frame reads (photon exchange, neglecting m):

$$\frac{d^2\sigma}{dE' d\Omega'} = \frac{\alpha_{\rm em}^2}{M 4E^2 \sin^4(\theta/2)}\left[2W_1(x,Q^2)\sin^2(\theta/2) + W_2(x,Q^2)\cos^2(\theta/2)\right]$$

The cross section for inclusive ep→eX

The DIS cross section in the nucleon rest frame reads (photon exchange, neglecting m):

$$\frac{d^2\sigma}{dE'd\Omega'} = \frac{\alpha_{\rm em}^2}{M 4E^2 \sin^4(\theta/2)}\left[2W_1(x,Q^2)\sin^2(\theta/2) + W_2(x,Q^2)\cos^2(\theta/2)\right]$$

It is customary to define "scaling" structure functions:

$$\left\{F_1, F_2, F_3\right\} = \left\{W_1, \frac{Q^2}{2xM^2}W_2, \frac{Q^2}{xM^2}W_3\right\}$$

Change of variables **(E',Ω')→(x,y)** and **W**$_i$ **→F**$_i$:

Show that $$\frac{d^2\sigma}{dxdy} = \frac{2\pi M y}{1-y}\frac{d^2\sigma}{dE'd\Omega'}$$ PDG'17, Eq. (19.1)

The DIS cross section in terms of Lorentz-invariants **x, y** (photon exchange, neglecting m):

$$\frac{d^2\sigma}{dxdy} = \frac{4\pi\alpha_{\rm em}^2 S}{Q^4}\left[xy^2F_1(x,Q^2) + (1-y-xyM^2/S)F_2(x,Q^2)\right]$$

PDG'17, Eq. (19.8)

Homework Problems

1. Show that the phase space for the outgoing lepton takes the following form in the variables **x** and **y** (without any approximation), where **F** is the flux and **S=2 p.l**:

$$\frac{d^3 l'}{(2\pi)^3 2E'} = \frac{2S^2 y}{(4\pi)^2 F} dx dy$$

2. Derive the following general expression for the doubly differential cross section:

$$\frac{d^2\sigma}{dx dy} = \frac{2S^2 y}{(4\pi)^2 F^2}\left[\frac{e^4}{Q^4} L_{\mu\nu} W^{\mu\nu} 4\pi\right] = \frac{4S^2}{F^2}\frac{2\pi\alpha^2}{Q^4} y\, L_{\mu\nu} W^{\mu\nu}$$

(Note that the factor **$4S^2/F^2 = 1 + O(m^2/S * M^2/S)$** and the mass term is negligibly small for incoming neutrinos, electrons, and muons even if the nucleon mass is taken into account.)

3. Show that the hadronic tensor in terms of the structure functions **F_1**, **F_2** is given by:

$$W^{\mu\nu} = -g_\perp^{\mu\nu} F_1(x, Q^2) + \frac{1}{p \cdot q} p_\perp^\mu p_\perp^\nu F_2(x, Q^2)$$

Homework Problems

Show that the hadronic tensor can be brought in the following forms:

$$4\pi W_{\mu\nu} = \sum_{\text{states } X} \int d\Phi_X (2\pi)^4 \delta^{(4)}(p+q-p_X) \Big\langle \langle N(p)|J_\nu^\dagger(0)|X\rangle\langle X|J_\mu(0)|N(p)\rangle \Big\rangle_{\text{spin}}$$

$$= \sum_{\text{states } X} \int d\Phi_X \int d^4y\; e^{iqy} \Big\langle \langle N(p)|J_\nu^\dagger(y)|X\rangle\langle X|J_\mu(0)|N(p)\rangle \Big\rangle_{\text{spin}}$$

$$= \int d^4y\; e^{iqy} \Big\langle \langle N(p)|J_\nu^\dagger(y) J_\mu(0)|N(p)\rangle \Big\rangle_{\text{spin}}$$

$$= \int d^4y\; e^{iqy} \Big\langle \langle N(p)|[J_\nu^\dagger(y), J_\mu(0)]|N(p)\rangle \Big\rangle_{\text{spin}}$$

- Use the integral representation for the delta-distribution:

$$(2\pi)^4 \delta^{(4)}(p+q-p_X) = \int dy\; e^{i(p+q-p_X)y} = \int dy\; e^{iqy}\; e^{i(p-p_X)y}$$

- The space-time translation of an operator in QM is generated by the 4-momentum operator:

$$\hat{O}(y) := e^{i\hat{P}\cdot y}\hat{O}(0)e^{-i\hat{P}\cdot y}$$

- Use the completeness relation:

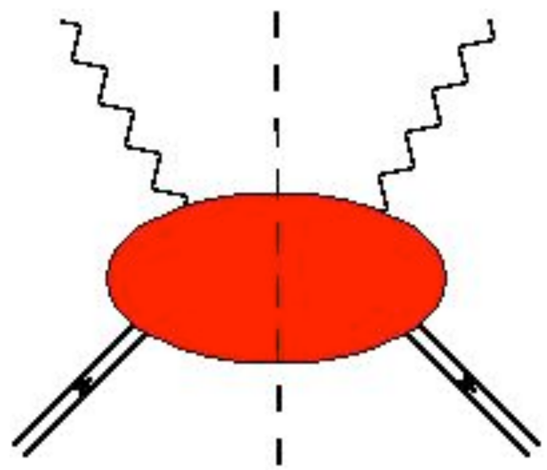
$$\sum_{\text{states } X} \int d\Phi_X |X\rangle\langle X| = \mathbf{1}$$

- The second term in the commutator leads to $q+p_X-p=0$ violating mom. cons. $q+p-p_X=0$!

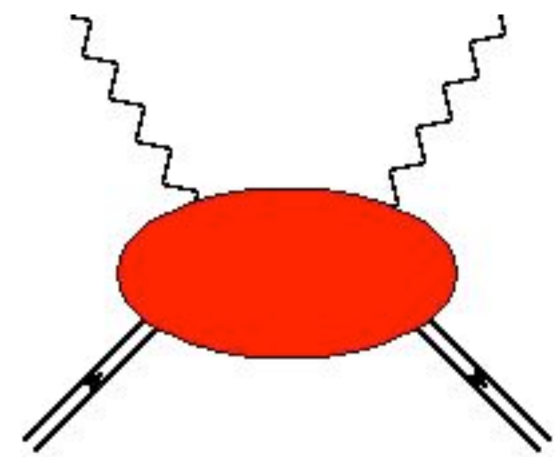
III. OPE

Hadronic tensor

Optical theorem: $W_{\mu\nu} \propto \operatorname{Im} T_{\mu\nu}$



$\propto \operatorname{Im}$



$$T_{\mu\nu} = i \int d^4x \; e^{iqx} \langle N | T[J_\mu^\dagger(x) J_\nu(0)] | N \rangle$$

OPE

Lit: Cheng, Li, pp. 298
Muta

$$T_{\mu\nu} = i \int d^4x \; e^{iqx} \langle N | T[J_\mu^\dagger(x) J_\nu(0)] | N \rangle$$

- $T_{\mu\nu}$ is dominated by contributions near the light cone (light cone dominance of DIS): **$x^2 \sim 0$**

- This is the starting point for the OPE (a sort of Taylor expansions of the product of currents):

$$T[J_\mu(x) J_\nu(0)] \underset{x^2 \sim 0}{\sim} \sum_{i,\tau,n} c_{\tau\mu\nu}^{i,\mu_1\cdots\mu_n}(x^2) O_{\mu_1\cdots\mu_n}^{i,\tau}(0)$$

Note:
a) independent of target
b) OPE implies factorization into Wilson coefficients and matrix elements of local operators

- Here $O^{i,\tau}$ are different local operators with the same twist $\tau = \mathrm{dim} - \mathrm{spin}$
Definite spin n $\leftrightarrow$ symmetric traceless tensors with n indices

- The c_τ^i are the Wilson coefficients. Naive dimensional counting: $c_\tau^i \underset{x^2 \to 0}{\to} (\sqrt{x^2})^{\tau - 2d_J} (\ln(x^2\mu^2))^p$
The leading term has lowest twist.

OPE

- After some manipulation ($e^{iqx}x^\mu = -i\partial/\partial q_\mu e^{iqx}, \partial/\partial q_\mu = q^\mu \partial/\partial q^2$) and in the approximation of keeping only the leading twist-2 operators one finds:

$$T^{\mu\nu} = \sum_{k=1}^{\infty} \Big(-g^{\mu\nu} q_{\mu_1} q_{\mu_2} C_{i1}^{2k} + g^\mu_{\mu_1} g^\nu_{\mu_2} Q^2 C_{i2}^{2k} - i\epsilon^{\mu\nu\alpha\beta} g_{\alpha\mu_1} q_\beta q_{\mu_2} C_{i3}^{2k}$$
$$+ \frac{q^\mu q^\nu}{Q^2} q_{\mu_1} q_{\mu_2} C_{i4}^{2k} + (g^\mu_{\mu_1} q^\nu q_{\mu_2} \pm g^\nu_{\mu_1} q^\mu q_{\mu_2}) C_{i5,6}^{2k} \Big) q_{\mu_3} \ldots q_{\mu_{2k}} \frac{2^{2k}}{Q^{4k}} A^i_{2k} \Pi^{\mu_1 \cdots \mu_{2k}}$$

- A_{2k} is the reduced matrix element of the twist-2 operator and $C_i^{2k}(q^2)$ is the (Fourier transformed) Wilson coefficient calculated using perturbation theory

- $$\Pi^{\mu_1 \ldots \mu_{2k}} = \sum_{j=0}^{k} (-1)^j \frac{(2k-j)!}{2^j (2k)!} \underbrace{\{g \ldots g\}}_{j\ g^{\mu_n \mu_m}\text{'s}} \quad \underbrace{\{p \ldots p\}}_{(2k-2j)\ p^{\mu_n}\text{'s}} (p^2)^j \underset{j=0}{=} p_{\mu_1} \cdots p_{\mu_{2k}}$$

  where $\{g \ldots g\}\ \{p_A \ldots p_A\}$abbreviates a sum over $(2k)!/[2^j j!(2k-2j)!]$ permutations of the indices.

OPE

- Covariant expansion of $T_{\mu\nu}$ similar to $W_{\mu\nu}$ (on page 19)

$$T_{\mu\nu}(p,q) = -\,g_{\mu\nu}T_1 + \frac{p_\mu p_\nu}{M^2}T_2 - i\epsilon_{\mu\nu\rho\sigma}\frac{p^\rho q^\sigma}{M^2}\tilde{T}_3 + \frac{q_\mu q_\nu}{M^2}T_4 + \frac{p_\mu q_\nu \pm p_\nu q_\mu}{M^2}T_{5,6}$$

- Neglecting target mass terms, things become much easier: **j=0**

- Working out the contractions and using the relation between $W_{\mu\nu}$ and $T_{\mu\nu}$ then relates Mellin moments of the structure functions to the reduced matrix elements:

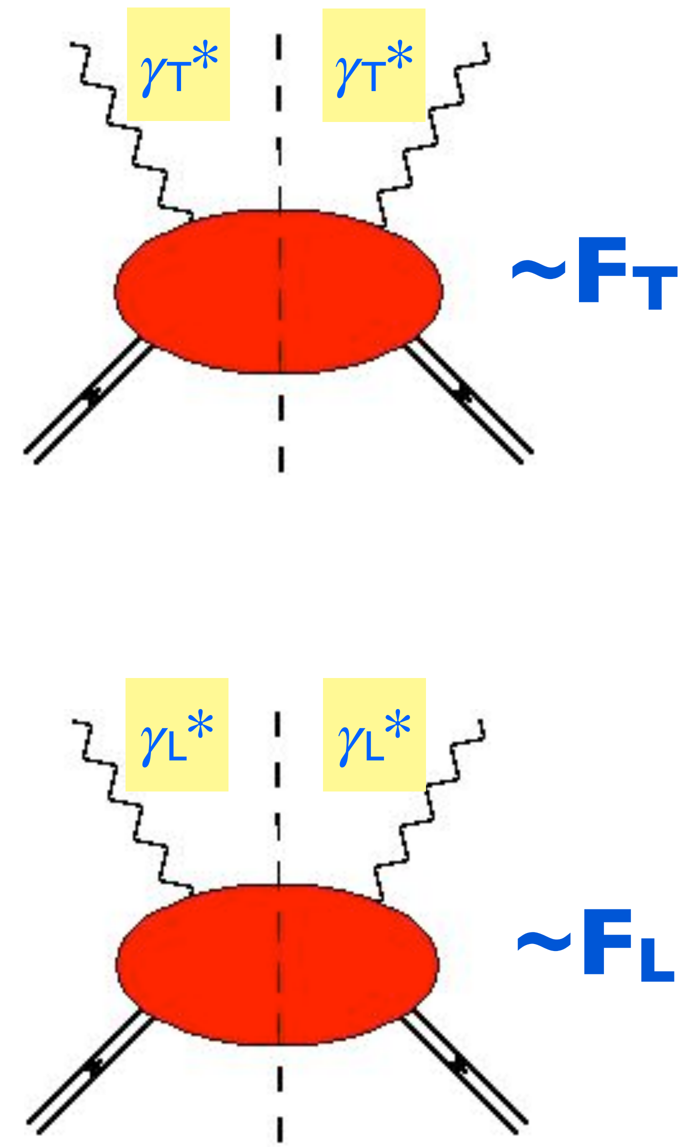
$$\int_0^1 dx x^{n-1}\frac{1}{x}F_2(x,Q^2) = C^n_{i2}A^i_n\,,\ \int_0^1 dx x^{n-1}F_1(x,Q^2) = C^n_{i1}A^i_n\,,\text{ETC}$$

IV. Longitudinal and transverse structure functions

Structure functions

$$W^{\mu\nu} = -g_\perp^{\mu\nu} F_1(x, Q^2) + \frac{1}{p \cdot q} p_\perp^\mu p_\perp^\nu F_2(x, Q^2)$$

- The sfs are non-perturbative objects which parameterize the structure of the target as 'seen' by virtual photons
- They are obtained with the help of projection operators: **$P_i^{\mu\nu} W_{\mu\nu} = F_i$**
- The projectors are rank-2 tensors formed out of the independent momenta **p, q** and the metric **g** (similar to **$W_{\mu\nu}$**)
- One can introduce transverse and longitudinal structure functions by contracting the hadronic tensor with the polarization vectors for transversely/longitudinally polarized virtual photons: **F_T, F_L**
- It turns out that: **$F_T = 2xF_1$, $F_2 = F_L + F_T$** (neglecting **M**)



Homework Problems

Halzen, Chap. 8.5

Chosing the z-axis along the three-momentum $\vec{q}$, such that $q^\mu = (q^0, 0, 0, |\vec{q}|)$, the polarisation vectors of *spacelike* photons with helicity $\lambda = 0, \pm 1$ can be written as:

$$\lambda = \pm 1 : \epsilon_\pm(q) = \mp \frac{1}{\sqrt{2}}(0, 1, \pm i, 0)$$

$$\lambda = 0 : \epsilon_\pm(q) = \frac{1}{\sqrt{-q^2}}(\sqrt{\nu^2 - q^2}, 0, 0, \nu)$$

1. Verify that $q \cdot \epsilon = 0$ for each λ, and show the following completeness relation for a space like photon ($q^2 < 0$):

$$\sum_{\lambda=0,\pm 1} (-1)^{\lambda+1} \epsilon^{*\mu}(q) \epsilon^\nu(q) = -g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}$$

2. Neglecting terms of order $O(M^2/Q^2)$ show that:

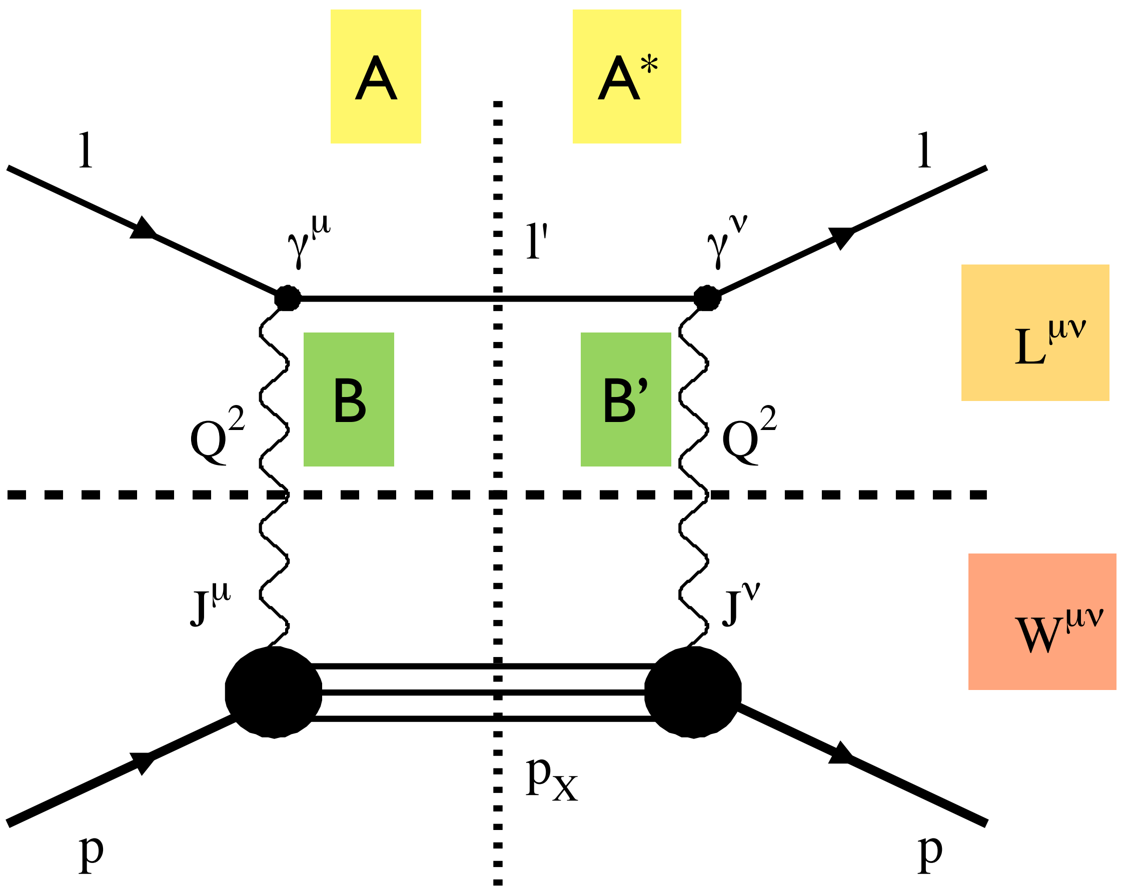
   a) $\epsilon_0^{*\mu}(q)\epsilon_0^\nu(q) W_{\mu\nu} = \frac{1}{2x} F_L$ with $F_L = F_2 - 2xF_1 = F_2 - F_T$

   b) $\frac{1}{2}[\epsilon_+^{*\mu}(q)\epsilon_+^\nu(q) + \epsilon_-^{*\mu}(q)\epsilon_-^\nu(q))] W_{\mu\nu} = \frac{1}{2x} F_T$ with $F_T = 2xF_1$

   It is useful to do the calculation in the nucleon rest frame $p = (M, 0, 0, 0)$.

V. CC and NC DIS

Cross section for CC and NC DIS



The differential cross section for DIS mediated by interfering gauge bosons **B**,**B'** can be written as:

$$\boxed{\frac{d^2\sigma}{dxdy} = \sum_{B.B'} \frac{d^2\sigma^{BB'}}{dxdy}}$$

- $B, B' \in \{\gamma, Z\}$ in the case of NC DIS
- $B = B' = W$ in the case of CC DIS

$$\boxed{d\sigma^{BB'} \sim L^{BB'}_{\mu\nu} W^{\mu\nu}_{BB'}}$$

Each of the terms **dσ^BB'** can be calculated from the general expression: PDG'17, Eq. (19.2)

$$\frac{d^2\sigma^{BB'}}{dxdy} = \frac{2S^2 y}{(4\pi)^2 F^2}\left[\frac{e^4}{Q^4}\chi_B\chi_{B'} L^{BB'}_{\mu\nu} W^{\mu\nu}_{BB'} 4\pi\right]$$
$$= \frac{4S^2}{F^2}\frac{2\pi\alpha^2}{Q^4} y \chi_B \chi_{B'} L^{BB'}_{\mu\nu} W^{\mu\nu}_{BB'}$$

$$\chi_\gamma(Q^2) = 1$$
$$\chi_Z(Q^2) = \frac{g^2}{(2\cos\theta_w)^2 e^2}\frac{Q^2}{Q^2+M_Z^2} = \frac{G_F}{\sqrt{2}}\frac{M_Z^2}{2\pi\alpha}\frac{Q^2}{Q^2+M_Z^2}$$
$$\chi_W(Q^2) = \frac{g^2}{(2\sqrt{2})^2 e^2}\frac{Q^2}{Q^2+M_W^2} = \frac{G_F}{\sqrt{2}}\frac{M_W^2}{4\pi\alpha}\frac{Q^2}{Q^2+M_W^2}$$

CC ν_τ-DIS

Albright, Jarlskog'75
Paschos, Yu'98
Kretzer, Reno'02

$$\frac{d^2\sigma^{\nu(\bar\nu)}}{dx\,dy} = \frac{G_F^2 M_N E_\nu}{\pi(1+Q^2/M_W^2)^2}\Big\{(y^2 x + \frac{m_\tau^2 y}{2E_\nu M_N})F_1^{W^\pm} + \Big[(1-\frac{m_\tau^2}{4E_\nu^2}) - (1+\frac{M_N x}{2E_\nu})y\Big]F_2^{W^\pm} \pm \Big[xy(1-\frac{y}{2}) - \frac{m_\tau^2 y}{4E_\nu M_N}\Big]F_3^{W^\pm} + \frac{\mathbf{m_\tau^2(m_\tau^2+Q^2)}}{\mathbf{4E_\nu^2 M_N^2 x}}\mathbf{F}_4^{W^\pm} - \frac{\mathbf{m_\tau^2}}{\mathbf{E_\nu M_N}}\mathbf{F}_5^{W^\pm}\Big\}$$

Albright-Jarlskog relations:

(derived at LO, extended by Kretzer, Reno)

$$F_4 = 0$$

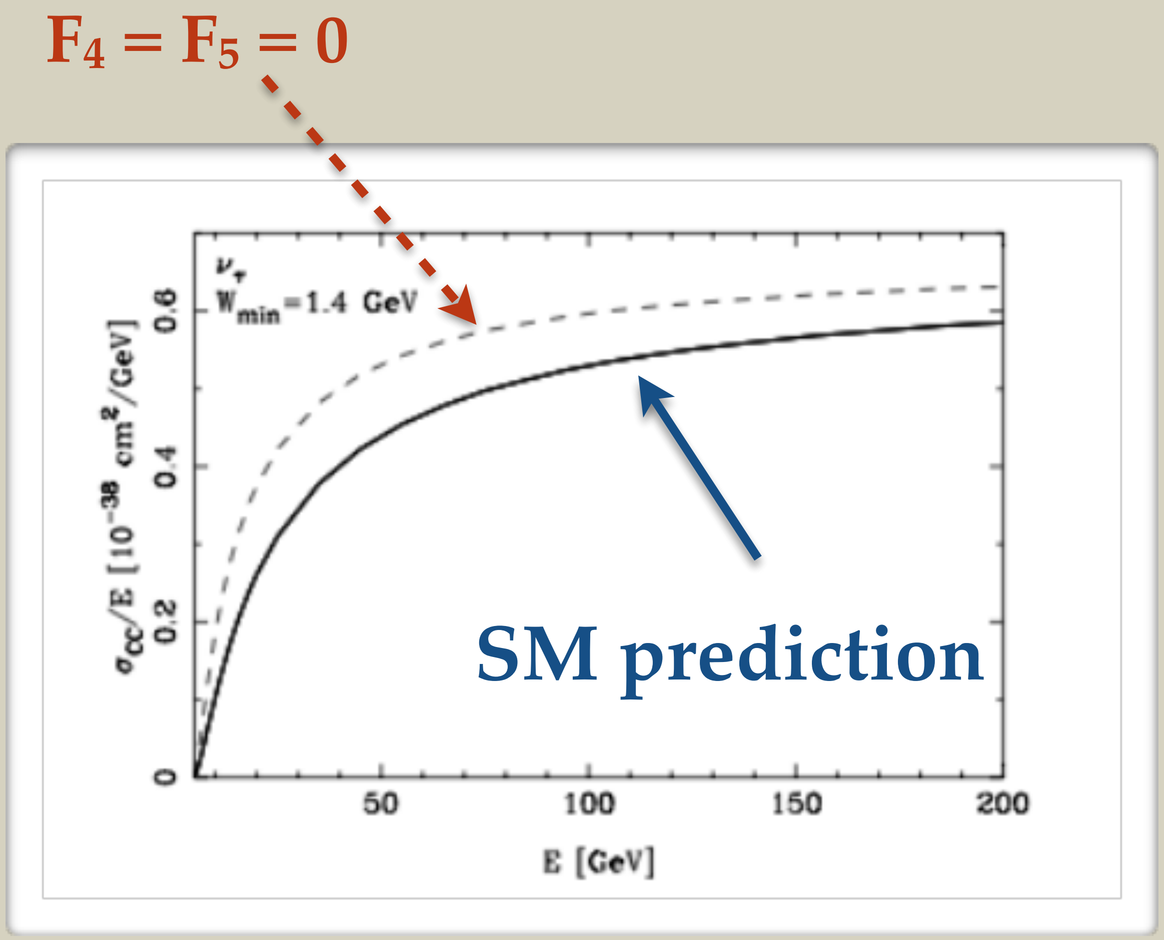
valid at LO $[\mathcal{O}(\alpha_s^0)]$, $M_N = 0$
(even for $m_c \neq 0$)

$$F_2 = 2xF_5$$

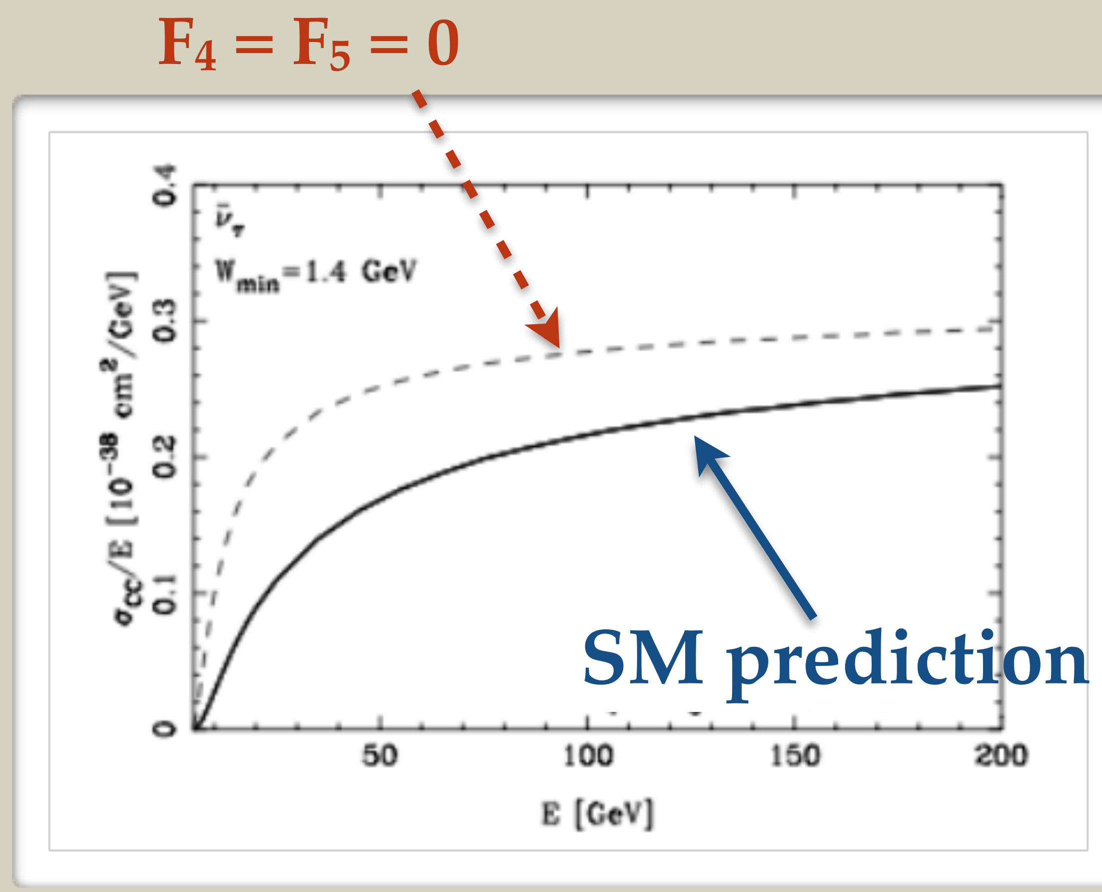
valid at **all orders** in α_s,
for $M_N = 0$, $m_q = 0$

Full NLO expressions ($M_N \neq 0, m_c \neq 0$): Kretzer, Reno'02

Sensitivity to F_4 and F_5



ν_τ CC DIS cross-section



$\bar{\nu}_\tau$ CC DIS cross-section

Homework Problems

* Little research project:

Work out the cross sections for NC and CC DIS

(Find typos in the following expressions,
Compare with expressions in PDF review)

Cross section for CC and NC DIS

Show that for an incoming electron with general $\gamma_\mu(V - A\gamma_5)$ current the leptonic tensor is given by (neglecting the lepton masses m_1 and m_2):

$$
\begin{aligned}
L_{\mu\nu}^{BB'} &= \frac{1}{2}\sum_{\lambda}\sum_{\lambda'} \bar{u}(l,\lambda)\Gamma_\nu^{B'} u(l',\lambda')\bar{u}(l',\lambda')\Gamma_\mu^{B} u(l,\lambda) \\
&= \frac{1}{2}\mathrm{Tr}[(\not{l} + m_1)\Gamma_\nu^{B'}\gamma^\beta(\not{l}' + m_2)\Gamma_\mu^{B}] \\
&= 2L_+[l^\mu l'^\nu + l^\nu l'^\mu - (l\cdot l')g^{\mu\nu}] + 4iR_{l,+}\epsilon_{\mu\nu\rho\sigma}l^\rho l'^\sigma
\end{aligned}
$$

Here $\Gamma_\mu^B = \gamma_\mu(V_e^B = A_e^B\gamma_5)$, $L_\pm = V_e^B V_e^{B'} \pm A_e^B A_e^{B'}$, $R_{e,\pm} = V_e^B A_e^{B'} \pm V_e^{B'} A_e^B$.

B	V_e^B	A_e^B
γ	-1	0
Z^0	$-1/2 + 2\sin^2\theta_w$	$-1/2$
W	1	1

see Halzen&Martin

Cross section for CC and NC DIS

The weak currents are *not* conserved (*) and parity is violated. Therefore, one has to assume the most general structure for the hadronic tensor. In particular one has to include a parity violating piece $\sim i\epsilon_{\mu\nu\rho\sigma}p^\rho q^\sigma$:

convention: **ε^{0123}=+1**

$$W_{\mu\nu}^{BB'} = -g_{\mu\nu}F_1^{BB'}(x,Q^2) + \frac{p_\mu p_\nu}{p\cdot q}F_2^{BB'}(x,Q^2) - i\epsilon_{\mu\nu\rho\sigma}\frac{p^\rho q^\sigma}{2p\cdot q}F_3^{BB'}(x,Q^2)$$
$$+ \frac{q_\mu q_\nu}{p\cdot q}F_4^{BB'}(x,Q^2) + \frac{p_\mu q_\nu + p_\nu q_\mu}{2p\cdot q}F_5^{BB'}(x,Q^2) + \frac{p_\mu q_\nu - p_\nu q_\mu}{2p\cdot q}F_6^{BB'}(x,Q^2)$$

The terms proportional to F_4, F_5 will be proportional to the lepton masses squared and are usually neglected (F_6 will not contribute to the cross section at all). Of course, these terms have to be kept in the hadronic tensor when projecting out structure functions.

(*) With $J_w^\mu = \bar{u}(p')\gamma_\mu(v - a\gamma_5)u(p)$ and using the Dirac equation one finds $q_\mu J_w^\mu \sim a(m+m')$ with $p^2 = m^2$, $p'^2 = m'^2$. Therefore $q_\mu L^{\mu\nu} \sim$ lepton mass.

Cross section for CC and NC DIS

We are now in a position to calculate the cross section:

$$\frac{d^2\sigma^{BB'}}{dxdy} = \frac{4\pi\alpha^2 S}{Q^4}\chi_B\chi'_B\left[xy^2 L_+ F_1^{BB'} + (1-y-xyM^2/S)L_+F_2^{BB'} - y(1-y/2)2R_{l,+}xF_3^{BB'}\right]$$

Introducing generalized structure functions we can form the Neutral Current (NC) cross section:

$$\frac{d^2\sigma^{NC}}{dxdy} = \frac{4\pi\alpha^2 S}{Q^4}\left[xy^2\mathcal{F}_1^{NC} + (1-y-xyM^2/S)\mathcal{F}_2^{NC} - y(1-y/2)x\mathcal{F}_3^{NC}\right]$$

with

$$\begin{aligned}
\mathcal{F}_{1,2}^{NC}(x,Q^2) &= F_{1,2}^{\gamma\gamma} + 2\chi_Z\left(-v_l^Z\right)F_{1,2}^{\gamma Z} + \chi_Z^2\left(\left(v_l^Z\right)^2 + \left(a_l^Z\right)^2\right)F_{1,2}^{ZZ} \\
\mathcal{F}_3^{NC}(x,Q^2) &= -2\chi_Z a_l^Z F_3^{\gamma Z} + \chi_Z^2 2v_l^Z a_l^Z F_3^{ZZ}\ .
\end{aligned}$$

VI. Bjorken scaling

Expectations from elastic ep scattering

Pointlike proton without spin, neglecting recoil:

$$\frac{d\sigma^{\mathrm{Mott}}}{d\Omega} = \frac{\alpha^2}{4E^2\sin^4(\theta/2)}\cos^2(\theta/2)$$

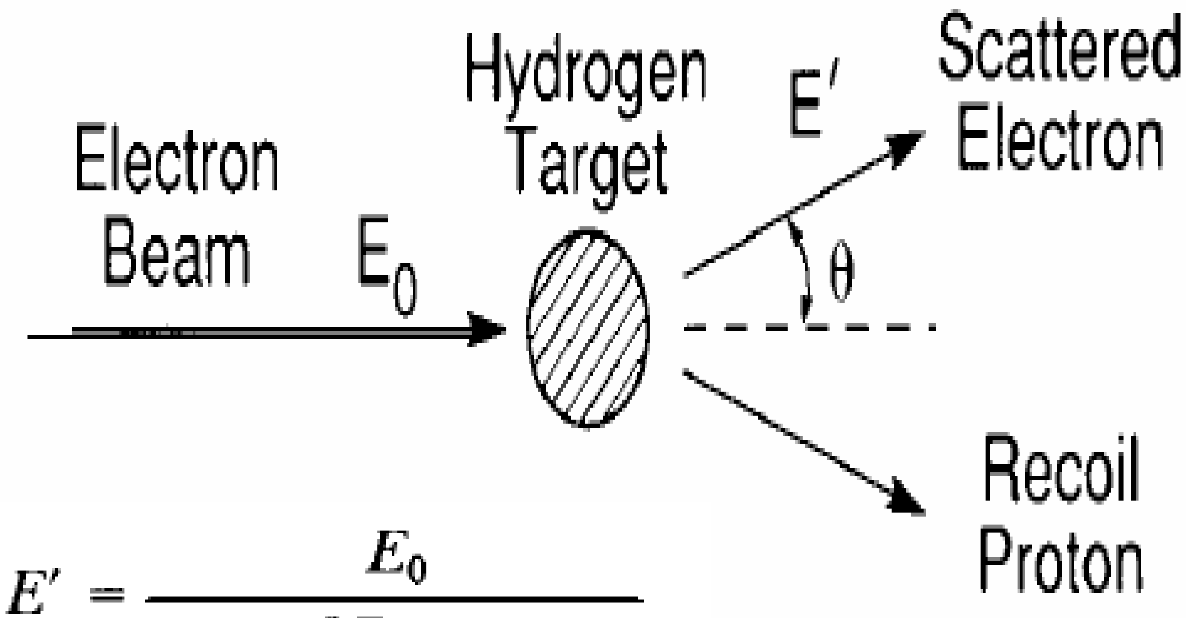
electron spin

Rutherford scattering

Expectations from elastic ep scattering

Pointlike proton without spin, neglecting recoil:

$$\frac{d\sigma^{\text{Mott}}}{d\Omega} = \frac{\alpha^2}{4E^2 \sin^4(\theta/2)} \cos^2(\theta/2)$$



$$E' = \frac{E_0}{1 + \frac{2E_0}{M} \sin^2 \theta/2}$$

Pointlike proton with spin:

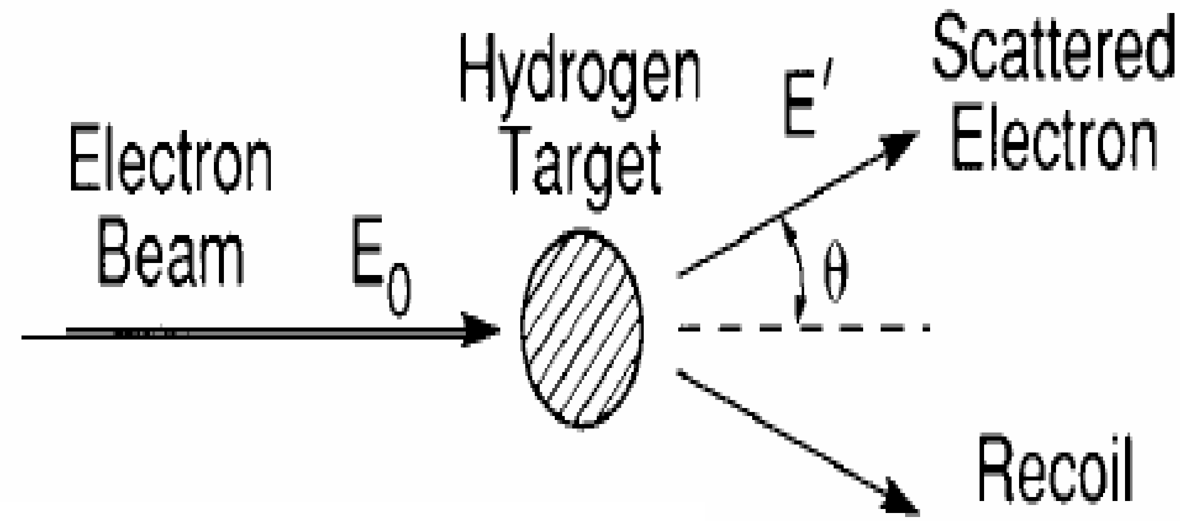
$$\frac{d\sigma}{d\Omega} = \frac{d\sigma^{\text{Mott}}}{d\Omega} \frac{E'}{E} [1 + 2\tau \tan^2(\theta/2)]$$

$$\tau = \frac{Q^2}{4M^2}\,,\; Q^2 = 4EE' \sin^2(\theta/2)$$

Expectations from elastic ep scattering

Pointlike proton without spin, neglecting recoil:

$$\frac{d\sigma^{\mathrm{Mott}}}{d\Omega} = \frac{\alpha^2}{4E^2 \sin^4(\theta/2)} \cos^2(\theta/2)$$



$$E' = \frac{E_0}{1 + \frac{2E_0}{M}\sin^2\theta/2}$$

Pointlike proton with spin:

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma^{\mathrm{Mott}}}{d\Omega} \frac{E'}{E} [1 + 2\tau \tan^2(\theta/2)]$$

Extended proton with spin (Rosenbluth formula):

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma^{\mathrm{Mott}}}{d\Omega} \frac{E'}{E} \left[\frac{G_E^2 + \tau G_M^2}{1+\tau} + 2\tau G_M^2 \tan^2(\theta/2)\right]$$

$$\tau = \frac{Q^2}{4M^2}\,,\; Q^2 = 4EE'\sin^2(\theta/2)$$

- Elastic form factor **$G_E(Q^2)$, $G_E(0)=1$**
- Magnetic form factor **$G_M(Q^2)$, $G_M(0)=\mu_p=2.79$**
  $\mu_p=2.79$: proton anomalous magnetic moment

Steeply falling form factors:

$$G_E(Q^2) = \frac{G_M(Q^2)}{\mu_p} = (1 + Q^2/a^2)^{-2}$$

$$a^2 = 0.71\ \mathrm{GeV}^2$$

Expectations from elastic ep scattering

Pointlike proton without spin, neglecting recoil:

$$\frac{d\sigma^{\mathrm{Mott}}}{d\Omega} = \frac{\alpha^2}{4E^2 \sin^4(\theta/2)} \cos^2(\theta/2)$$

Pointlike proton with spin:

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma^{\mathrm{Mott}}}{d\Omega} \frac{E'}{E} [1 + 2\tau \tan^2(\theta/2)]$$

Note that the idea of a **point-like strongly interacting** particle is rather **academic**!

Due to **quantum corrections** we **have to generalize** the 'point-like current' by the most general current respecting all symmetries of the interaction and **introduce form factors**.

This is even the case in QED. However, here the Dirac and Pauli form factors are **calculable in perturbation theory**.

Extended proton with spin (Rosenbluth formula):

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma^{\mathrm{Mott}}}{d\Omega} \frac{E'}{E} \left[\frac{G_E^2 + \tau G_M^2}{1+\tau} + 2\tau G_M^2 \tan^2(\theta/2) \right]$$

$$\tau = \frac{Q^2}{4M^2}, \; Q^2 = 4EE' \sin^2(\theta/2)$$

- Elastic form factor **$G_E(Q^2)$, $G_E(0)=1$**
- Magnetic form factor **$G_M(Q^2)$, $G_M(0)=\mu_p=2.79$**
  $\mu_p=2.79$: proton anomalous magnetic moment

Steeply falling form factors:

$$G_E(Q^2) = \frac{G_M(Q^2)}{\mu_p} = (1 + Q^2/a^2)^{-2}$$

$$a^2 = 0.71 \ \mathrm{GeV}^2$$

What do we expect for a point-like particle?

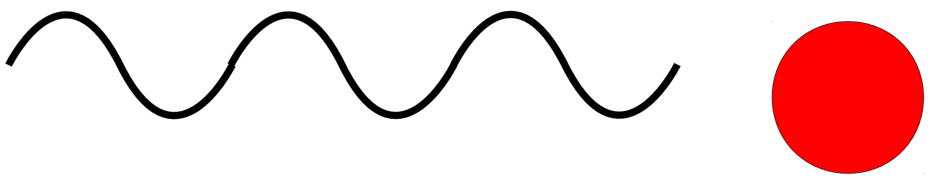
Point-like proton without spin, neglecting recoil:

$$\frac{d\sigma^{\mathrm{Mott}}}{d\Omega} = \frac{\alpha^2}{4E^2 \sin^4(\theta/2)} \cos^2(\theta/2)$$

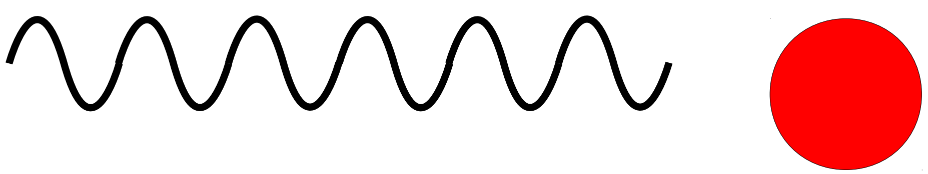
Point-like proton/muon with spin:

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma^{\mathrm{Mott}}}{d\Omega} \frac{E'}{E} [1 + 2\tau \tan^2(\theta/2)]$$

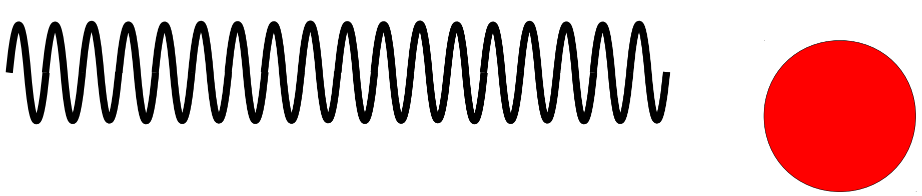
Fred Olness, CTEQ school 2012



$$d\sigma \sim \frac{4\pi\alpha^2}{Q^2} \times 1$$



$$d\sigma \sim \frac{4\pi\alpha^2}{Q^2} \times 1$$



$$d\sigma \sim \frac{4\pi\alpha^2}{Q^2} \times 1$$

Dimensional considerations

Structure Function

Expectations from elastic ep scattering

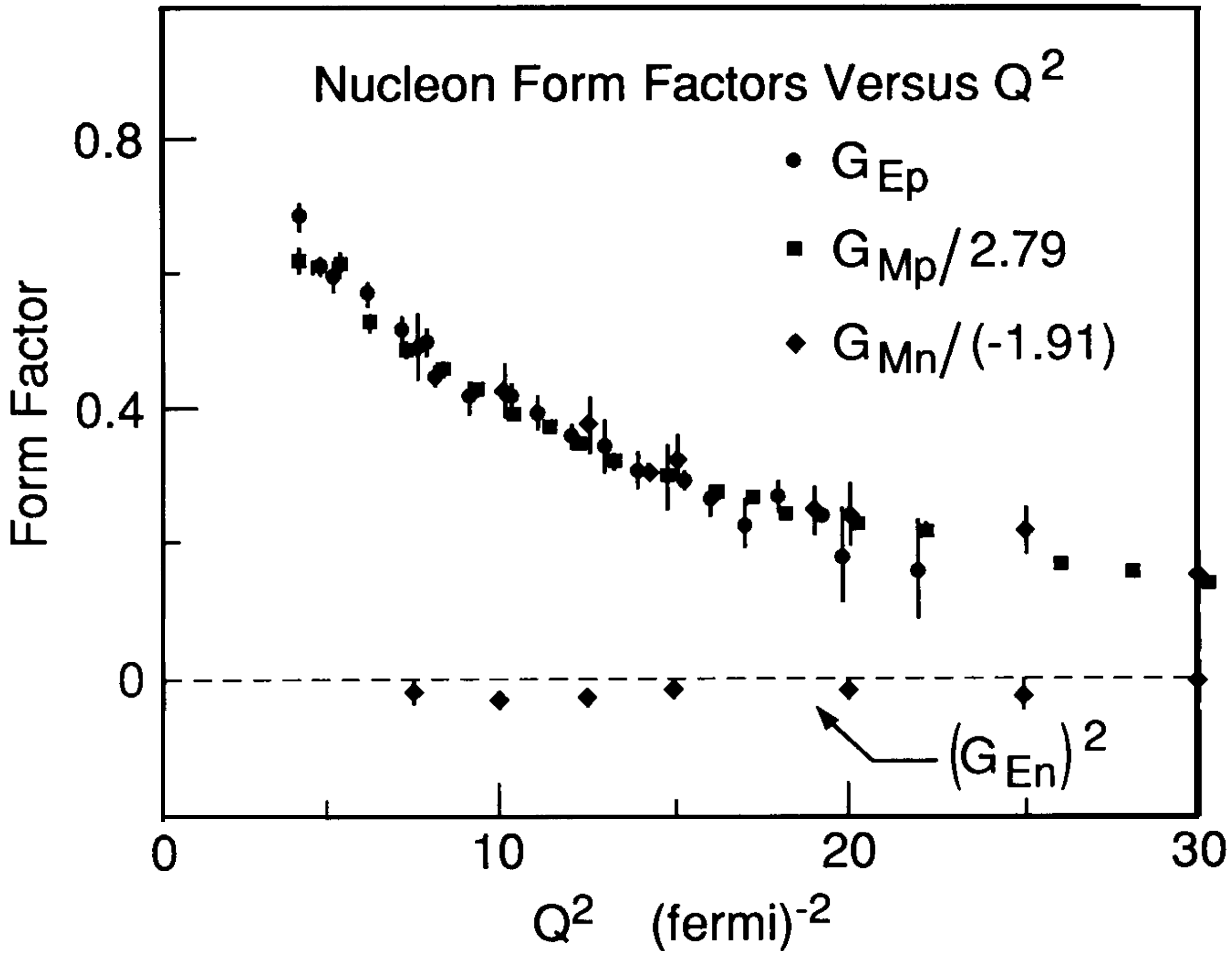


Fig. 23. Summary of results on nuclear form factors presented by the Stanford group at the 1965 "International Symposium on Electron and Photon Interactions at High Energies". (A momentum transfer of 1 GeV^2 is equivalent to 26 Fermis^{-2}.)

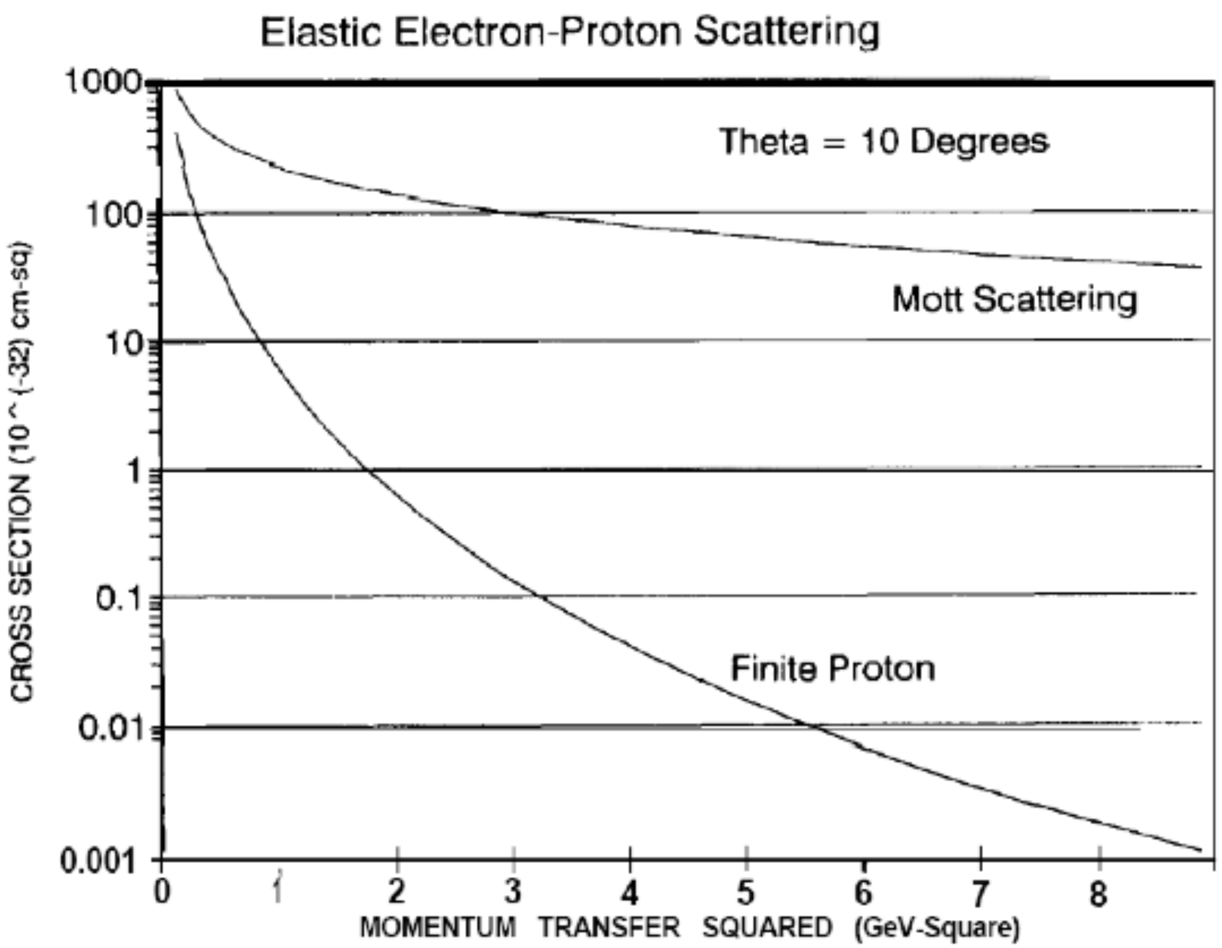


Fig. 4. Elastic scattering cross sections for electrons from a "point" proton and for the actual proton. The differences are attributable to the finite sire of the proton.

The results formed the prejudice that the proton was a soft "mushy" extended object, possibly with a hard core surrounded by a cloud of mesons, mainly pions.

The SLAC-MIT team saw its objective in searching for the hard core of the proton.
First DIS experiments (>=1967).

Bjorken scaling

Elastic scattering (Rosenbluth formula):

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma^{\text{Mott}}}{d\Omega}\frac{E'}{E}\left[\frac{G_E^2+\tau G_M^2}{1+\tau}+2\tau G_M^2\tan^2(\theta/2)\right]$$

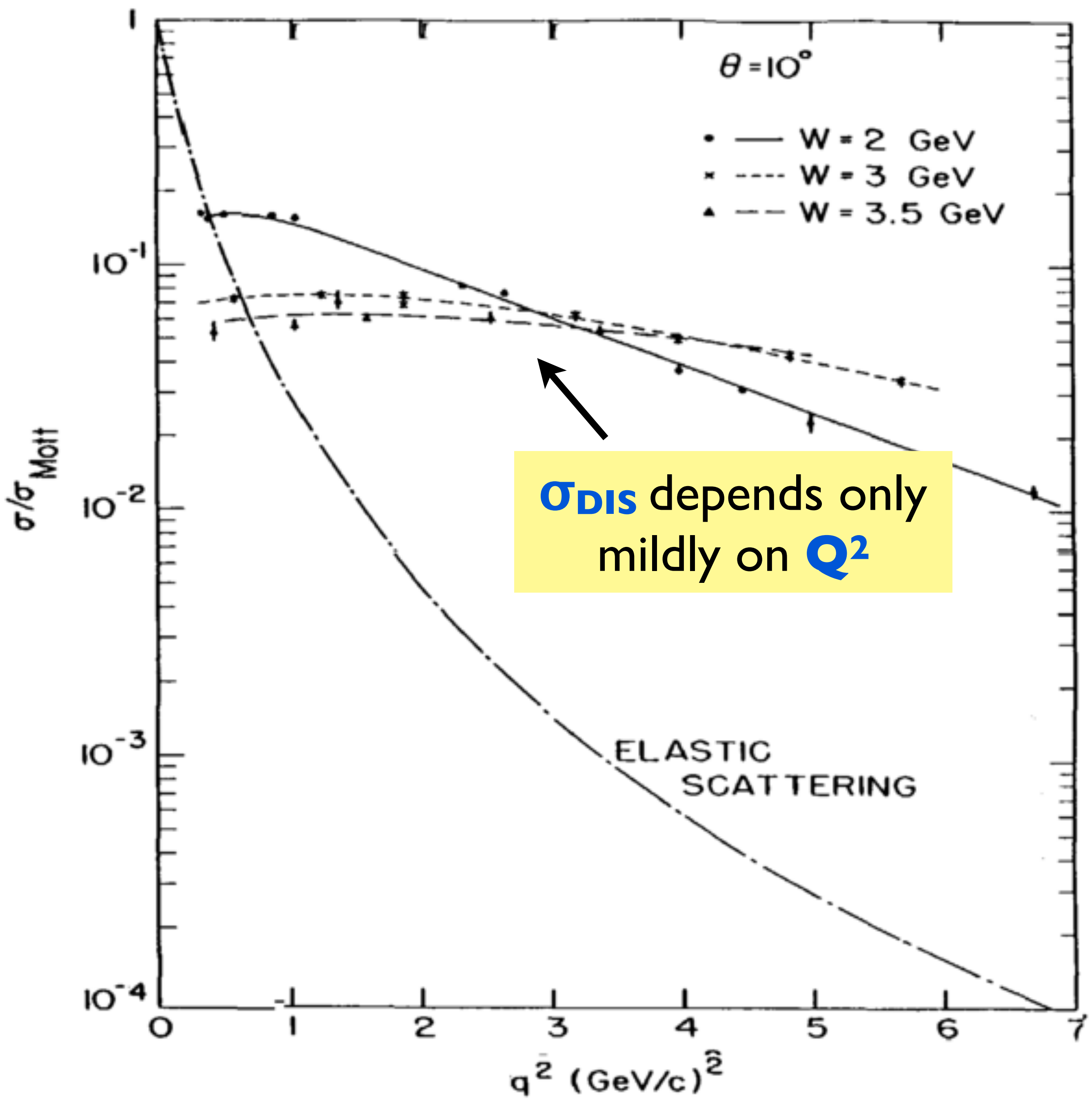
The DIS cross section resembles the elastic one:

$$d\sigma_{\text{DIS}} \sim d\sigma^{\text{Mott}}[W_2+2W_1\tan^2(\theta/2)]$$

The form factors had been know to fall rapidly as a function of **Q^2**.

Therefore, the general expectation for **σ_{DIS}** before its measurement was that it also would be a fast falling function of **Q^2**.

Early data on DIS from the SLAC-MIT experiment [PRL23(1969)935]



σ_{DIS} depends only mildly on **Q^2**

Bjorken scaling

Scaling hypothesis (Bjorken 1968):

In the limit $Q^2 \rightarrow \infty$, $\nu \rightarrow \infty$, such that $x = Q^2/(2M\nu)$ is fixed ('Bjorken limit') the structure functions $F_i(x,Q^2)$ are insensitive to Q^2: $F_i = F_i(x)$

This behaviour is called scaling and x is called the scaling variable

Bjorken scaling

Scaling hypothesis (Bjorken 1968):

In the limit $Q^2 \rightarrow \infty$, $\nu \rightarrow \infty$, such that $x = Q^2/(2M\nu)$ is fixed ('Bjorken limit') the structure functions $F_i(x,Q^2)$ are insensitive to Q^2: $F_i = F_i(x)$

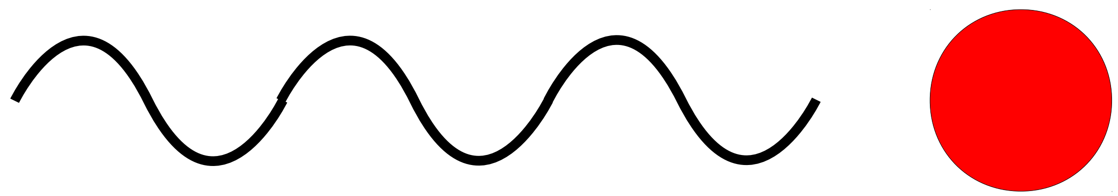
This behaviour is called scaling and x is called the scaling variable

Scaling implies that the nucleon appears as a collection of point-like constituents when probed at very high energies (Q^2 large).

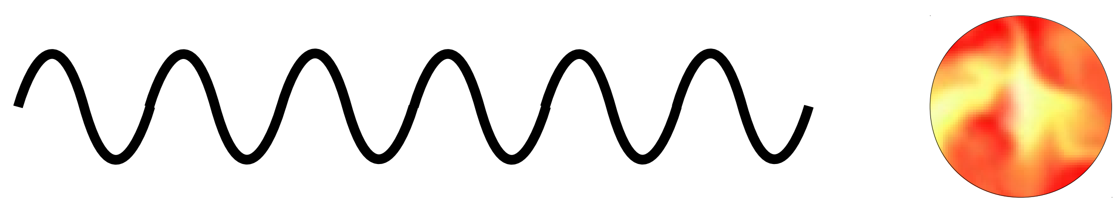
The possible existence of such point-like constituents was also proposed by Feynman from a different theoretical perspective and he gave them the name 'partons'.

Structure of the proton

Fred Olness,
CTEQ school 2012

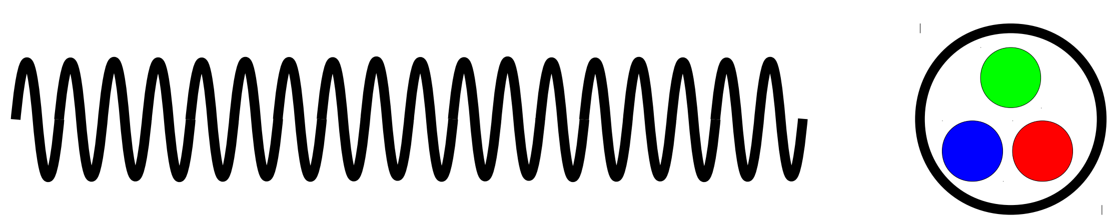


$$d\sigma \sim \frac{4\pi\alpha^2}{Q^2} \times 1$$



$$d\sigma \sim \frac{4\pi\alpha^2}{Q^2} \times F\left(\frac{Q^2}{\Lambda^2}\right)$$

Λ of order of the proton mass scale



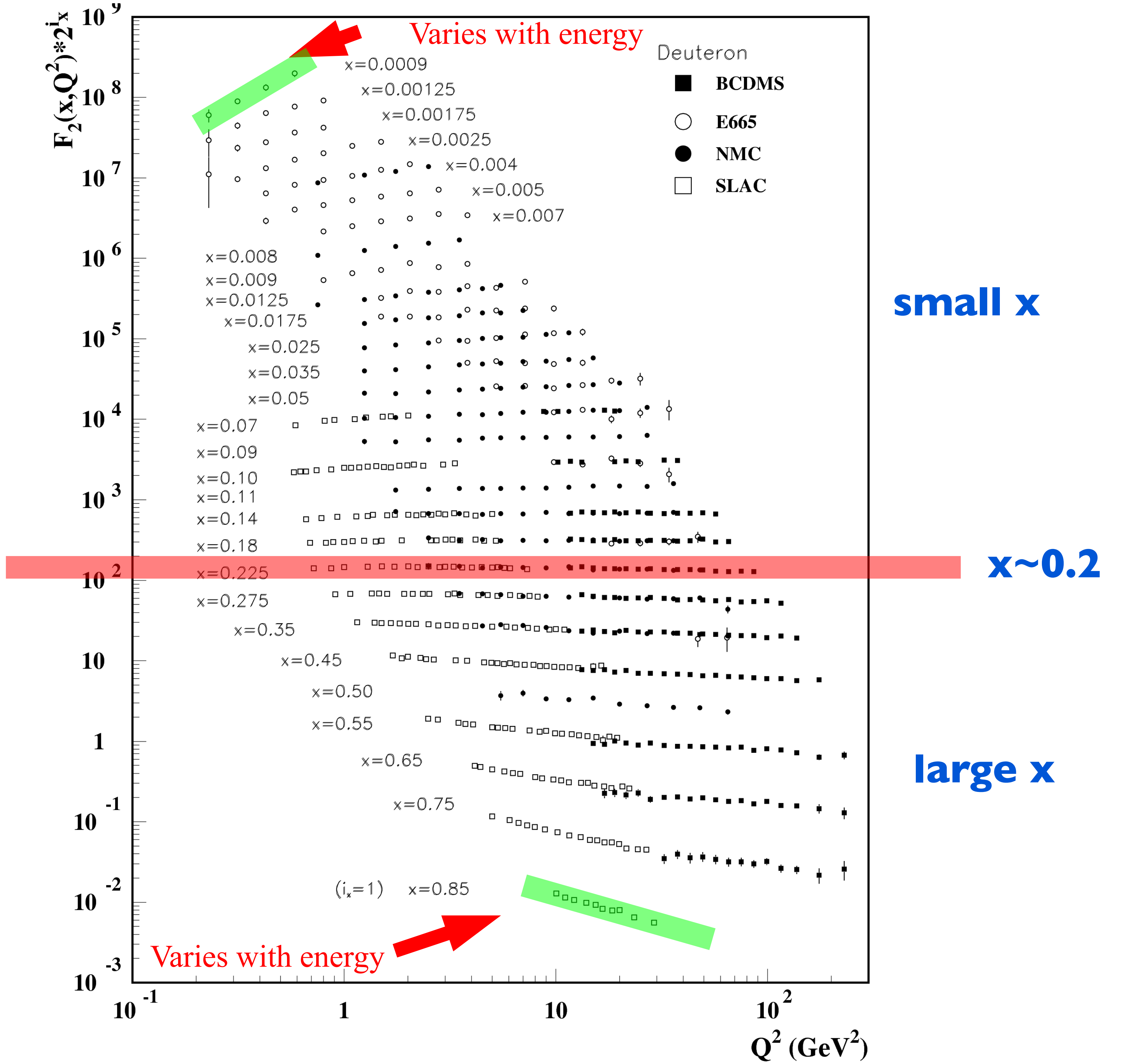
$$d\sigma \sim \frac{4\pi\alpha^2}{Q^2} \times \sum_i e_i^2$$

Bjorken scaling for F_2

Fred Olness,
CTEQ school 2012

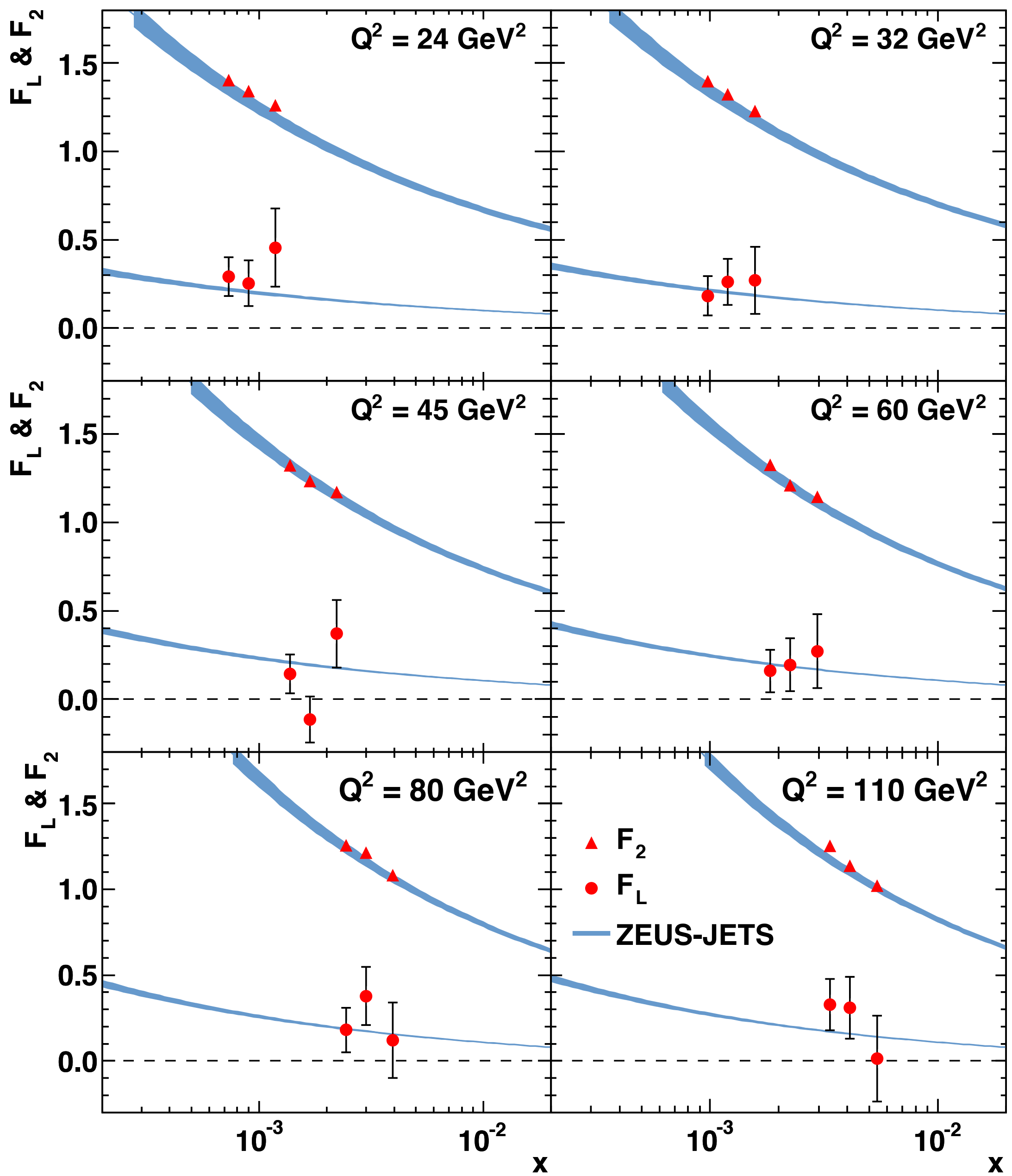
Data is (relatively) independent of energy

Scaling Violations observed at extreme x values

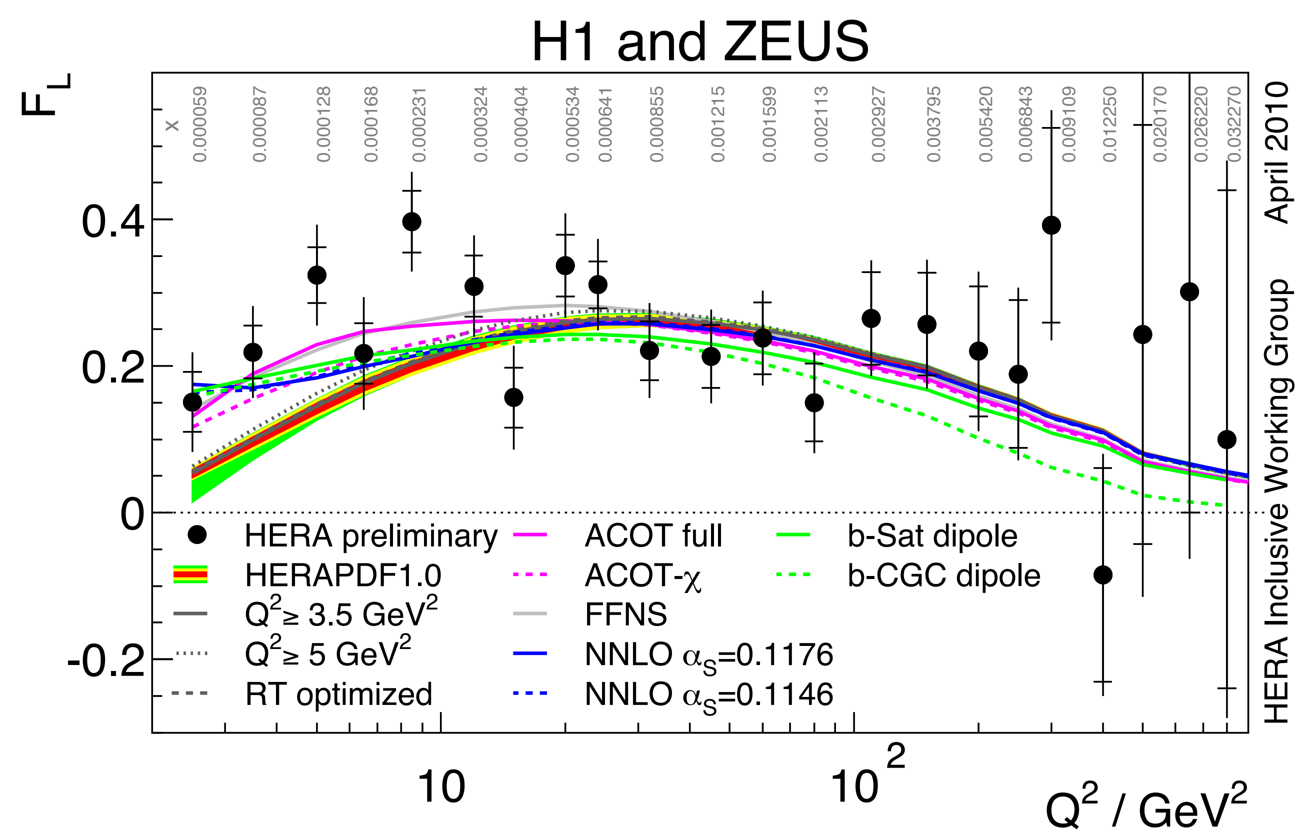


Bjorken scaling for F_L

ZEUS collab, arXiv:0904.1092



The HERA combined measurement of **F_L** is compatible with scaling



We note that **F_L** is quite smaller than **F_2**.

The End of Lecture 1