Lecture 1: Theoretical introduction to Drell-Yan production in QCD

Valerio Bertone
IRFU, CEA, Université Paris-Saclay

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Outline

- Historical introduction:
    - Drell-Yan production as a testing ground for the Parton Model.
- The advent of QCD:
    - the need for a factorisation theorem,
    - higher-order corrections and the limitations of fixed-order calculations.
- Resummation:
    - the origin of large logarithms,
    - resumming the leading logarithms,
    - the Collins-Soper-Sterman formalism and TMD factorisation,
    - (transverse non-perturbative effects).

The Parton Model was born as a leap of faith…

Excerpt from R. P. Feynman [*Phys. Rev. Lett.* 23, 1415 (1969)]:

VERY HIGH-ENERGY COLLISIONS OF HADRONS

Richard P. Feynman
California Institute of Technology, Pasadena, California
(Received 20 October 1969)

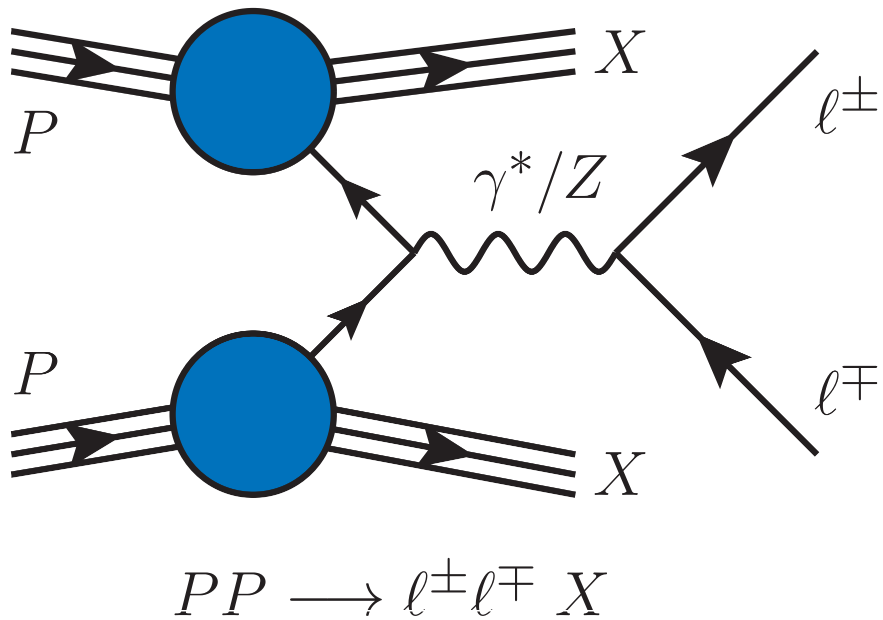
I have difficulty in writing this note because it is not in the nature of a deductive paper, but is the result of an induction. I am more sure of the conclusions than of any single argument which suggested them to me for they have an internal consistency which surprises me and exceeds the consistency of my deductive arguments which hinted at their existence.

The Parton Model is based on an intuition that even Richard Feynman could not give a solid argument to (an enlightening reading of only 2 pages that I would suggest to anyone).

Core of the Parton Model: hadrons are made of *partons*, free elementary particles that undergo an instantaneous interaction with a projectile carrying a *large* energy. Then one can neglect binding effects during the interaction and treat the collision as between free particles.

Drell-Yan in the Parton Model

One year later Drell and Yan applied the Parton Model to the inclusive production of massive lepton pairs in hadron-hadron collision (called the Drell-Yan process ever since) [*Phys. Rev. Lett.* 25, 316 (1970)]:



Q: invariant mass of the lepton pair

$\sqrt{s}$: collision center-of-mass energy

The result of their calculation assuming photon exchange was:

$$\frac{d\sigma^{\mathrm{PM}}_{PP\to\ell\bar{\ell}+X}}{dQ^2} = \frac{4\pi\alpha^2_{\mathrm{em}}}{3sQ^2(N_c)} \sum_q e_q^2 \int_{Q^2/s}^1 \frac{dy}{y} f_q(y) f_{\overline{q}}\left(\frac{Q^2}{sy}\right)$$

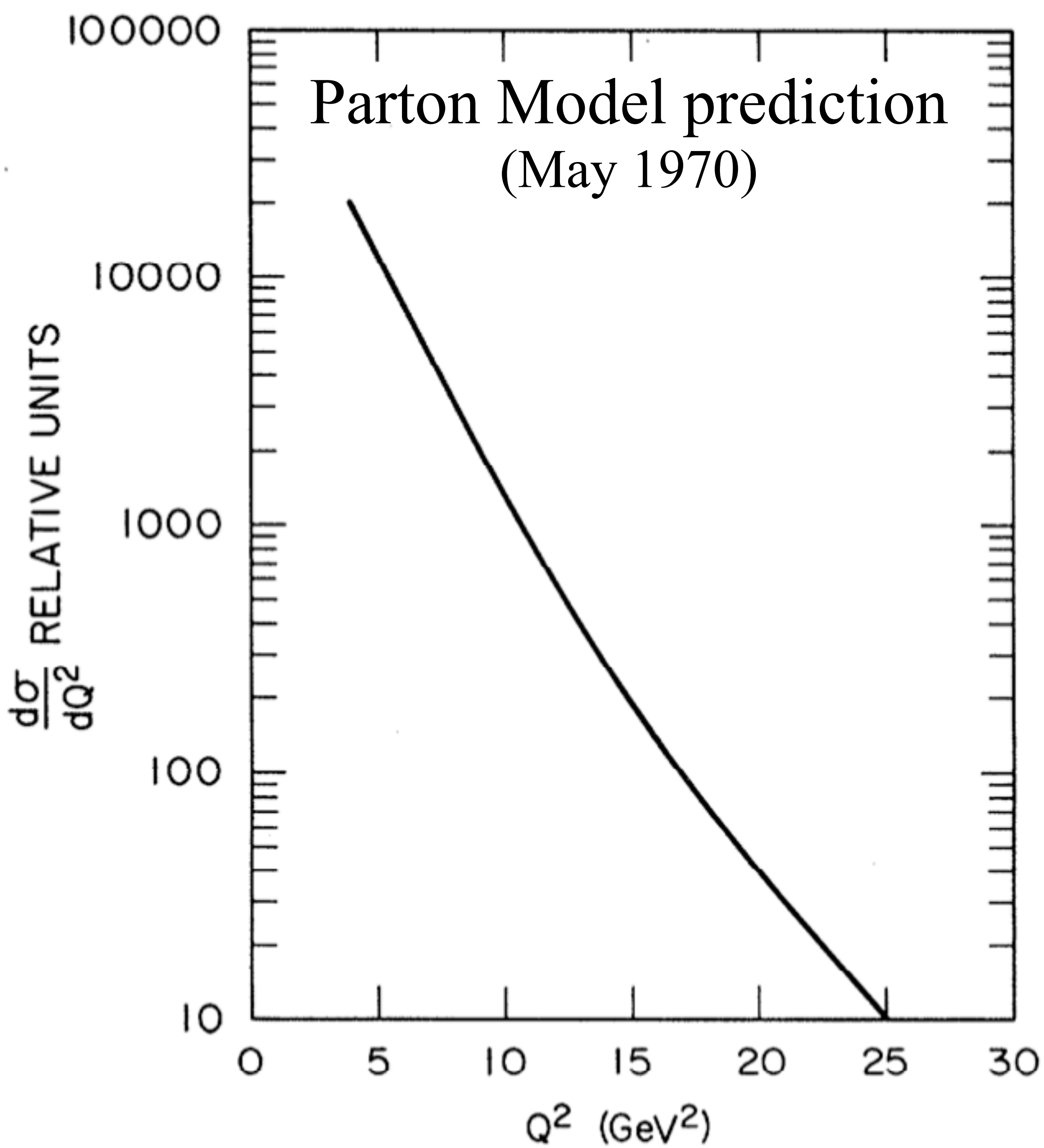
As a pre-QCD result, the colour factor N_c was not included but the kinematic fall-off with Q^2 was predicted.

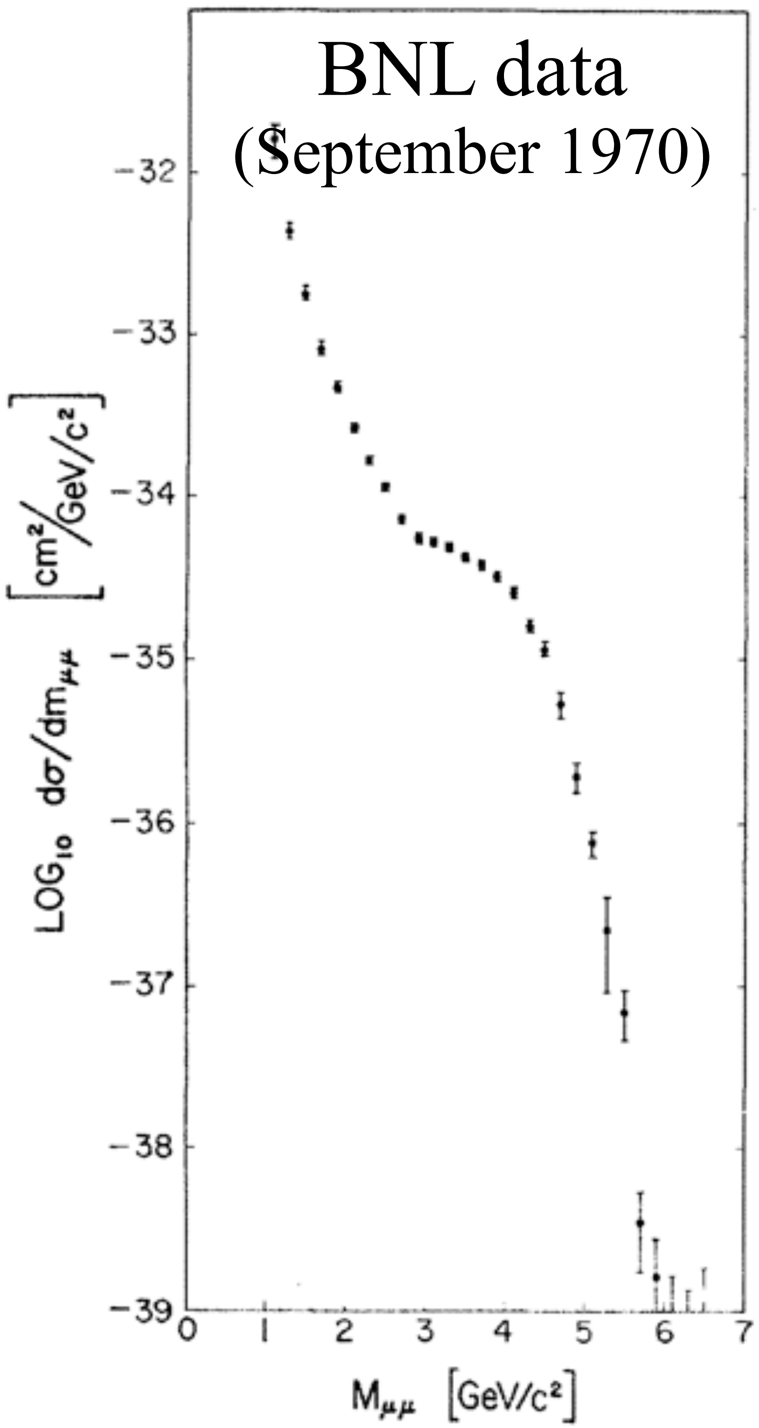
The functions $f_{q(\overline{q})}$ are the **parton distribution functions** (PDFs) that in this context can be interpreted as **probability** distribution functions.

Drell-Yan in the Parton Model

The Parton Model was tested in $p + U \rightarrow \mu^+ + \mu^- + X$ at BNL in 1970.

Parton Model "[…] in **rough** shape agreement with the data […]." (citation from "Observation of massive muon pairs in hadron collisions" [*Phys. Rev. Lett.* 25 (1970) 1523-1526]).





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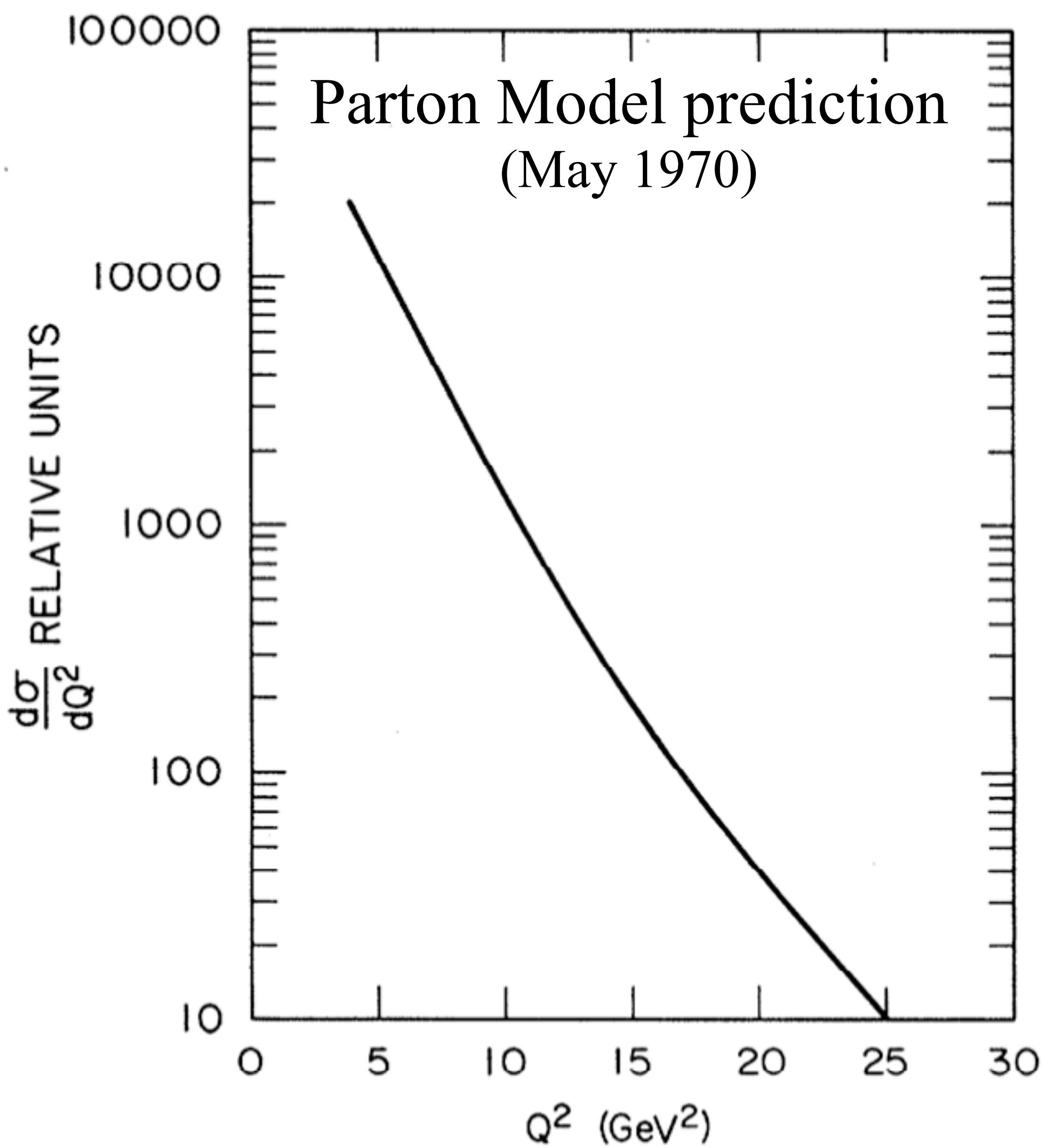
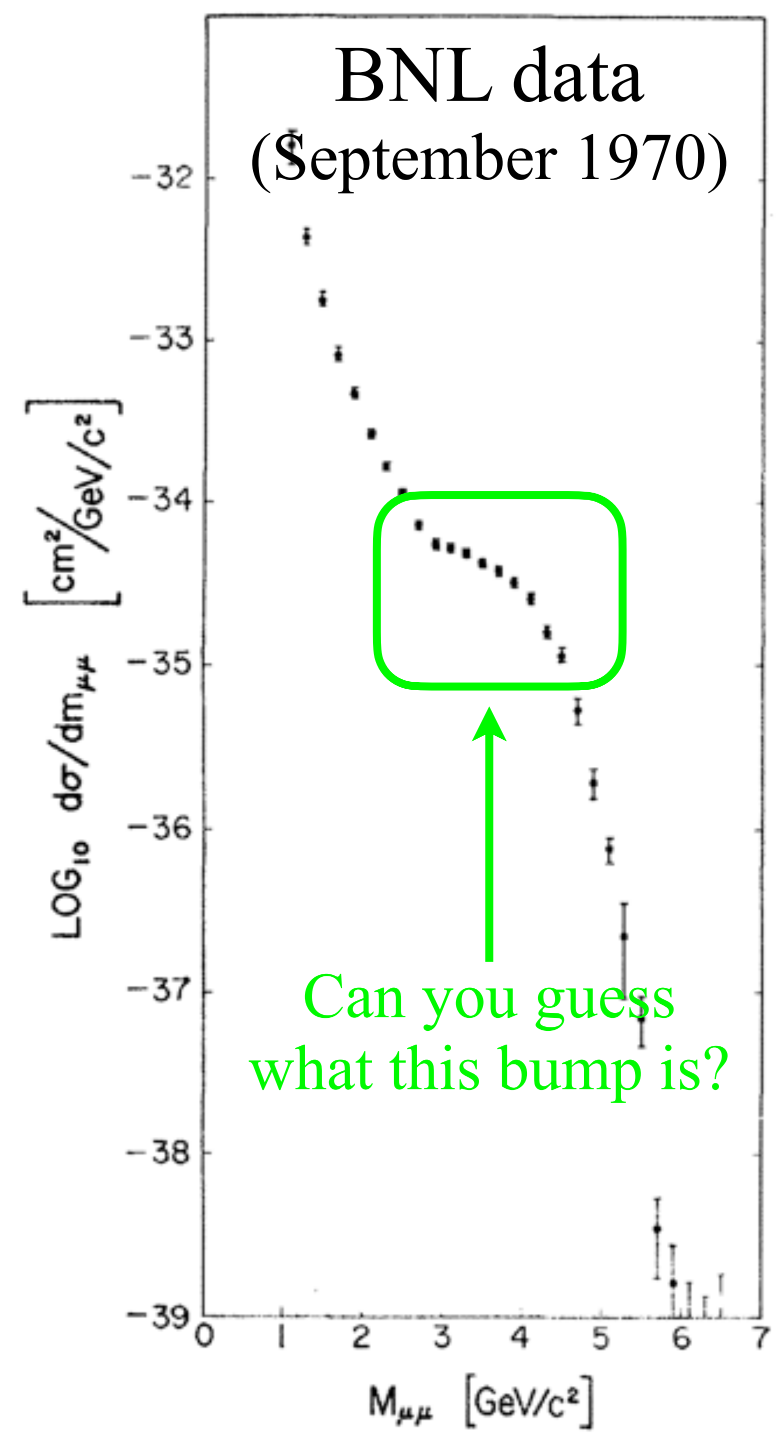




The advent of QCD

I am not going to (re)introduce you QCD!

All I want to stress here is that Parton Model and QCD are *fundamentally different*:
- the former is a (semi)classical model,
- the latter is a sound quantum-field theory that assumes a given matter content (quarks) and a non-abelian gauge symmetry (SU(3)).

Yet, when it comes to Drell-Yan, the Parton Model can be regarded as the *skeleton* of QCD. How comes?
1. QCD is **renormaliseable** in four dimensions (dimensionless coupling),
2. the Drell-Yan cross section in QCD enjoys leading-power **factorisation** of low- and high-energy contributions:

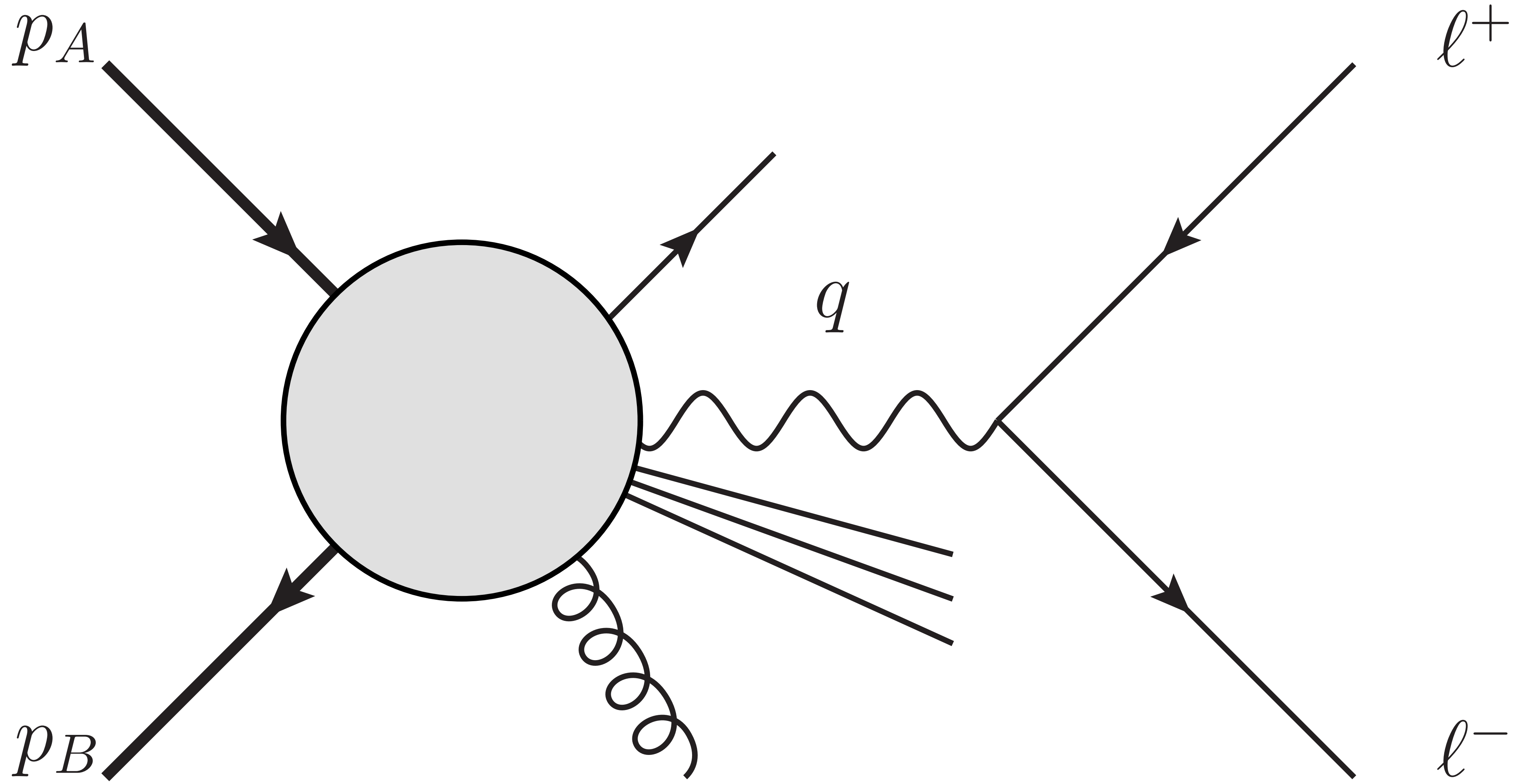
$$\sigma = \hat{\sigma} \otimes f$$

“Byproduct” of factorisation: **operator definition of PDFs** f.

Moreover, the non-abelian nature of QCD is responsible for the **asymptotic freedom** of the coupling (α_s) that ultimately enables us to compute the high-energy contribution $\hat{\sigma}$ in **perturbation theory**. In other words, QCD allows us to systematically improve on the Parton Model.

Factorisation in Drell-Yan

To understand how QCD "contains" the Parton Model, let us sketch the main steps of **leading-power** factorisation in Drell-Yan production:



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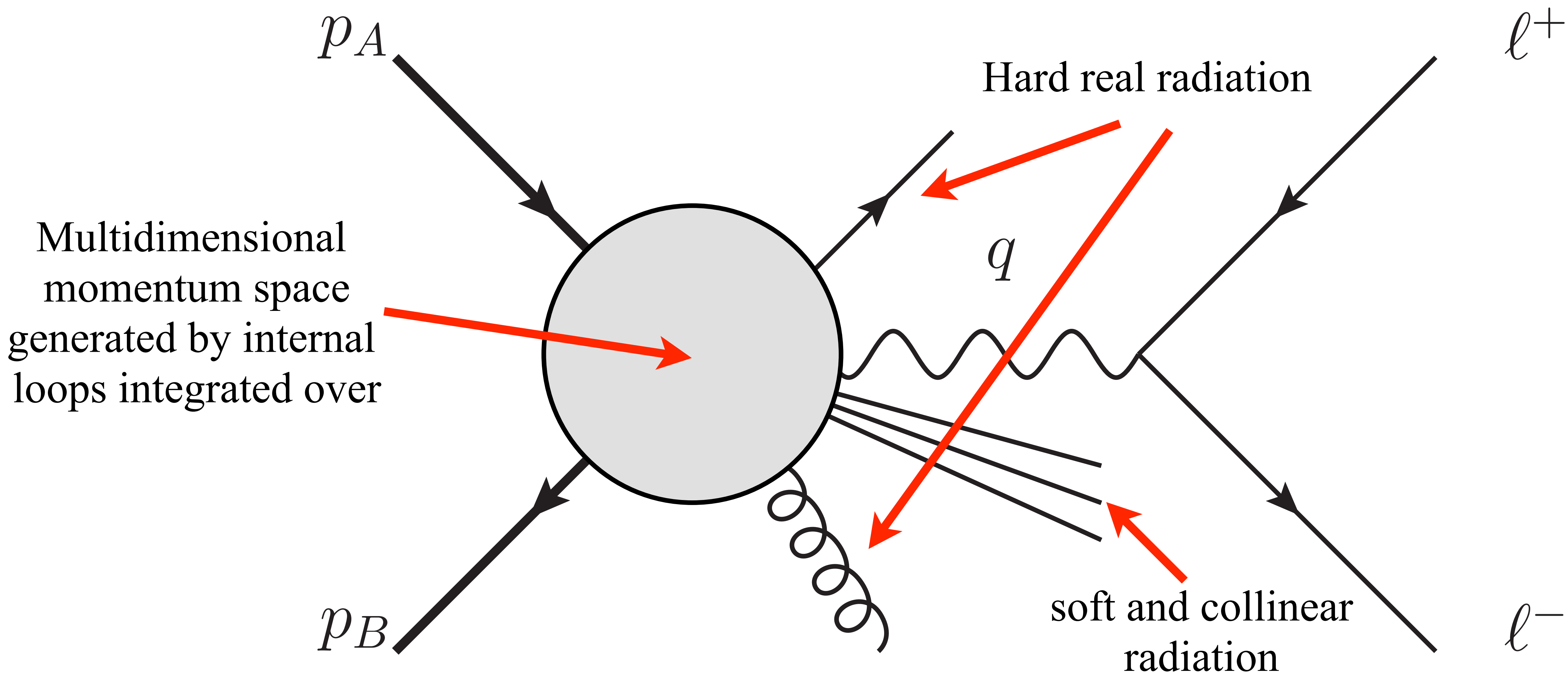
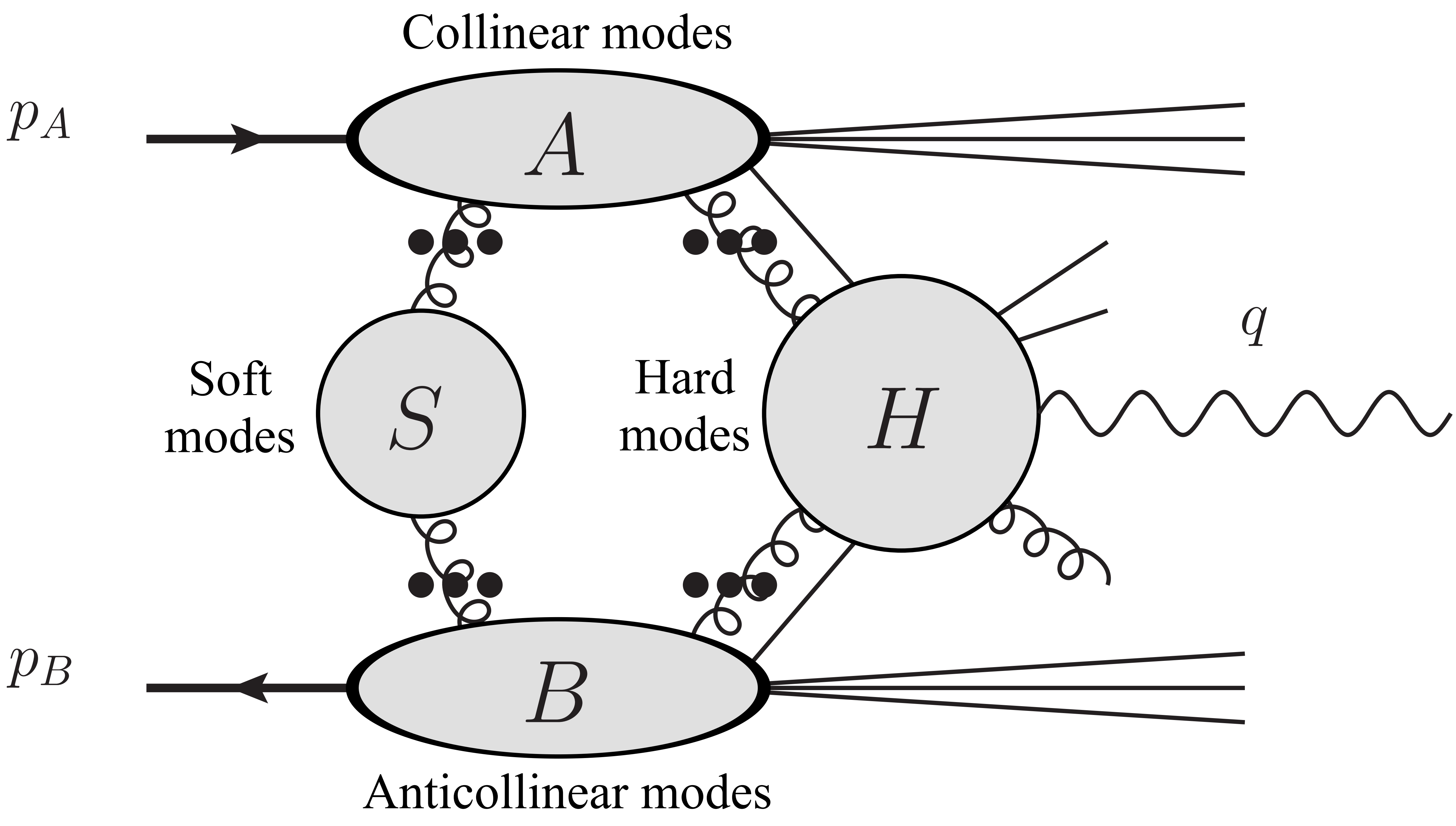


Factorisation in Drell-Yan

Apply Libby-Sterman power counting to the scattering amplitude to identify the large-Q **asymptote** [*Phys.Rev.D* 18 (1978) 4737].

The well-known result (in a **covariant gauge**) is:



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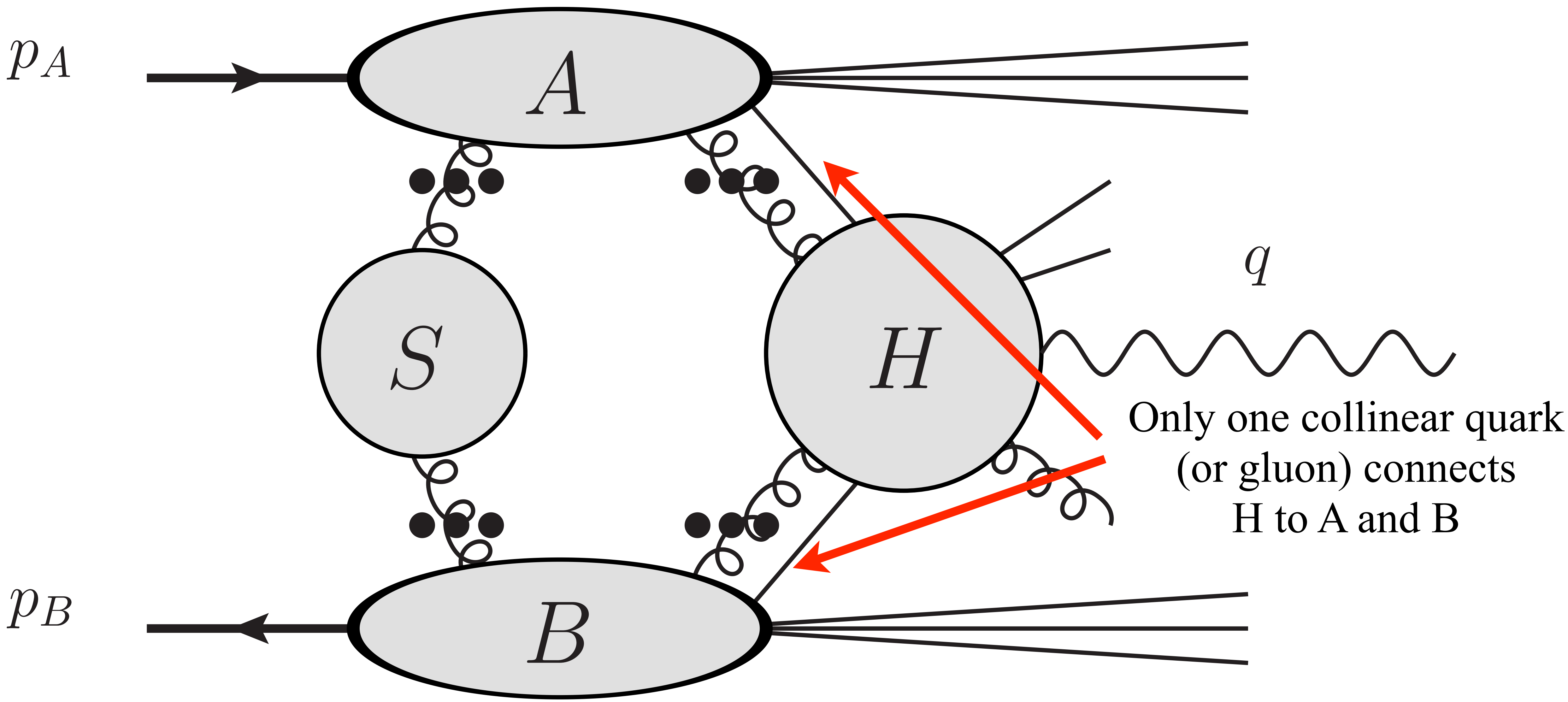

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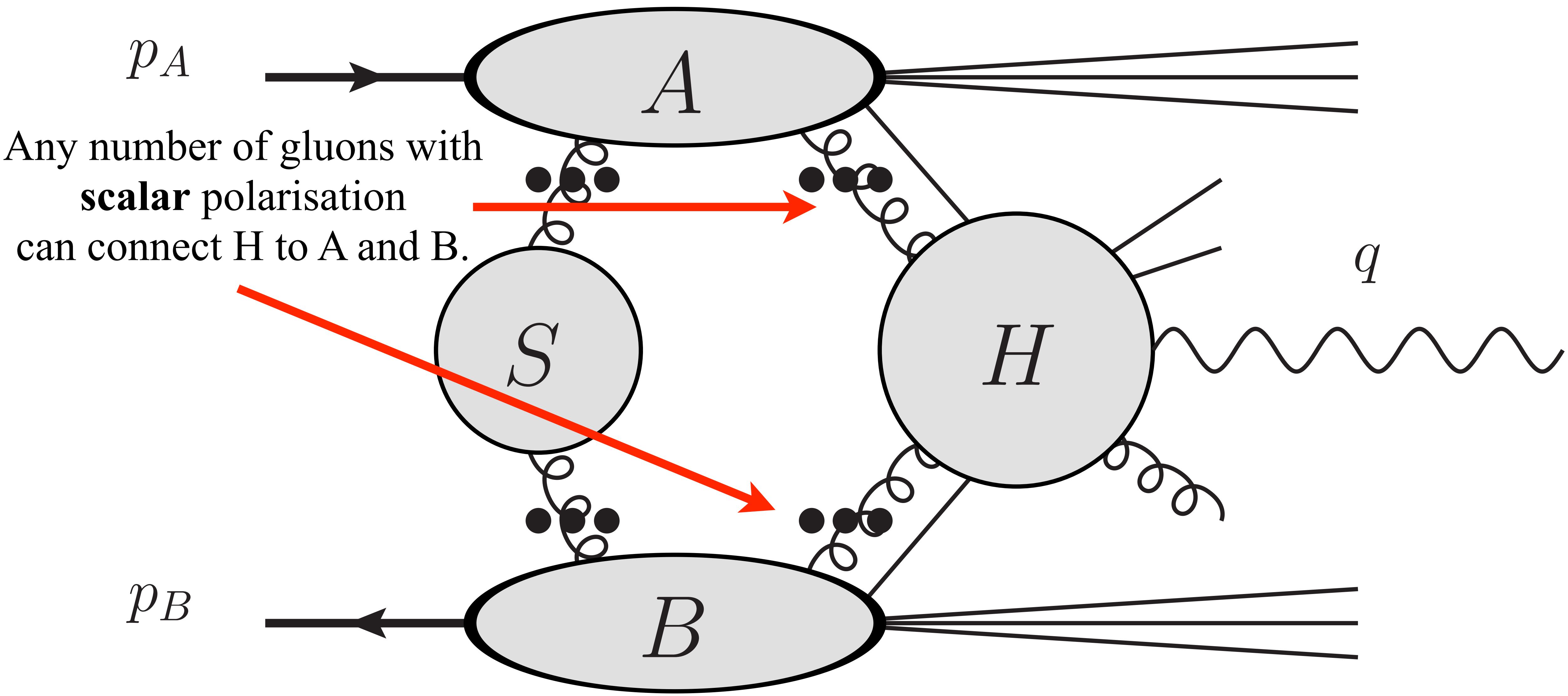

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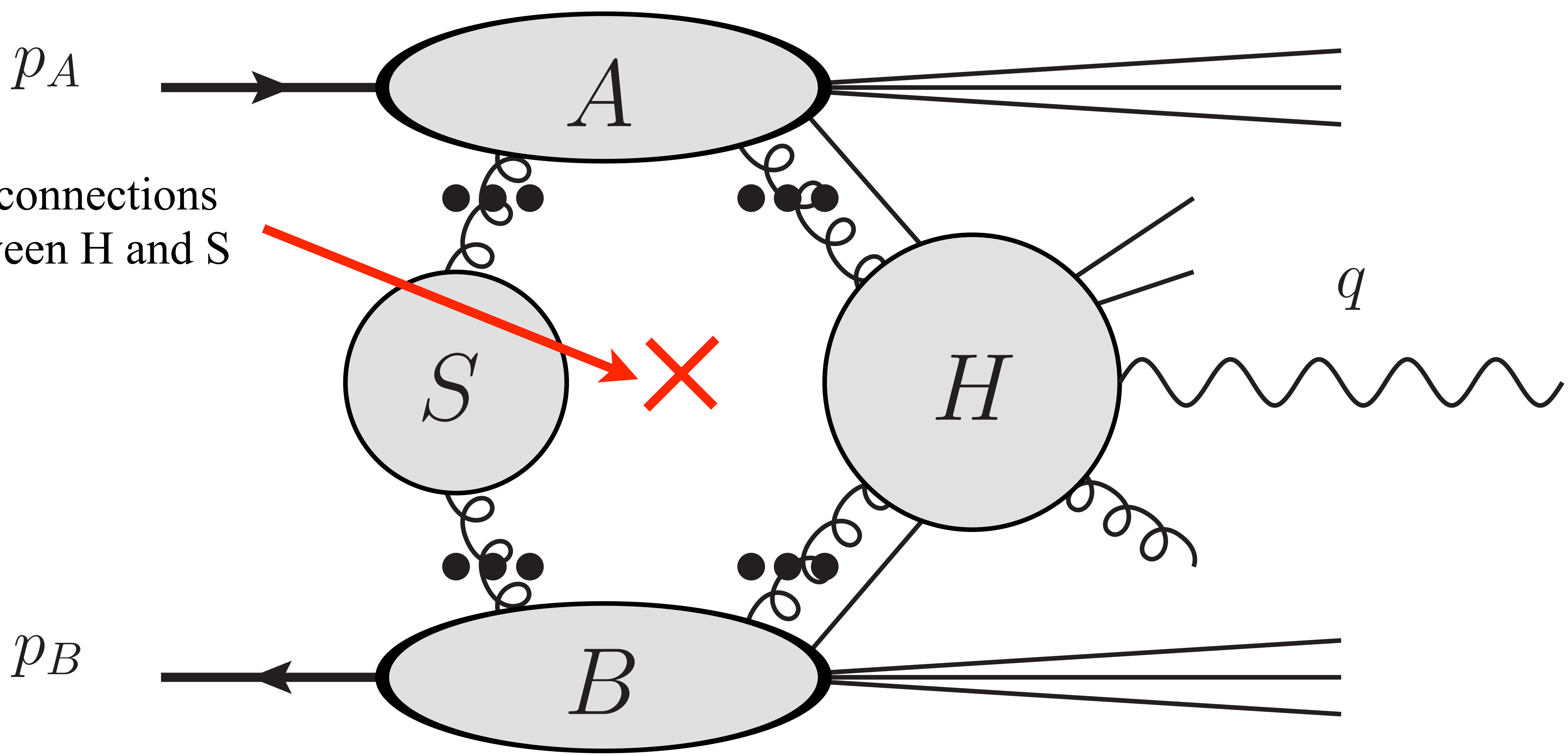
No connections between H and S



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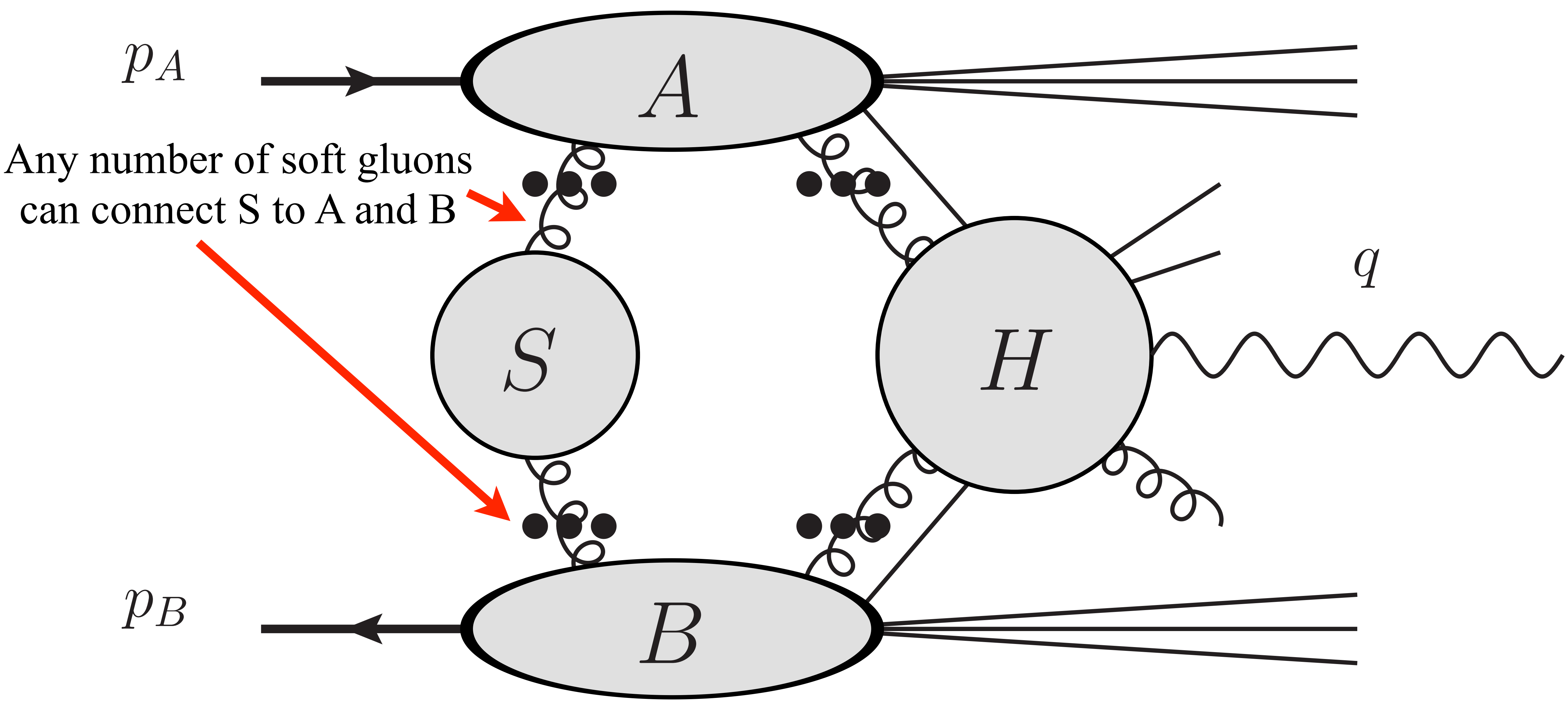

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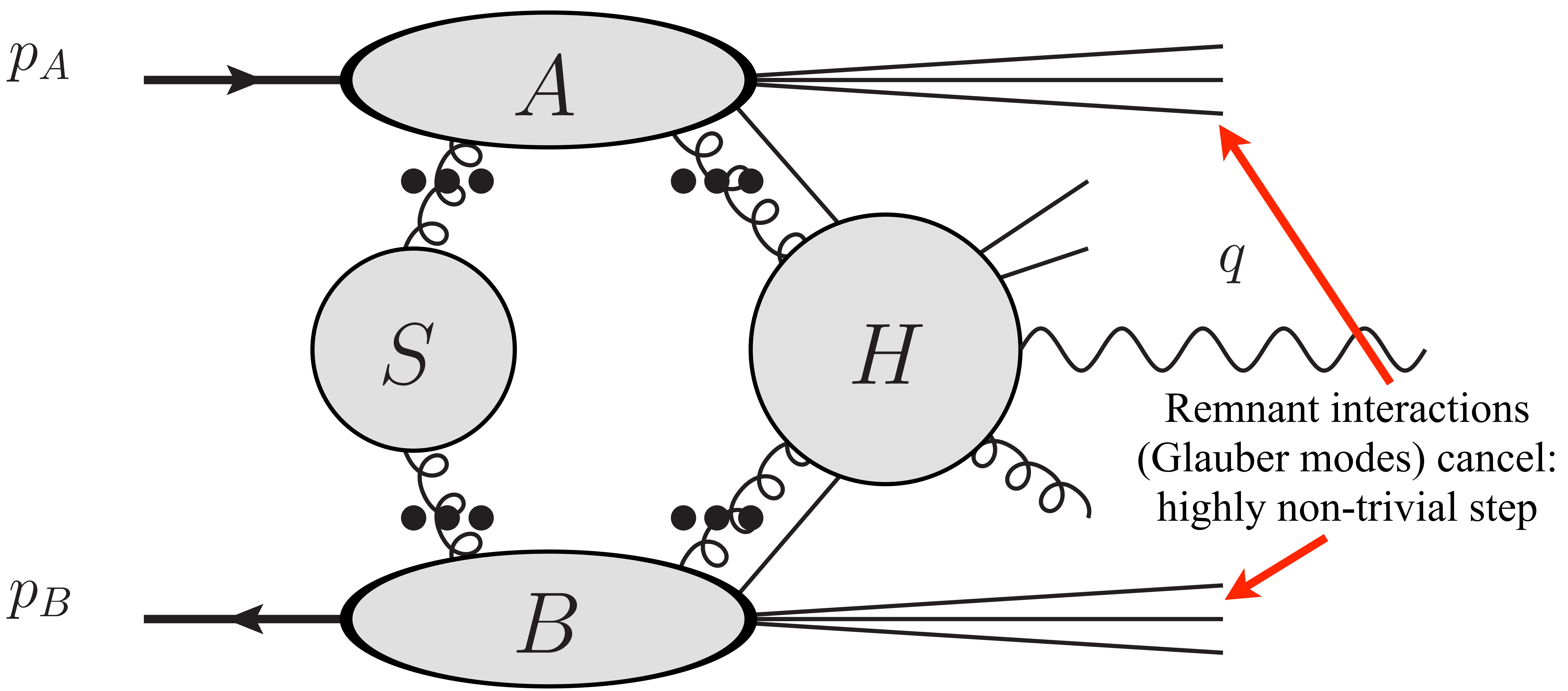


Factorisation in Drell-Yan

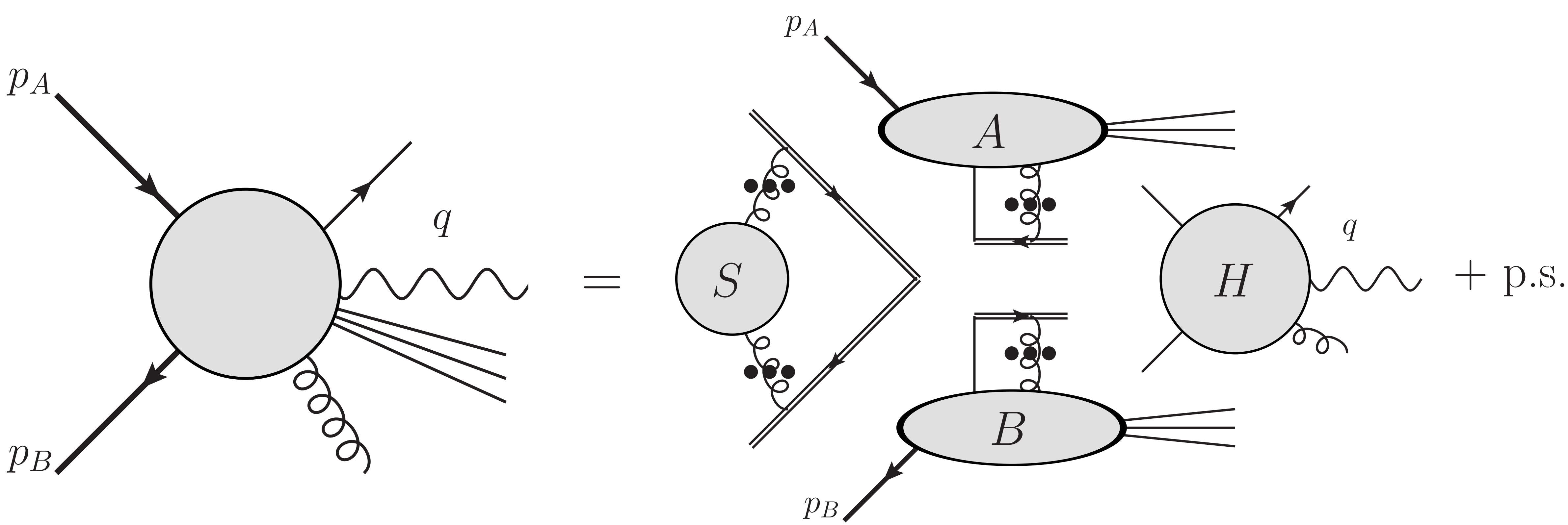
In brief, factorisation is finally achieved by:

- using Grammer-Yennie-like approximations for each soft/collinear gluon with momentum k to write, for example:

$$S^{\mu\cdots} g_{\mu\nu} A^{\nu\cdots} \sim (k \cdot S^{\cdots}) \frac{1}{n \cdot k} (n \cdot A^{\cdots})$$

- this allows one to use **Ward identities** and introduces **Wilson lines**.

A recursive application of this argument leads to **factorisation** of the amplitude:



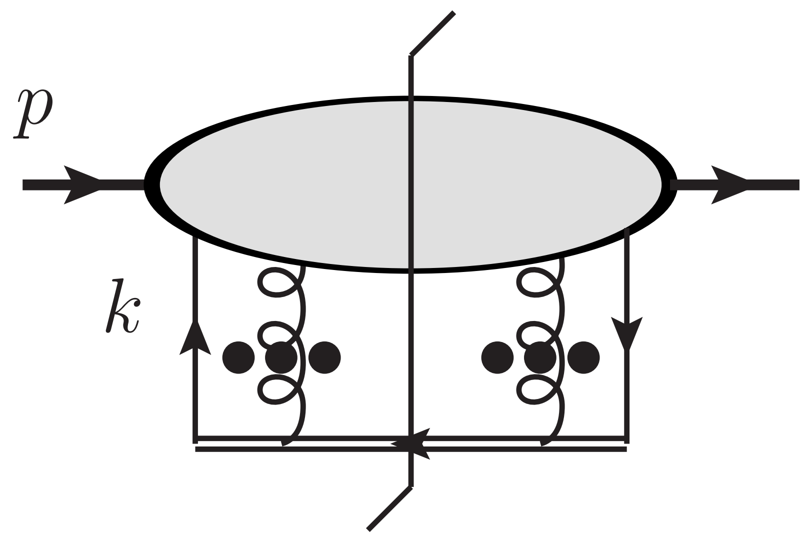
$$\boxed{\mathcal{A} \sim \boldsymbol{H} \cdot \boldsymbol{A} \cdot \boldsymbol{B} \cdot \boldsymbol{S}}$$

Factorisation in Drell-Yan

Upon squaring, summing over the unobserved radiation and integrating over loop momenta, factorisation leads to the operator definition of:

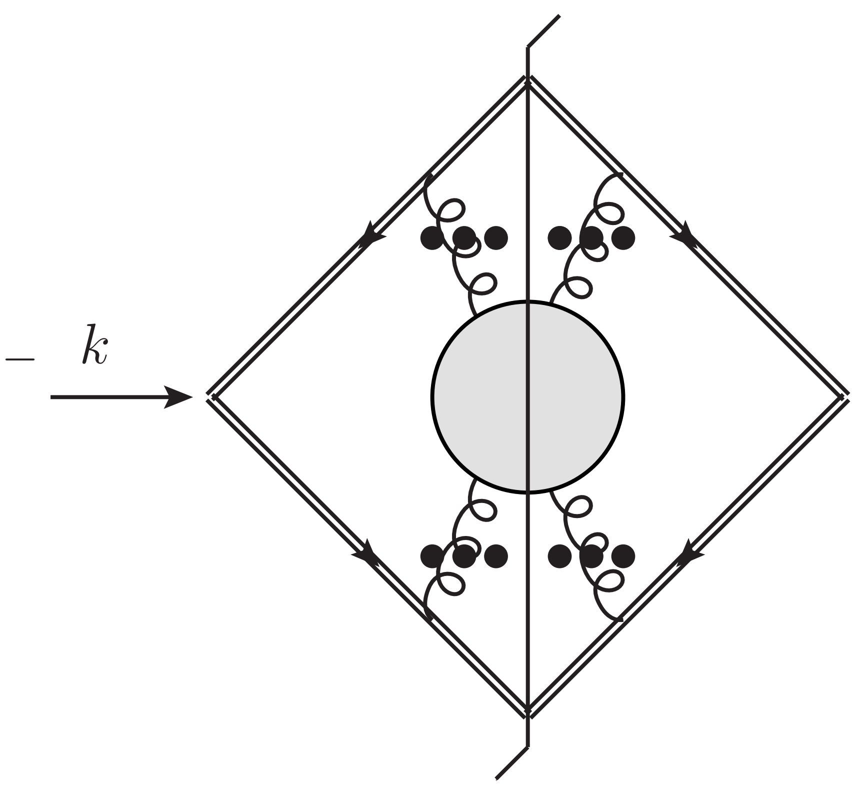
- gauge invariant **jet** (or **beam**) **function**:

$$f^{(0)}(x, \mathbf{k}_T) \propto \int dk^-$$



- and **soft function:**

$$S^{(0)}(\mathbf{k}_T) \propto \int dk^+ dk^-$$



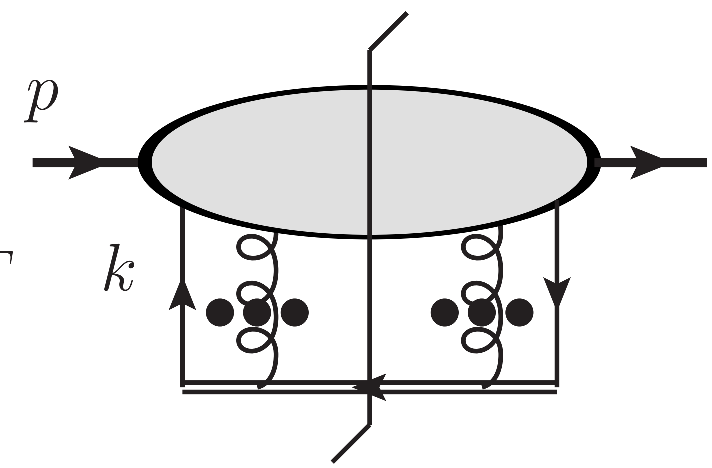
Factorisation in Drell-Yan

If $q_T \sim Q \gg \Lambda_{\text{QCD}}$ ($q_T \ll Q$ later), one can neglect the partonic traverse momenta k_{T}'s in the hard function H (**collinear factorisation**).

The integration over the k_{T}'s is then **short-circuited** over the jet and soft functions:

1. the contributions from the **soft function cancels**,
2. we are left with the k_{T}-integrated jet function that ultimately defines the **PDF**:

$$f^{(0)}(x) \propto \int dk^- d^{2-2\epsilon}\mathbf{k}_T$$



The gauge-invariant definition of the quark PDF in terms of the field $\psi(x)$ reads:

$$f^{(0)}(x) = \int \frac{dy^-}{2\pi} e^{-ixp^+y^-} \left\langle p \left| \overline{\psi}(0, y^-, \mathbf{0}_T) \frac{\gamma^+}{2} W[y^-, 0] \psi(0, 0, \mathbf{0}_T) \right| p \right\rangle$$

Gauge invariance guaranteed by the the **Wilson line** (or **gauge link**):

$$W[y^-, 0] = \mathcal{P} \exp \left[i g t_a \int_0^{y^-} dx^- A_a^+(0, x^-, \mathbf{0}_T) \right]$$

A similar operator definition exists for the gluon PDF.

Factorisation in Drell-Yan

After integrating over rapidity and transverse momentum of the lepton pair, the net result of QCD collinear factorisation for Drell-Yan is:

$$\frac{d\sigma^{\mathrm{QCD}}_{PP\to\ell\bar{\ell}+X}}{dQ^2} = \frac{4\pi\alpha_{\mathrm{em}}^2}{3sQ^2N_c}\sum_{i,j=q,\bar{q},g}\int_{Q^2/s}^{1}\frac{dt}{t}\hat{H}_{ij}(t,\alpha_s(Q))\int_{Q^2/st}^{1}\frac{dy}{y}f_i^{(0)}(y)f_j^{(0)}\left(\frac{Q^2}{syt}\right)$$

where $\hat{H}_{ij}$ is the so-called **partonic cross section** that, being only sensitive to large scales, is computable perturbatively and at N^{p}LO is:

$$\hat{H}_{ij}(t,\alpha_s(Q)) = e_q^2\delta_{iq}\delta_{j\bar{q}}\delta(1-t) + \sum_{n=1}^{p}\alpha_s^n(Q)\hat{H}_{ij}^{[n]}(t)$$

Truncating to **leading order** one finds back the **Parton Model** result!

Unfortunately, this is not quite as "simple" as that if one wants to go **beyond LO**.

Assuming to work in **massless** QCD in 4 dimensions (a common configuration):

1. $\hat{H}_{ij}$ beyond LO is affected by **collinear divergences** (k_{T} of the radiation $\to 0$) ,

2. the operator definition of **PDF** $f_i^{(0)}$ **evaluates to zero**: UV and IR divergences cancel.

Factorisation in Drell-Yan

At one loop and in $4-2\epsilon$ dimensions the situation can be exemplified as follows:

$$\frac{d\sigma^{[1]}}{dQ^2} \propto \alpha_s \left[\underbrace{\cancel{\frac{1}{\epsilon_{\rm IR}}}}_{H} + \underbrace{\left(\frac{1}{\epsilon_{\rm UV}} - \cancel{\frac{1}{\epsilon_{\rm IR}}} \right)}_{f^{(0)} f^{(0)}} + \text{finite} \right]$$

$\frac{1}{\epsilon_{\rm IR}}$ pole for $k_T \to 0$

$\frac{1}{\epsilon_{\rm UV}}$ pole for $k_T \to \infty$

Therefore the IR (collinear) divergences **cancel** between H and the PDF leaving only the **UV divergence of the PDFs**.

This UV divergence can be removed defining appropriate **renormalisation constants**:

$$f_i^{(0)}(x) = \sum_j \boxed{Z_{ij}(x, \alpha_s(\mu), \epsilon)} \otimes f_j(x, \mu)$$

The $\overline{\rm MS}$ renormalisation constants contain the UV divergences of PDFs and are computable order by order in perturbation theory:

$$Z_{ij}(x, \alpha_s(\mu), \epsilon) = \delta(1-x) + \sum_{n=1}^{\infty} \alpha_s^n(\mu) \sum_{k=1}^{n} \frac{Z_{ij}^{[n,k]}(x)}{\epsilon^k}$$

Factorisation in Drell-Yan

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Renormalised PDF

Factorisation in Drell-Yan

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$$f_i^{(0)}(x) = \sum_j Z_{ij}(x, \alpha_s(\mu), \epsilon) \otimes f_j(x, \mu)$$

The scale μ is introduced as usual to keep the coupling dimensionless. It is often referred to **factorisation** scale but it is in fact a standard renormalisation scale

Factorisation in Drell-Yan

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This UV divergence can be removed defining appropriate **renormalisation constants**:

$$f_i^{(0)}(x) = \sum_j Z_{ij}(x, \alpha_s(\mu), \epsilon) \otimes f_j(x, \mu)$$

Finally, exploiting the **independence** of $f_i^{(0)}$ from μ, one can derive a RGE:

Can you prove it?

$$\frac{df_i(x,\mu)}{d\ln\mu^2} = \sum_j P_{ij}(x, \alpha_s(\mu)) \otimes f_j(x,\mu) \quad \text{with} \quad P_{ij} = -\sum_k Z_{ik}^{-1} \otimes \frac{dZ_{kj}}{d\ln\mu^2}$$

the famous **DGLAP equation**. The evolution kernels P_{ij} are finite perturbative objects:

$$P_{ij}(x, \alpha_s(\mu)) = \sum_{n=0}^{\infty} \alpha_s^{n+1}(Q) P_{ij}^{[n]}(x)$$

Factorisation in Drell-Yan

After the cancellation of the IR divergencies and the renormalisation of the PDFs, the factorised Drell-Yan cross section in QCD is finite and reads:

$$\frac{d\sigma^{\rm QCD}_{PP\to\ell\bar{\ell}+X}}{dQ^2} = \frac{4\pi\alpha_{\rm em}^2}{3sQ^2N_c}\sum_{i,j=q,\overline{q},g}\int_{Q^2/s}^{1}\frac{dt}{t}H_{ij}\left(t,\alpha_s(Q),\frac{\mu}{Q}\right)\int_{Q^2/st}^{1}\frac{dy}{y}\boxed{f_i(y,\mu)f_j\left(\frac{Q^2}{syt},\mu\right)}$$

Renormalised PDFs

Factorisation in Drell-Yan

After the cancellation of the IR divergences and the renormalisation of the PDFs, the factorised Drell-Yan cross section in QCD is finite and reads:

$$\frac{d\sigma^{\mathrm{QCD}}_{PP\to\ell\bar{\ell}+X}}{dQ^2} = \frac{4\pi\alpha^2_{\mathrm{em}}}{3sQ^2N_c}\sum_{i,j=q,\bar{q},g}\int_{Q^2/s}^{1}\frac{dt}{t}\boxed{H_{ij}\left(t,\alpha_s(Q),\frac{\mu}{Q}\right)}\int_{Q^2/st}^{1}\frac{dy}{y}f_i(y,\mu)f_j\left(\frac{Q^2}{syt},\mu\right)$$

Subtracted partonic cross sections

$$H_{ij}\left(t,\alpha_s(Q),\frac{\mu}{Q}\right) = \sum_{n=0}^{\infty}\alpha_s^n(Q)\sum_{k=0}^{n}H_{ij}^{[n,k]}(t)\ln^k\left(\frac{\mu}{Q}\right)$$

Despite the scale μ is in principle **arbitrary**, in order for the series in α_s to be convergent, the truncation to fixed order of H_{ij} requires all the coefficients of the expansion be of the **same order**. A necessary condition for this to happen is:

$$\ln\left(\frac{\mu}{Q}\right) \lesssim 1 \quad \Rightarrow \quad \boxed{\mu \simeq Q}$$

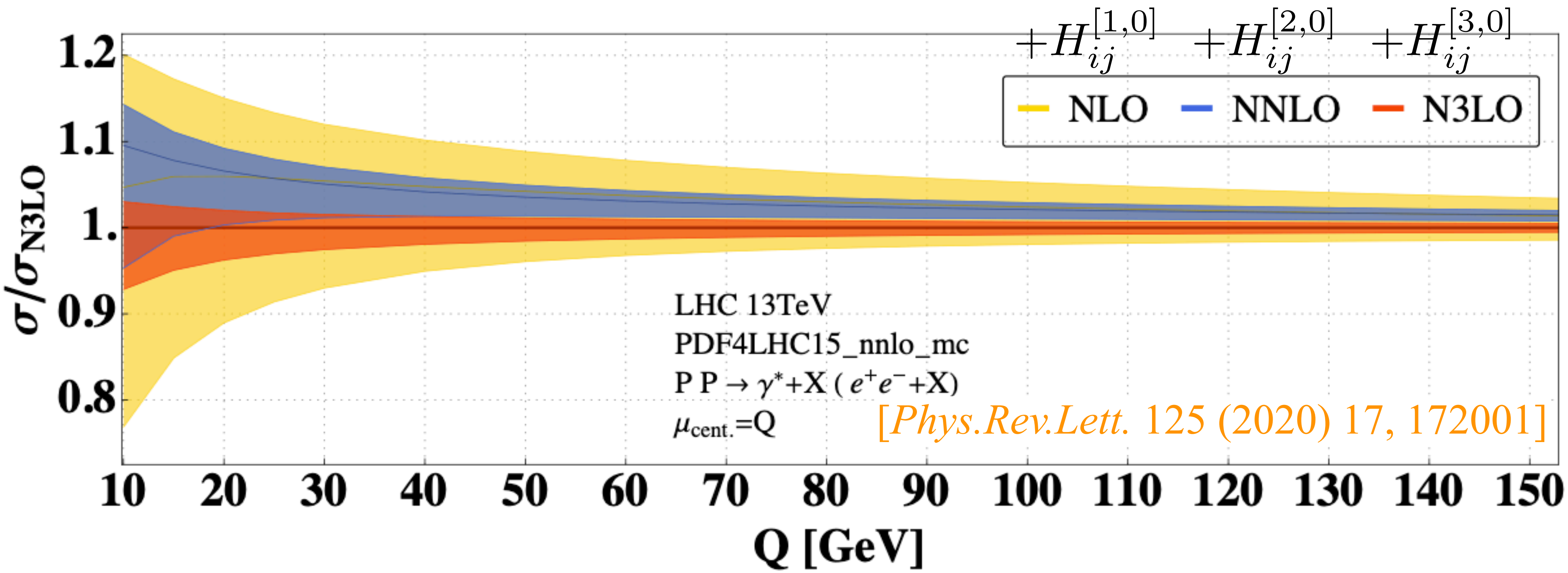
Natural factorisation-scale choice.
Variations by moderate factors give an estimate of higher-order corrections.
Can you show why?

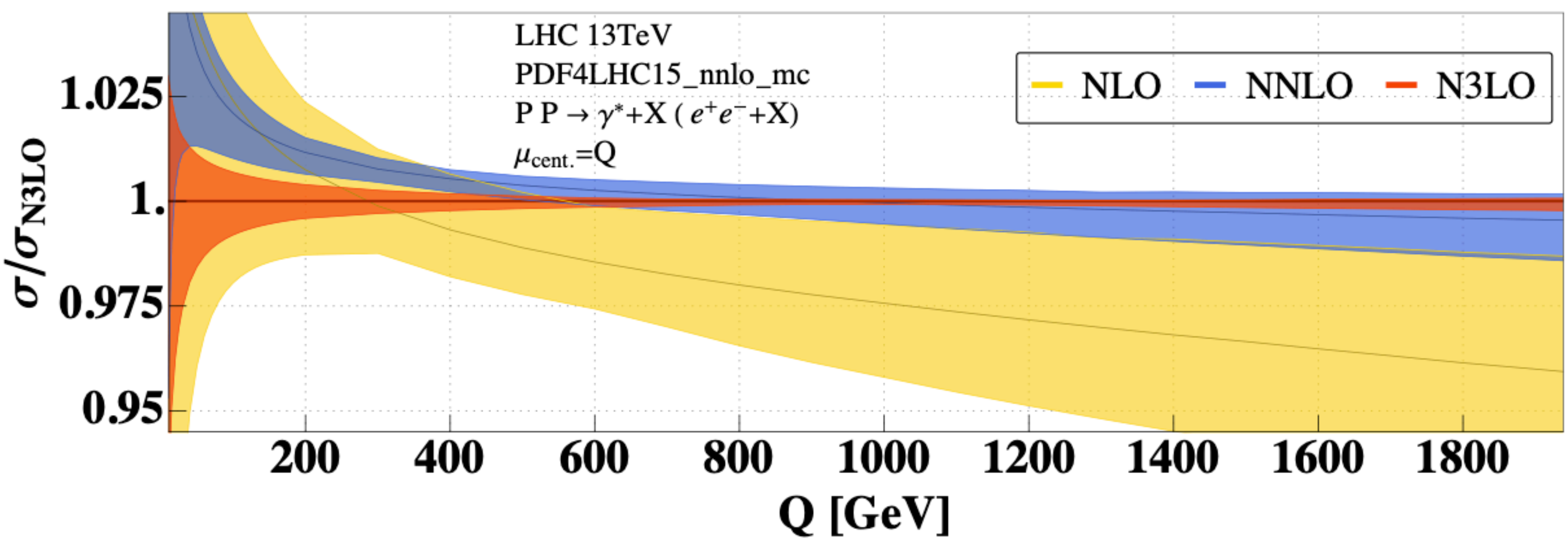
While the x dependence of PDFs is not perturbatively accessible, their μ dependence is governed by the DGLAP equation.

Once PDFs at some reference scale μ_0 are known, that can be **evolved** to any other scale.

Higher-order corrections in Drell-Yan

Here is what happens when perturbative corrections are included into the partonic cross section H_{ij}.





Higher-order corrections in Drell-Yan

The fully inclusive cross section $d\sigma/dQ^2$ is theoretically interesting but would it be possible to go **more differential**?

Sure, for example collinear factorisation works just as well for the **rapidity distribution** y of the lepton pair:

$$y = \frac{1}{2}\ln\left(\frac{E+q_z}{E-q_z}\right) = \frac{1}{2}\ln\left(\frac{q^+}{q^-}\right)$$

$$\frac{d\sigma^{\rm QCD}_{PP\to\ell\bar{\ell}+X}}{dQ^2dy} = \frac{4\pi\alpha_{\rm em}^2}{3sQ^2N_c}\sum_{i,j=q,\bar{q},g}\int_{Q/\sqrt{s}e^y}^1\frac{dy_1}{y_1}\int_{Q/\sqrt{s}e^{-y}}^1\frac{dy_2}{y_2}G_{ij}\left(\frac{x_1}{y_1},\frac{x_2}{y_2},\alpha_s(Q),\frac{\mu}{Q}\right)f_i(y_1,\mu)f_j(y_2,\mu)$$

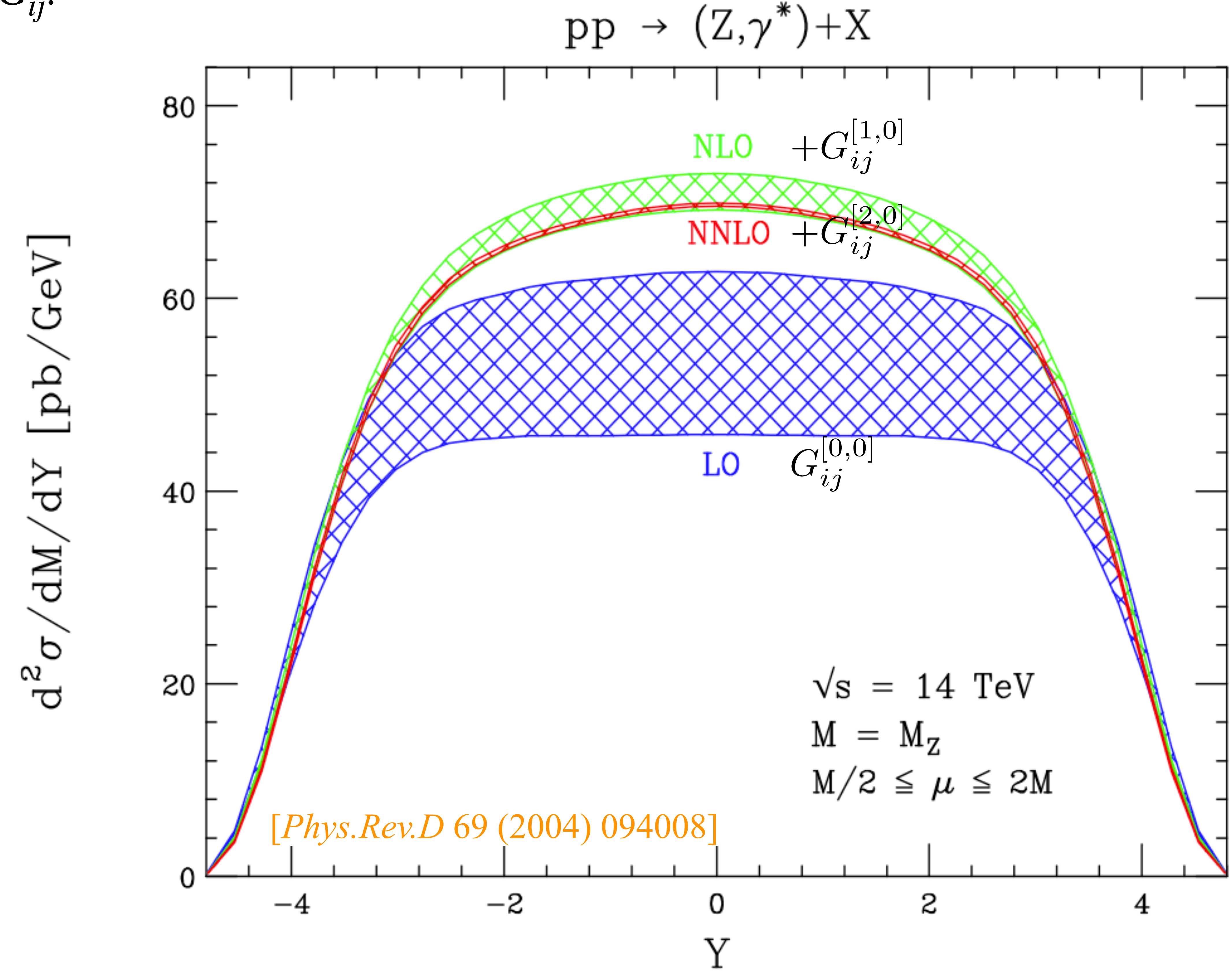
Also here the **subtracted** partonic cross sections admit a perturbative expansion:

$$G_{ij}\left(y_1,y_2,\alpha_s(Q),\frac{\mu}{Q}\right) = e_q^2\delta_{iq}\delta_{j\bar{q}}\delta(1-y_1)\delta(1-y_2)+\sum_{n=1}^{\infty}\alpha_s^n(Q)\sum_{k=0}^{n}G_{ij}^{[n,k]}(y_1,y_2)\ln^k\left(\frac{\mu}{Q}\right)$$

The **cancellation of IR divergences** between partonic cross sections and PDFs takes place exactly like in the y-inclusive case and logs of μ/μ_0 are resummed via DGLAP.

Higher-order corrections in Drell-Yan

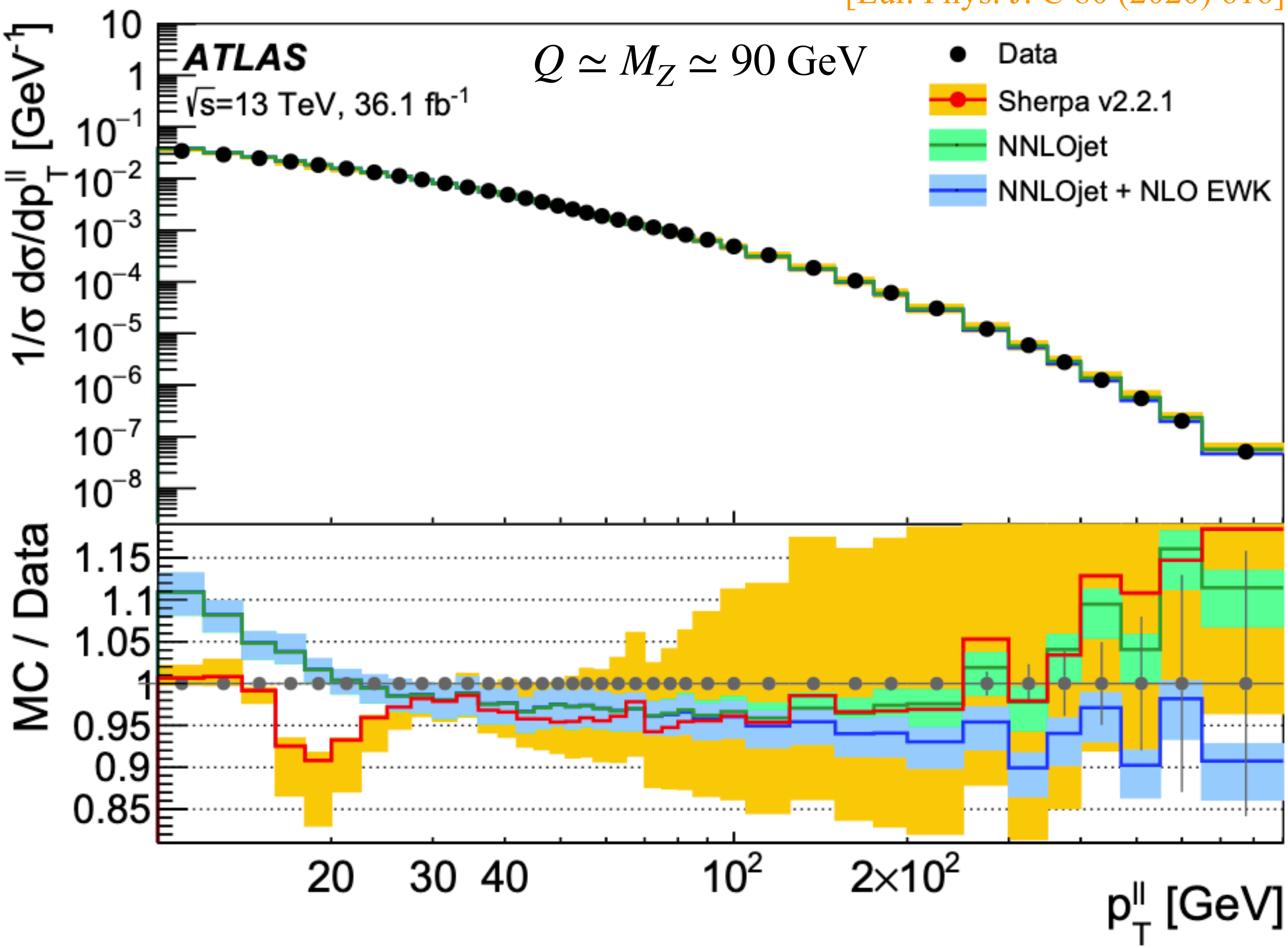
Here is what happens when perturbative corrections are included into the partonic cross section G_{ij}.



q_T distribution at fixed order

So far so good! Can we try to be even bolder and use **collinear factorisation** for the differential cross section $d\sigma/dQ^2dydq_T$? Why not!

[Eur. Phys. J. C 80 (2020) 616]



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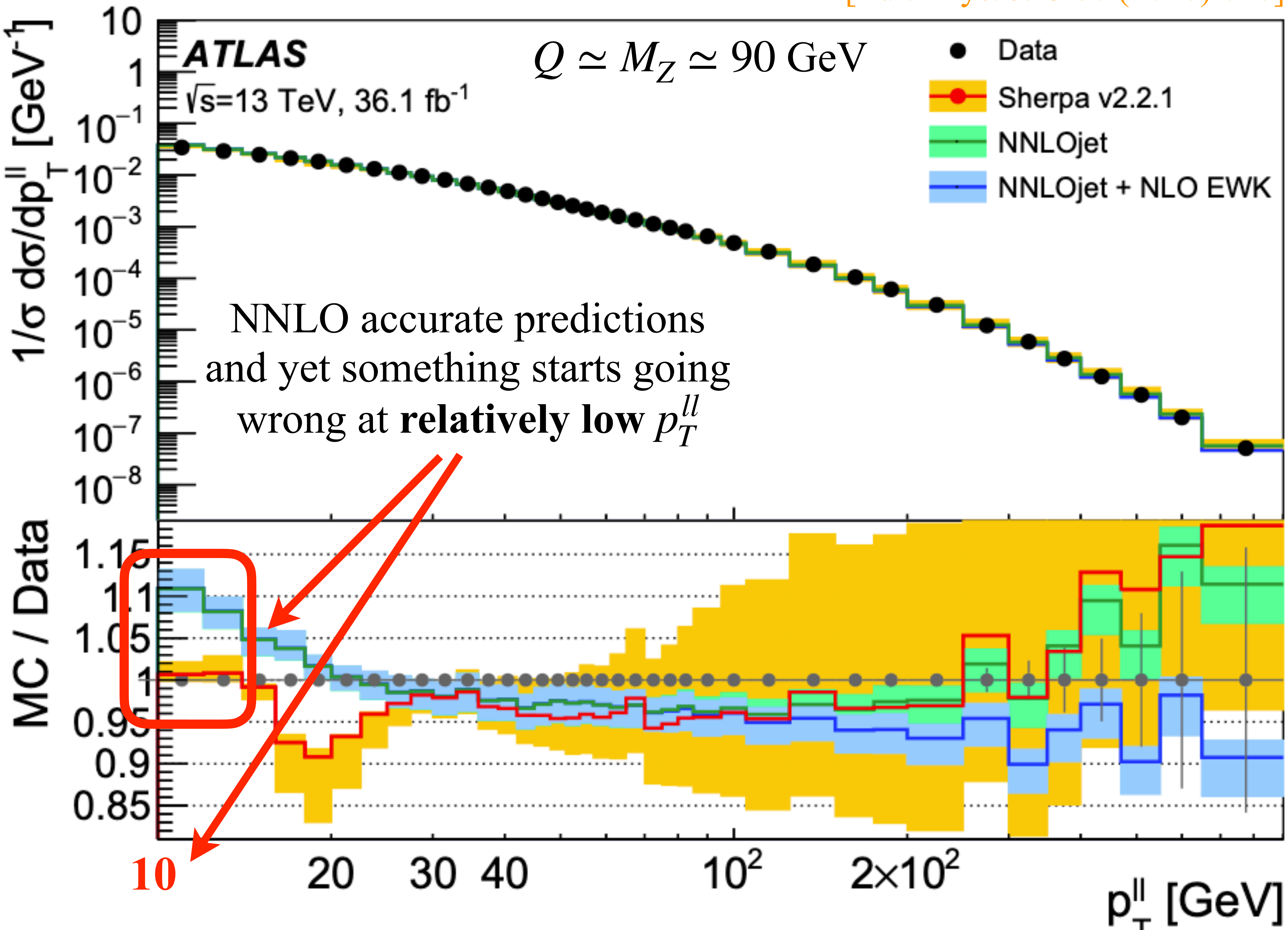

q_T distribution at fixed order

We derived collinear factorisation by requiring $q_T \sim Q \gg \Lambda_{\rm QCD}$. No wonder this formalism breaks down if $q_T \ll Q$. An analysis of the cross section reveals that for $q_T \neq 0$ the N^{p}LO collinear calculation has the following structure:

$$\frac{d\sigma^{\rm QCD}_{PP\to \ell\bar{\ell}+X}}{dQ^2 dy dq_T^2} \underset{q_T \ll Q}{\propto} \sum_{n=1}^{p+1} \alpha_s^n(Q) \sum_{k=0}^{2n-1} I^{[n,k]} \frac{1}{q_T^2} \ln^k\left(\frac{Q}{q_T}\right) + \mathcal{O}\left(\frac{q_T^2}{Q^2}\right)$$

q_T distribution at fixed order

We derived collinear factorisation by requiring $q_T \sim Q \gg \Lambda_{\text{QCD}}$, no wonder it breaks down if $q_T \ll Q$. Indeed, for $q_T \neq 0$ the N^{p}LO calculation has the following structure:

$$\frac{d\sigma^{\text{QCD}}_{PP\to\ell\bar{\ell}+X}}{dQ^2 dy dq_T^2} \underset{q_T \ll Q}{\propto} \sum_{n=1}^{p+1} \alpha_s^n(Q) \sum_{k=0}^{2n-1} I^{[n,k]} \frac{1}{q_T^2} \ln^k\left(\frac{Q}{q_T}\right) + \mathcal{O}\left(\frac{q_T^2}{Q^2}\right)$$

No $\mathcal{O}(1)$ contribution.
It only appears at $q_T = 0$,
i.e. it is proportional to $\delta(q_T)$.

There can be terms proportional
to $\delta(q_T)$ also beyond $\mathcal{O}(1)$?
If so, where do they come from?

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Non-integrable singularities at $q_T = 0$.

q_T distribution at fixed order

We derived collinear factorisation by requiring $q_T \sim Q \gg \Lambda_{\rm QCD}$, no wonder it breaks down if $q_T \ll Q$. Indeed, for $q_T \neq 0$ the N^{p}LO calculation has the following structure:

$$\frac{d\sigma^{\rm QCD}_{PP\to\ell\bar{\ell}+X}}{dQ^2 dy dq_T^2} \underset{q_T \ll Q}{\propto} \sum_{n=1}^{p+1} \alpha_s^n(Q) \sum_{k=0}^{\boxed{2n-1}} I^{[n,k]} \frac{1}{q_T^2} \ln^k\left(\frac{Q}{q_T}\right) + \mathcal{O}\left(\frac{q_T^2}{Q^2}\right)$$

Up to two logs per power of α_s
(double logs)

q_T distribution at fixed order

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Power corrections

q_T distribution at fixed order

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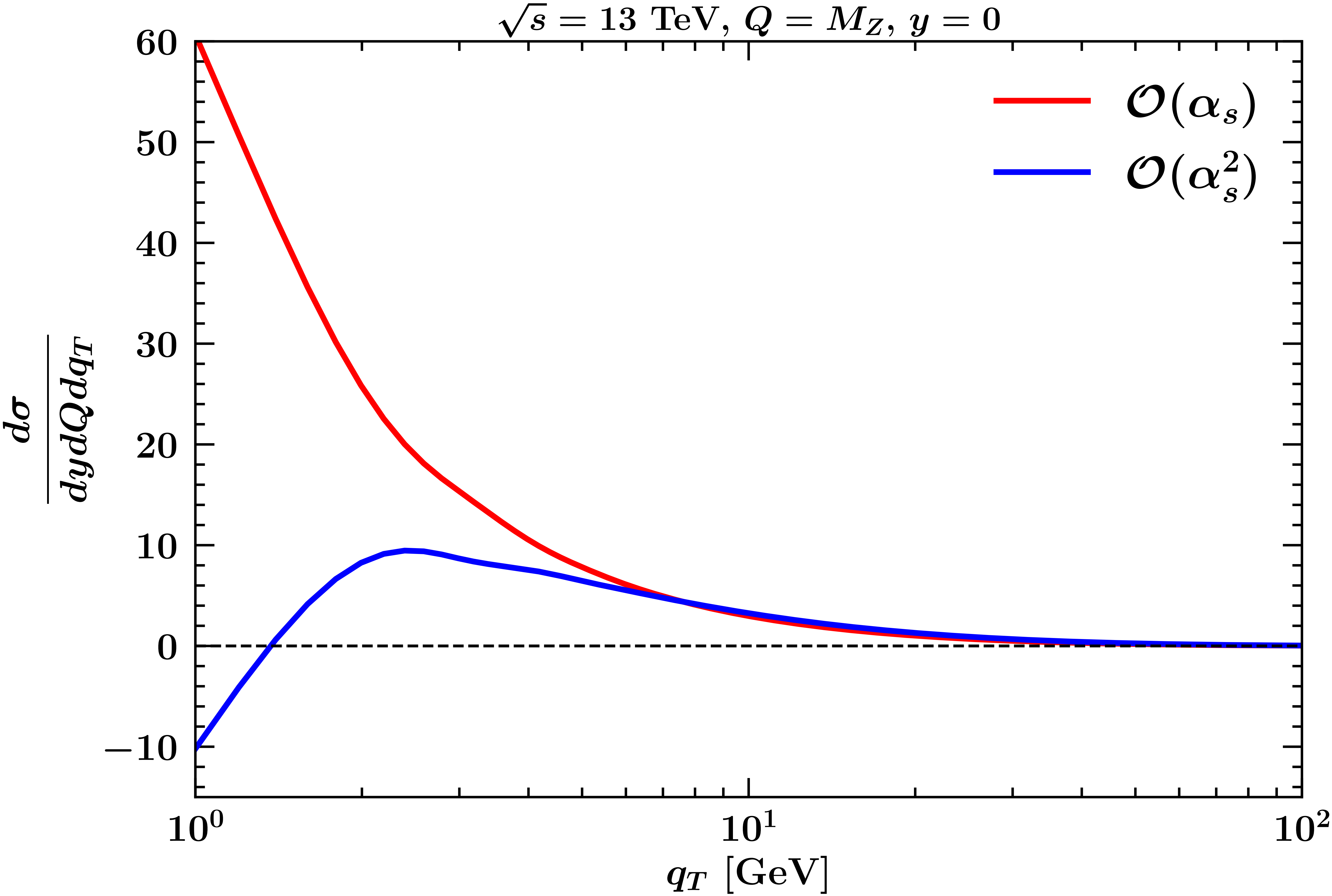
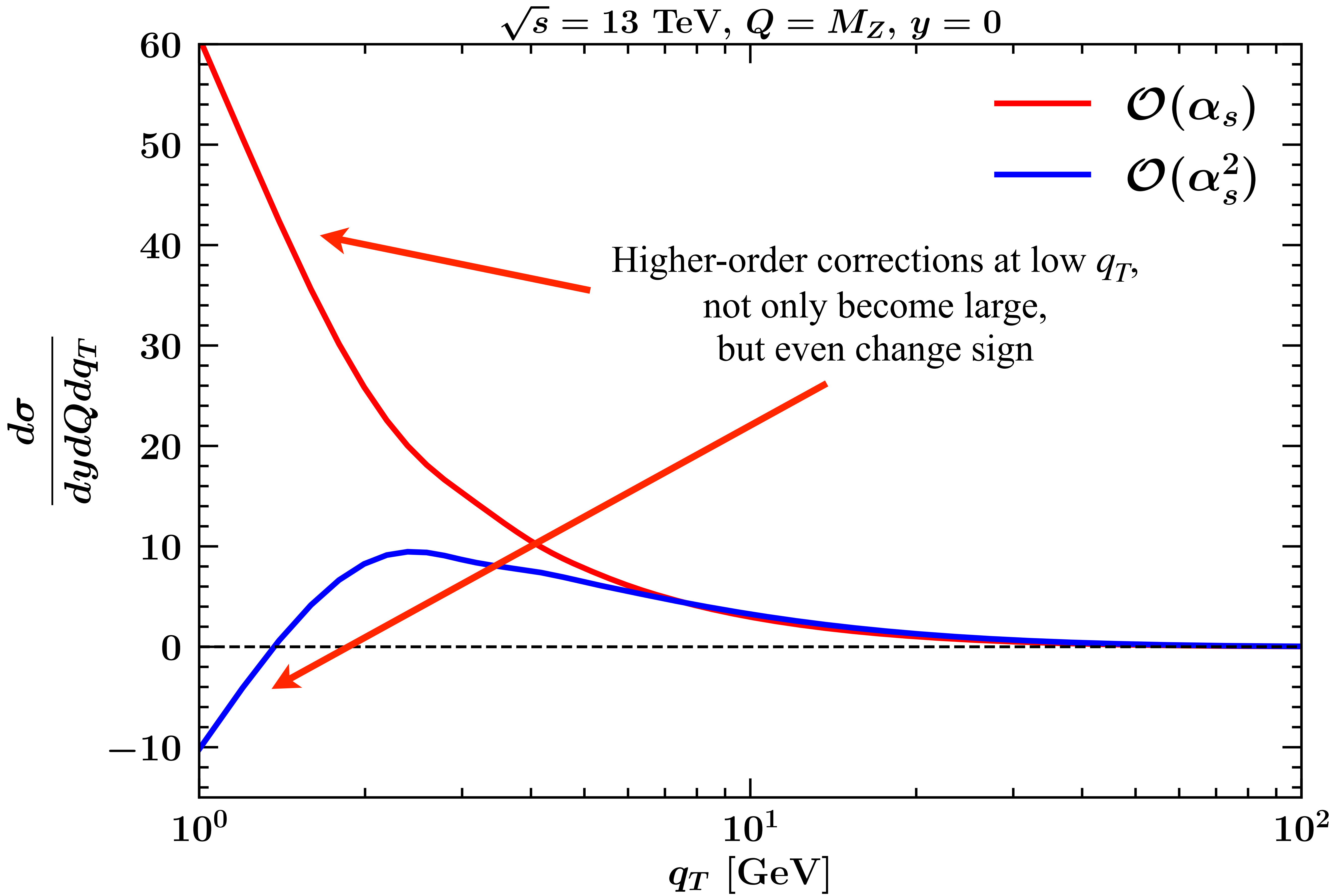
For $q_T \ll Q$ any truncation of this series is inaccurate because the **logs enhance higher-order corrections** and spoils the convergence. In fact, any truncation diverges at $q_T = 0$.

q_T distribution at fixed order



q_T distribution at fixed order



q_T distribution at fixed order

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For $q_T \ll Q$ any truncation of this series is inaccurate because the **logs enhance higher-order corrections** and spoils the convergence. In fact, any truncation diverges at $q_T = 0$.

In addition, this cross section is not even integrable at $q_T = 0$, this means that:

$$\int_0^{k_T^2} dq_T^2 \frac{d\sigma^{\rm QCD}_{PP\to\ell\bar{\ell}+X}}{dQ^2 dy dq_T^2} = \infty \neq \frac{d\sigma^{\rm QCD}_{PP\to\ell\bar{\ell}+X}}{dQ^2 dy}$$

Something is missing (can you tell what?).

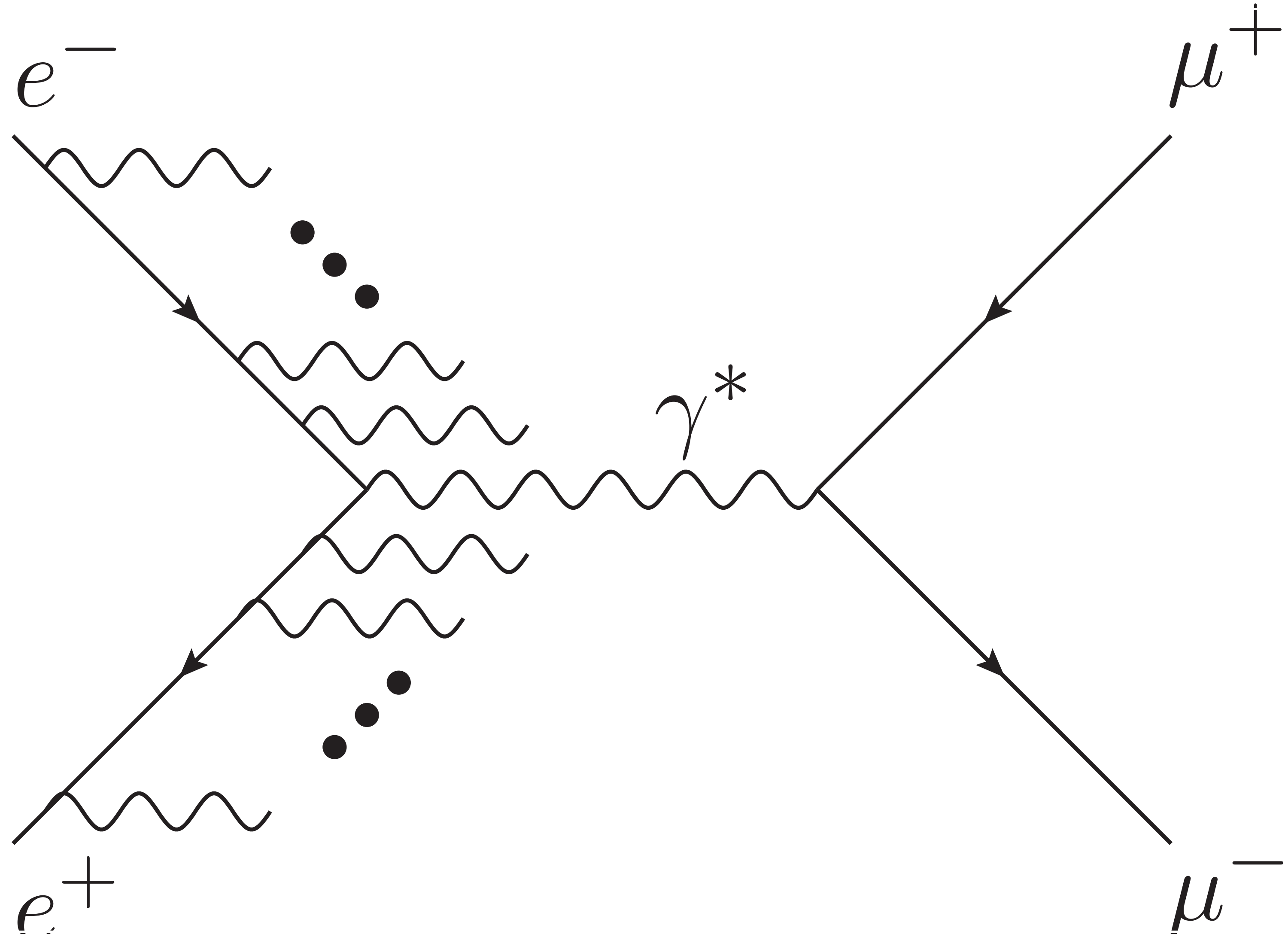
1. Do we understand the origin of these logs?

2. Can we resum them to all orders in α_s?

Resummation of the q_T distribution

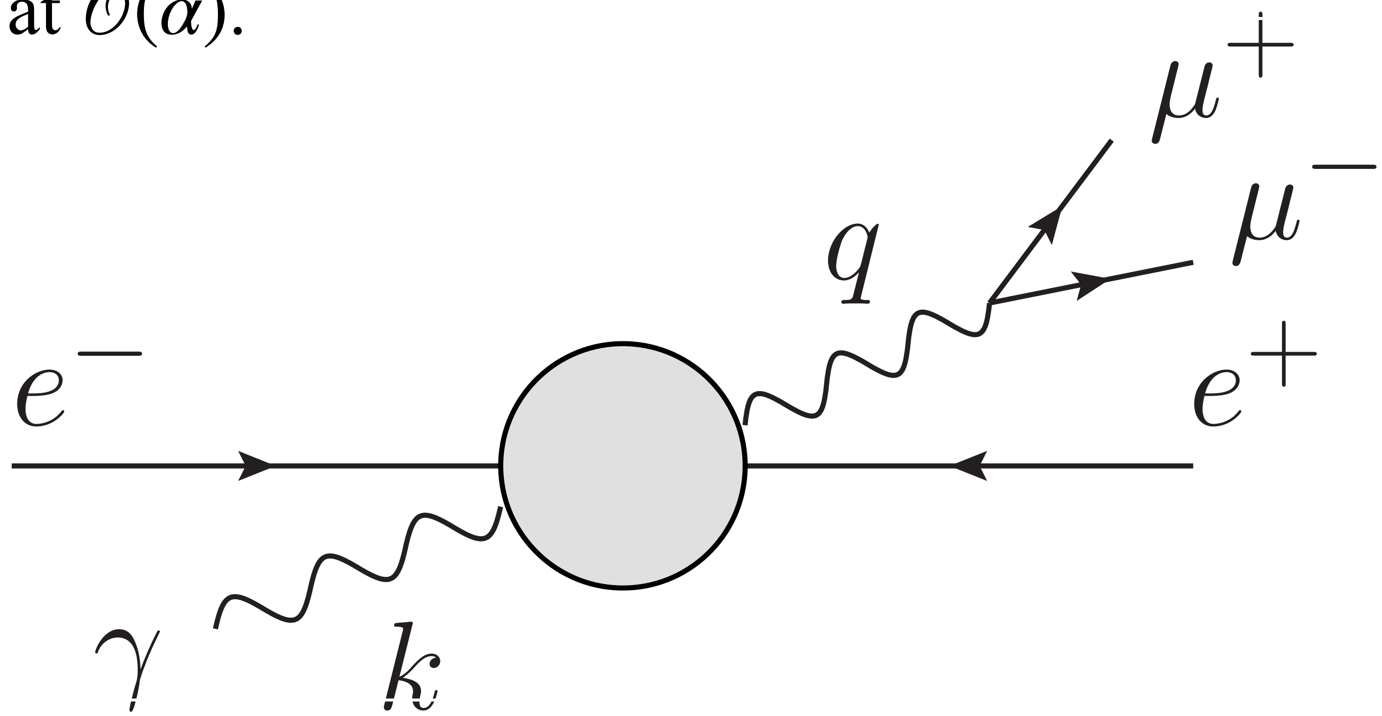
A classical reference that discusses the origin of the small-q_T logs is [*Nucl.Phys.B* 154 (1979) 427-440] by G. Parisi and R. Petronzio.

In order to avoid the complications of QCD being non-abelian, they considered the q_T distribution of a muon pair produced in the purely QED process $e^+e^- \to \mu^+\mu^- + n\gamma$ via photon exchange with n photons emitted by e^+e^- (not $\mu^+\mu^-$).



Resummation of the q_T distribution

In order to have a q_T of the muon pair different from zero, we need to emit **at least** one additional photon against which the muon pair recoils, *i.e.* $e^+e^- \to \mu^+\mu^- + \gamma$, hence the cross sections starts at $\mathcal{O}(\alpha)$.



Since $k_T = q_T$, if $q_T \ll Q$ (and $Q \lesssim \sqrt{s}$ what if $Q \ll \sqrt{s}$?) the additional photon with momentum k has to be **soft** ($k^\mu \ll Q$).

In this case the q_T-differential cross section at small q_T but $q_T \neq 0$ evaluates to:

$$\left.\frac{d\sigma}{dq_T^2}\right|_{q_T \neq 0} = \sigma_0(Q)\frac{\alpha}{\pi}\left(\frac{\ln(Q^2/q_T^2)}{q_T^2} + \text{less singular terms}\right)$$

where σ_0 is the **total cross section** for $e^+e^- \to \mu^+\mu^- + X$.

Resummation of the q_T distribution

It is easy to extend the result to $q_T = 0$ by requiring:

$$\int_0^{Q^2} dq_T^2 \frac{d\sigma}{dq_T^2} = \sigma_0(Q) \quad \Rightarrow \quad \frac{d\sigma}{dq_T^2} = \sigma_0(Q) \left[\delta(q_T^2) + \alpha \underbrace{\left(\frac{\ln(Q^2/q_T^2)}{\pi q_T^2} \right)_+}_{\nu(k_T)} \right] + \text{subleading}$$

Can you prove that this integrates to σ_0 up to subleading terms?

where:

$$\int_0^{Q^2} dq_T^2 f(q_T)[H(q_T)]_+ = \int_0^{Q^2} dq_T^2 H(q_T)(f(q_T) - f(1))$$

Now let us consider the **cumulant** cross section:

$$\Sigma(k_T) = \frac{1}{\sigma_0} \int_0^{k_T^2} dq_T^2 \frac{d\sigma}{dq_T^2} = \frac{1}{\sigma_0} \int_0^{Q^2} dq_T^2 \frac{d\sigma}{dq_T^2} - \frac{1}{\sigma_0} \int_{k_T^2}^{Q^2} dq_T^2 \frac{d\sigma}{dq_T^2}$$

$$= 1 - \alpha \int_{k_T^2}^{Q^2} dq_T^2 \nu(q_T) = 1 - \frac{\alpha}{2\pi} \ln^2 \left(\frac{Q^2}{k_T^2} \right)$$

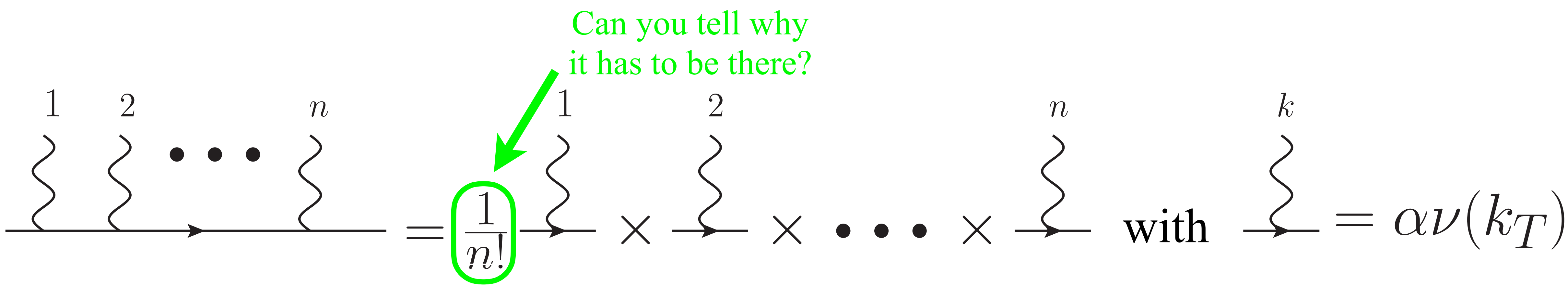
where we have neglected subdominant terms. Notice that $\Sigma(k_T)$ is engineered such that:

$$\frac{d}{dk_T^2} \Sigma(k_T) = \frac{1}{\sigma_0} \frac{d\sigma}{dk_T^2}$$

What happens if there are n soft photons in the final state?

Resummation of the q_T distribution

It is "well known" that soft photons factorise in QED (**eikonal approximation**):



(Fortunately, this remains true in QCD for the emission of **soft gluons** but it is much harder to prove (see *e.g. Nucl. Phys. B* 327 (1989) 323-352)).

Therefore, the cumulant cross section for any number of soft-gluon emissions is:

$$\begin{aligned}\Sigma(k_T) &= 1+\alpha\int_0^{k_T^2} dq_{T,1}^2\nu(q_{T,1}) + \frac{\alpha^2}{2}\int_0^{k_T^2} dq_{T,1}^2 dq_{T,2}^2\nu(q_{T,1})\nu(q_{T,2}) + \dots \\ &= \sum_{n=0}^{\infty}\frac{\alpha^n}{n!}\prod_{k=1}^{n}\int_0^{k_T^2} dq_{T,k}^2\nu(q_{T,k}) = \boxed{\exp\left[-\frac{\alpha}{2\pi}\ln^2\left(\frac{Q^2}{k_T^2}\right)\right]}\end{aligned}$$

Sudakov form factor

Finally, soft gluon emissions **exponentiate** such that the **resummed** cross section reads:

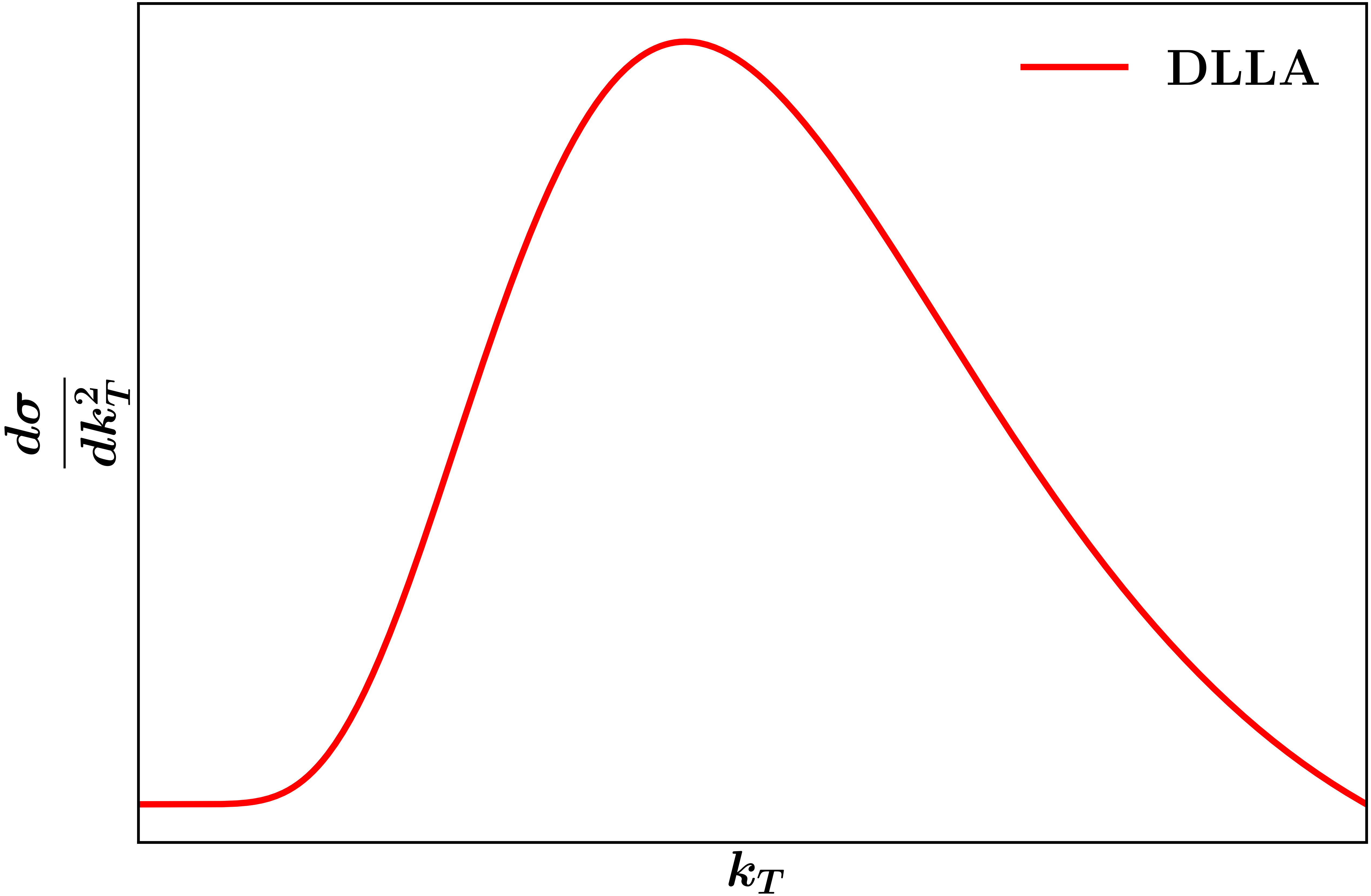
$$\frac{1}{\sigma_0}\frac{d\sigma}{dk_T^2} = \frac{d}{dk_T^2}\Sigma(k_T) = \frac{\alpha}{\pi}\frac{\ln(Q^2/k_T^2)}{k_T^2}\exp\left[-\frac{\alpha}{2\pi}\ln^2\left(\frac{Q^2}{k_T^2}\right)\right]$$

This result, that works also in QCD, is the Double-Leading-Log Approximation (DLLA)

The result is that the q_T-differential cross section, does not diverge anymore for $q_T \to 0$.

In fact, it tends to **zero** exponentially in this limit.

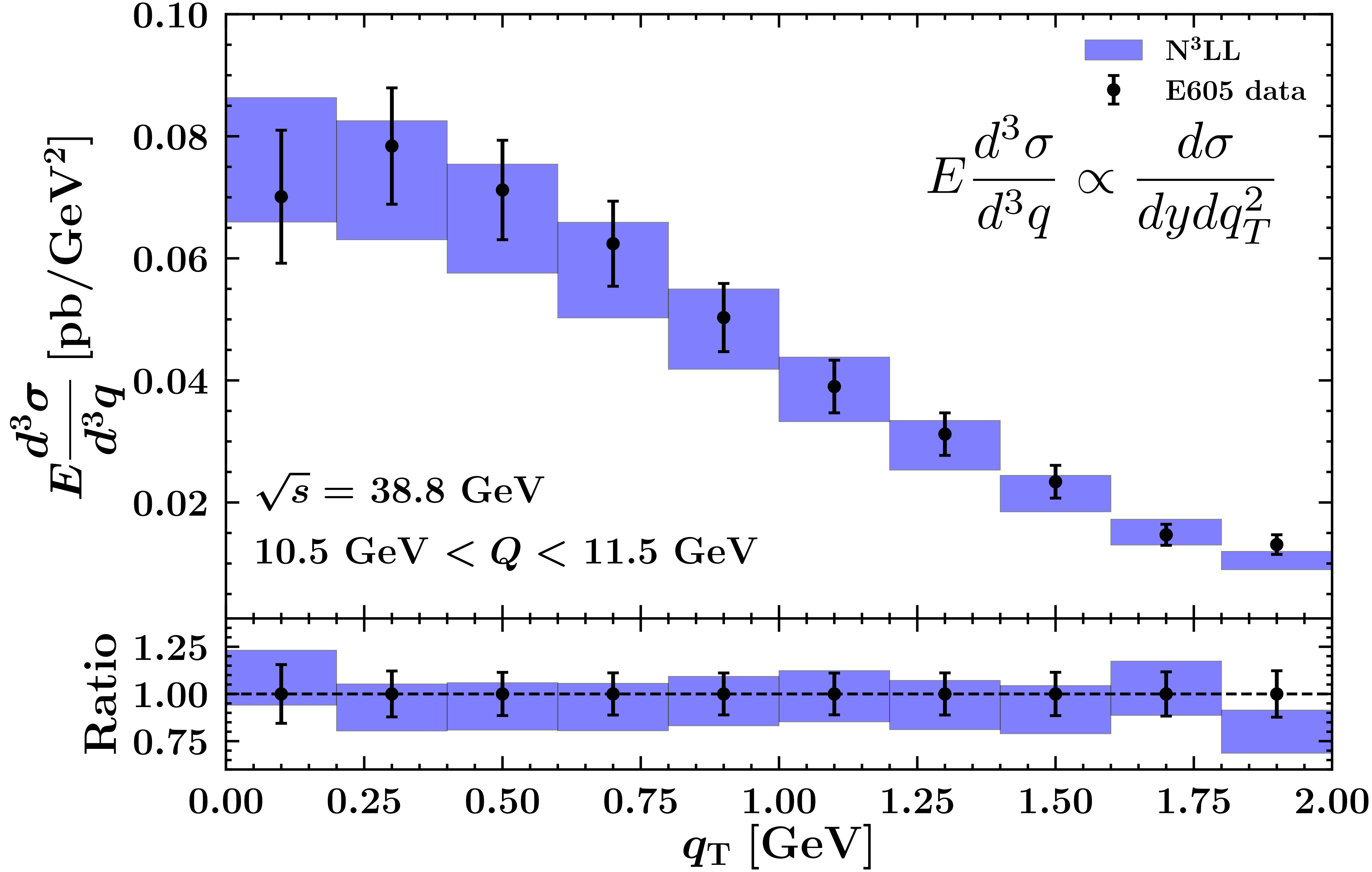
Resummation of the q_T distribution



Resummation of the q_T distribution

Unfortunately, the **DLLA** is **too suppressed** for $q_T \to 0$. Experimentally one observes:

$$\frac{d\sigma}{dq_T^2} \xrightarrow[q_T \to 0]{} \text{constant}$$



What's wrong with DLLA?

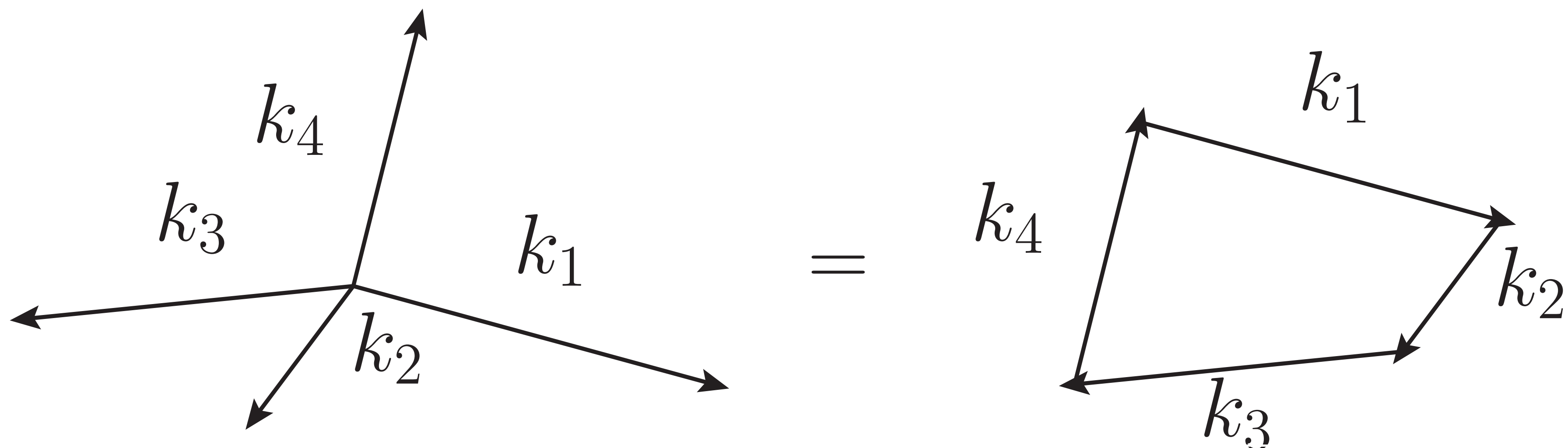
Resummation of the q_T distribution

Let us examine DLLA more in depth:

- emitted soft photons are all **independent**.

- In this configuration, the only possibility to get $q_T = 0$ is to **veto all photons**. Clearly, the probability of emitting no photons is zero and so is the cross section.

- But in reality, if more than a single photon is emitted, it is possible to obtain $q_T = 0$ through **vectorial sum** of transverse momenta:



- Therefore, DLLA **overly constraints** the phase space of emitted gluons around $q_T = 0$.

- For $q_T \to 0$ the **leading contribution vanishes** and subleading terms become important.

- In fact, the main shortcoming of DLLA is that it neglects **momentum conservation**.

Resummation of the q_T distribution

Let us enforce momentum conservation on the soft gluon emissions:

$$\Sigma(k_T) = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \int \delta^{(2)}(\mathbf{k}_T - \sum_{k=1}^{n} \mathbf{q}_{T,k}) \prod_{k=1}^{n} dq_{T,k}^2 \nu(q_{T,k}) = \text{and now what?}$$

The **δ-function** seemingly **spoils the exponentiation** because it entangles all momenta.

Let us consider the Fourier representation of the δ-function:

$$\delta^{(2)}(\mathbf{k}_T - \sum_{k=1}^{n} \mathbf{q}_{T,k}) = \int \frac{d^2\mathbf{b}}{(2\pi)^2} e^{-i\mathbf{k}_T\cdot\mathbf{b}} \prod_{k=1}^{n} e^{i\mathbf{q}_{T,k}\cdot\mathbf{b}}$$

plugged into Σ, this gives:

$$\Sigma(k_T) = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \int \frac{d^2\mathbf{b}}{(2\pi)^2} e^{-i\mathbf{k}_T\cdot\mathbf{b}} \prod_{k=1}^{n} \underbrace{\int dq_{T,k}^2 e^{i\mathbf{q}_{T,k}\cdot\mathbf{b}} \nu(q_{T,k})}_{\widetilde{\nu}(\mathbf{b})}$$

$$= \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \int \frac{d^2\mathbf{b}}{(2\pi)^2} e^{-i\mathbf{k}_T\cdot\mathbf{b}} \left[\widetilde{\nu}(\mathbf{b})\right]^n = \int \frac{d^2\mathbf{b}}{(2\pi)^2} e^{-i\mathbf{k}_T\cdot\mathbf{b}} \exp\left[\alpha\widetilde{\nu}(\mathbf{b})\right]$$

Soft gluon emissions with momentum conservation exponentiate in ***b*** space (often called impact parameter space) allowing to resum them.

Momentum conservation ensures that $\frac{d\sigma}{dq_T^2} \underset{q_T\to 0}{\longrightarrow} \text{constant}$

Resummation of the q_T distribution

A full-fledged extension of this formalism to Drell-Yan in QCD was carried out by Collins, Soper and Sterman [Nucl. Phys. B250, 199 (1985)]:

$$\begin{aligned}
\frac{d\sigma^{\mathrm{CSS}}_{PP\to\ell\bar{\ell}+X}}{dQ^2\,dy\,dq_T^2} &= \frac{2\pi^2\alpha_{\mathrm{em}}^2}{3sQ^2N_c}\sum_{a,b=q,\bar{q}} H_{ab}(\alpha_s(Q))\int_0^\infty db\, b J_0(bq_T) \\
&\times \sum_i \int_{x_1}^1 \frac{dy_1}{y_1} C_{ai}(y_1,\alpha_s(\mu_b)) f_i\left(\frac{x_1}{y_1},\mu_b\right) \\
&\times \sum_j \int_{x_2}^1 \frac{dy_2}{y_2} C_{bj}(y_2,\alpha_s(\mu_b)) f_j\left(\frac{x_2}{y_2},\mu_b\right) \\
&\times \exp\left\{-\int_{\mu_b^2}^{Q^2}\frac{d\mu^2}{\mu^2}\left[A(\alpha_s(\mu))\ln\left(\frac{Q^2}{\mu^2}\right)+B(\alpha_s(\mu))\right]\right\} \\
&+ Y(Q,y,q_T)
\end{aligned}$$

$$\mu_b = \frac{b_0}{b} = \frac{2e^{-\gamma_{\mathrm{E}}}}{b}$$

$$x_{1,2} = \frac{Q}{\sqrt{s}}e^{\pm y}$$

Functions, ***A***, ***B***, ***C***, ***H*** are all perturbative.

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Fourier tranform

Can you prove how the Fourier transform reduces to this?

$$\times \sum_i \int_{x_1}^1 \frac{dy_1}{y_1} C_{ai}(y_1,\alpha_s(\mu_b)) f_i\left(\frac{x_1}{y_1},\mu_b\right)$$

$$\times \sum_j \int_{x_2}^1 \frac{dy_2}{y_2} C_{bj}(y_2,\alpha_s(\mu_b)) f_j\left(\frac{x_2}{y_2},\mu_b\right)$$

$$\times \exp\left\{-\int_{\mu_b^2}^{Q^2}\frac{d\mu^2}{\mu^2}\left[A(\alpha_s(\mu))\ln\left(\frac{Q^2}{\mu^2}\right)+B(\alpha_s(\mu))\right]\right\}$$

$$+ \; Y(Q,y,q_T)$$

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$$\boxed{+ \quad Y(Q,y,q_T)}$$

Correction when $q_T \sim Q$
(power corrections of q_T/Q)

Dominant for $q_T \ll Q$
(often referred to as W term)

$$\mu_b = \frac{b_0}{b} = \frac{2e^{-\gamma_{\mathrm{E}}}}{b}$$

$$x_{1,2} = \frac{Q}{\sqrt{s}}e^{\pm y}$$

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& + \; Y(Q,y,q_T)
\end{aligned}
$$

Hard function (virtual emissions)

Matching on to collinear PDFs (unresolved real emissions)

Sudakov form factor (resummation of large logarithms coming from real emissions)

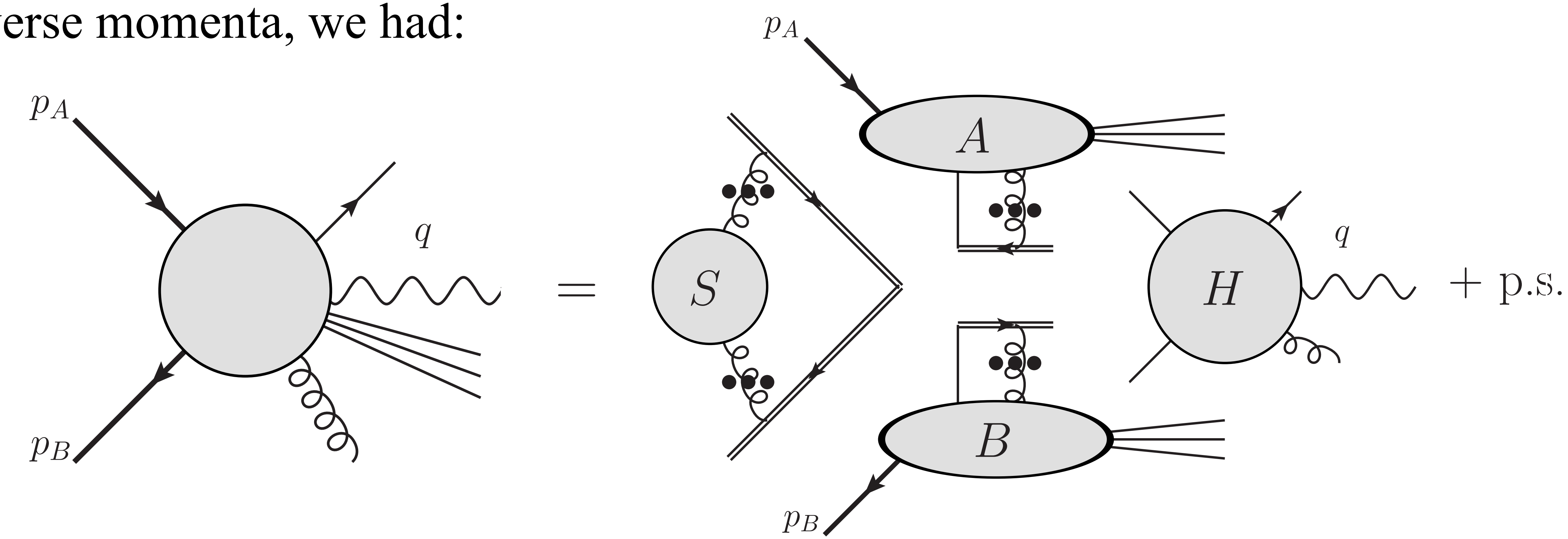
$$\mu_b = \frac{b_0}{b} = \frac{2e^{-\gamma_{\mathrm{E}}}}{b}$$

$$x_{1,2} = \frac{Q}{\sqrt{s}}e^{\pm y}$$

Functions, ***A***, ***B***, ***C***, ***H*** are all perturbative.

q_T resummation *à la* CSS and TMD factorisation

We saw a very sketchy derivation of factorisation. Before integrating over the partonic traverse momenta, we had:



$$d\sigma \sim H \cdot A \cdot B \cdot S$$

where the **dots** indicate integration over the internal transverse momenta.

Since the soft function ***S*** is **not observable** in this process, one can define:

$$F_a = A \cdot \sqrt{S} \qquad F_b = B \cdot \sqrt{S}$$

While ***A***, ***B***, and ***S*** are separately affected by the so-called rapidity divergences, their combination in ***F*₁** and ***F*₂** is finite and defines the **TMD PDFs**. One finally finds:

$$d\sigma \sim H \cdot F_a \cdot F_b$$

that is the essence of TMD factorisation.

q_T resummation *à la* CSS and TMD factorisation

Perhaps unsurprisingly, in impact-parameter space the convolutions over the internal transverse momenta become simple products and the **TMD factorisation** formula reads:

$$\frac{d\sigma}{dQ^2 dy dq_T^2} = \frac{2\pi\alpha_{\rm em}^2}{3sQ^2N_c}\sum_q H_{ab}\left(\alpha_s(Q),\frac{\mu}{Q}\right)\int_0^\infty db\, bJ_0(bq_T)F_a(x_1,b;\mu,\zeta_a)F_b(x_2,b;\mu,\zeta_b)$$

Each TMD PDF depends on **two scales**:

1. the **renormalisation scale** $\boldsymbol{\mu}$ is the consequence of the UV renormalisation of the Lagrangian. The only constraint on μ is $\mu \simeq Q$.

2. the **rapidity scales** $\boldsymbol{\zeta_{a,b}}$ originate from the cancellation of the rapidity divergence between soft function (S) and beam functions (A and B). The rapidity scales are kinematically constraint to be $\zeta_a\zeta_b = Q^4$.

The dependence of TMD PDFs on μ and ζ is computable in perturbation theory. Indeed, TMD PDFs obey **evolution equations** whose solution gives the dependence on μ and ζ.

q_T resummation *à la* CSS and TMD factorisation

The TMD evolution equations read:

$$\frac{\partial \ln F}{\partial \ln \sqrt{\zeta}} = K(\mu)$$

$$\frac{\partial \ln F}{\partial \ln \mu} = \gamma_F(\alpha_s(\mu)) - \gamma_K(\alpha_s(\mu)) \ln \frac{\sqrt{\zeta}}{\mu}$$

with: $$\frac{\partial K}{\partial \ln \mu} = -\gamma_K(\alpha_s(\mu))$$

In addition, for small values of b, the TMD PDFs can be **matched** on to the collinear PDFs by means of a perturbative coefficient (effect of unresolved radiation):

$$F_a(x, b, \mu, \zeta) = \sum_i \int_x^1 \frac{dy}{y} C_{ai}(y, \alpha_s(\mu), \mu/\mu_b, \zeta/\mu_b^2) f_i\left(\frac{x}{y}\mu\right) \equiv F(\mu, \zeta) \otimes f(\mu)$$

The solution of the evolution equations between the **initial** scales (μ_0, ζ_0) and the **final** scales (μ, ζ) is:

Can you prove it given the differential equations above?

$$\boxed{F(\mu, \zeta) = \exp\left\{K(\mu_0) \ln \frac{\sqrt{\zeta}}{\sqrt{\zeta_0}} + \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \left[\gamma_F(\alpha_s(\mu')) - \gamma_K(\alpha_s(\mu')) \ln \frac{\sqrt{\zeta}}{\mu'}\right]\right\} C(\mu_0, \zeta_0) \otimes f(\mu_0)}$$

For applications in Drell-Yan, the set of initial/final scales that nullify all the unresummed logs (central scales) is:

$$\mu_0 = \sqrt{\zeta_0} = \mu_b \qquad \mu = \sqrt{\zeta} = Q$$

q_T resummation *à la* CSS and TMD factorisation

One can relate the anomalous dimension γ_K, γ_F, and K to the functions A and B finding:

$$F(\mu,\zeta) = \exp\left\{-\frac{1}{2}\int_{\mu_b^2}^{Q^2}\frac{d\mu^2}{\mu^2}\left[A(\alpha_s(\mu))\ln\frac{Q^2}{\mu^2}+B(\alpha_s(\mu))\right]\right\}C(\mu_0,\zeta_0)\otimes f(\mu_0)$$

Plugging this result into the TMD factorisation formula one finds back the **CSS formula**.

Of course, this is no surprise: after all CSS and TMD factorisation *are* the same thing. The scope was to illustrate that there are **two** different ways of resumming large logs:

1. The **diagrammatic** approach *à la* Parisi-Petronzio (that we used to illustrate CSS) is based on factorising emissions in specific limits and exponentiating them. On top of being used to obtain analytic resummation formulas, it can also be used to perform resummation *directly* in momentum space. This approach is used by some specific Monte Carlo generators and has the advantage of being exclusive w.r.t. additional radiation, allowing one to study more exclusive observables.

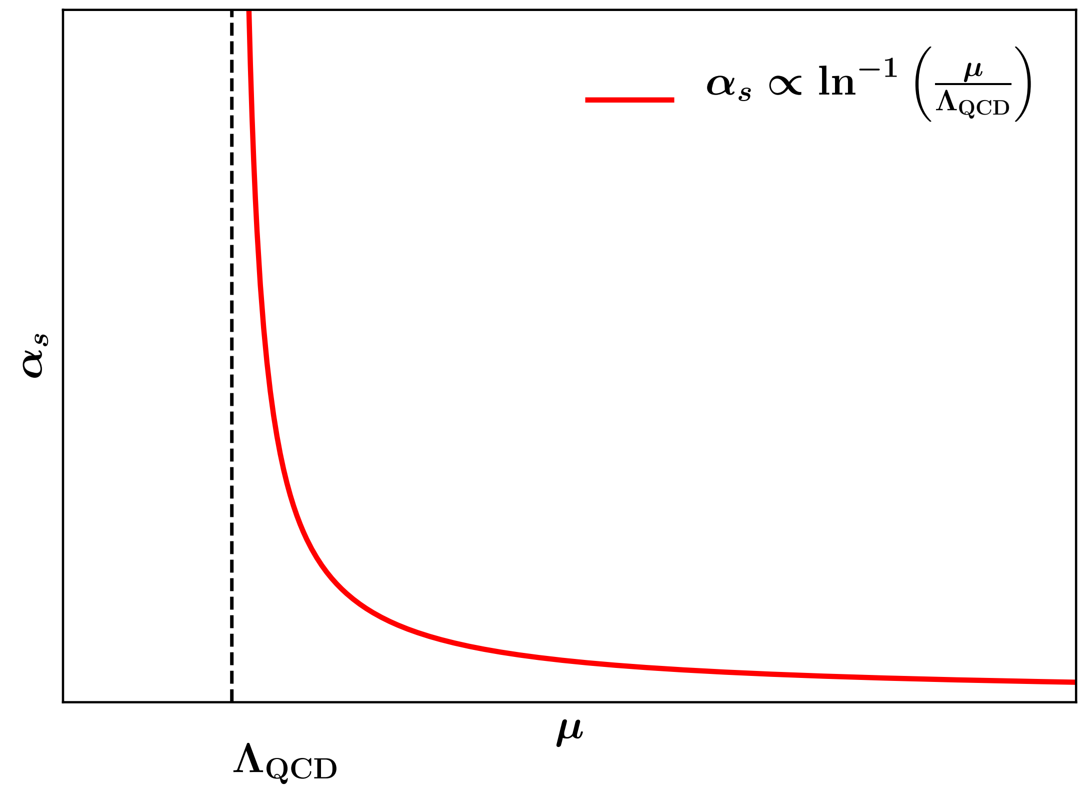
2. The renormalisation-group-equation (**RGE**) approach *à la* TMD factorisation are based on solving evolution equations derived by identifying the singular behaviours. Their advantage is that they often allow one to derive analytic formulas that are thus easier to implement.

Non-perturbative corrections

So far, I omitted the fact that, as it is, the CSS formula gives a **divergent** result.
The origin of the divergence is the fact that we are dealing with integrals of this kind:

$$\sigma \propto \int_0^\infty db\, \alpha_s^p\left(\frac{1}{b}\right)\ldots\ldots$$

therefore when $b \to \infty$ we are forced to compute α_s at very low scales eventually hitting the **Landau pole** at $b = 1/\Lambda_{\rm QCD}$ that is a non-integrable singularity:



This divergence signals of our lack of control on **non-perturbative** contributions.

The fact that in impact-parameter space non-perturbative corrections take place around $b \gtrsim \Lambda_{\rm QCD}^{-1}$, immediately tells us that in momentum space this corrections become relevant at around $q_T \lesssim \Lambda_{\rm QCD} \sim 1$ GeV.

Non-perturbative corrections

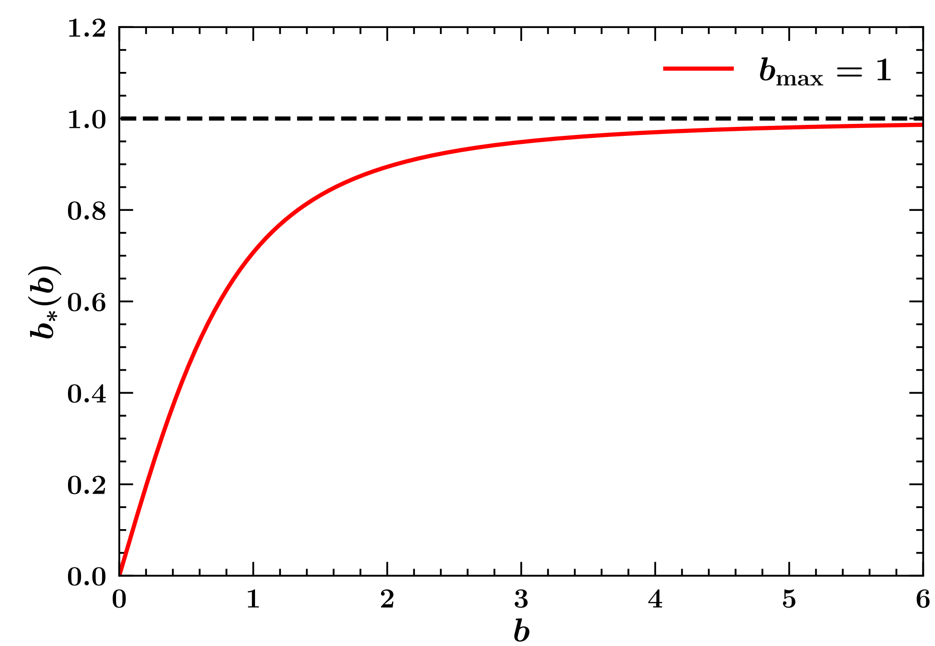
To make sense of the CSS formula, it is necessary to **regularise** this divergence.

Different recipes exist but I find that the TMD factorisation "view" provides a particularly transparent way of separating perturbative from non-perturbative effects.

1. First one fixes a value of b, say $b_{\max}$, above which non-perturbative domain become significant. The requirement is $1/b_{\max} \gtrsim \Lambda_{\rm QCD}$. A reasonable value is $b_{\max} = 1\ \text{GeV}^{-1}$.

2. Then one introduces a monotonic function $b_*(b)$ such that:

$$\begin{array}{lll} b_*(b) \rightarrow b & \text{for} & b \rightarrow 0\,, \\ b_*(b) \rightarrow b_{\max} & \text{for} & b \rightarrow \infty \end{array}$$



An example of a function as such was introduced by CSS:

$$b_*(b) = \frac{b}{\sqrt{1 + b^2/b_{\max}^2}}$$

3. Finally, one rewrites the TMD PDFs as follows:

$$F(x, b; \mu, \zeta) = \left[\frac{F(x, b; \mu, \zeta)}{F(x, b_*(b); \mu, \zeta)}\right] F(x, b_*(b); \mu, \zeta) = f_{\rm NP}(x, b, \zeta) F(x, b_*(b); \mu, \zeta)$$

In this way, F is always computed in the perturbative domain avoiding the Landau pole and all non-perturbative effects are relegated into $f_{\rm NP}$ that can be determined from data.

Non-perturbative corrections

Properties of $f_{\rm NP}$:

1. It has to go to *one* as b goes to zero to reproduce the full perturbative regime.

2. It has to go to *zero* as b becomes large to mimic the Sudakov suppression.

A popular (educated) parameterisation of $f_{\rm NP}$ is:

$$f_{\rm NP}(x,b,\zeta) = \exp\left[g_1(b)\ln\left(\frac{\zeta}{Q_0^2}\right) + g_2(x,b)\right]$$

Can you tell why $f_{\rm NP}$ does not depend on μ?

1. The scaling with ζ is fully determined by TMD evolution. The function $\mathbf{g_1}$ can only depend on b and can be regarded as the anomalous dimension of the non-perturbative evolution. It is is *flavour independent*.

2. The function $\mathbf{g_2}$ can depend on both x and b and parametrises the non-perturbative effects due to the transverse momentum of partons inside the hadron. For this reason it is sometimes called intrinsic or primordial k_T. It can be *flavour dependent*.

3. Q_0 is an arbitrary parameter with mass dimensions.

Despite this parameterisation of $f_{\rm NP}$ suggests a physical interpretation to the functions g_1 and g_2, $f_{\rm NP}$ is tightly connected to the **arbitrary** choice of the b_* function, including the choice of $b_{\rm max}$. As a consequence, no actual physical meaning can be given to it.

Non-perturbative corrections

Before concluding, it is worth mentioning that the q_T spectrum for $q_T \to 0$ as derived by Parisi and Petronzio, and consistently with CSS, converges to an asymptotic value that behaves as a power of $\Lambda_{\rm QCD}/Q$.

Hard question: can you prove this statement?

This confirms that a proper treatment of momentum conservation cures the DLLA over-suppression at $q_T \to 0$ and also tells us that non-perturbative effects, that are also powers of $\Lambda_{\rm QCD}/Q$, are expected to be relative important in that region.

End of lecture 1

Questions for the discussion

1. Predictivity of QCD heavily relies on leading-power factorisation:

   - What is meant by "leading-power"?
   - Do all processes factorise in QCD?
   - If not, what prevents factorisation from happening?

2. Can you say why the solution of the DGLAP equation resums terms like $\alpha_s^n \ln^m(\mu/\mu_0)$ with $m \leq n$.

3. In slide 25, can you show that $H^{[n,k]}$ for $k \neq 0$ can be written in terms of $H^{[m<n,0]}$ and the the DGLAP splitting kernels $P^{[m<n]}$? Try it at one loop ($n = 1$).

4. Can you think about another example of resummation even more fundamental than the DGLAP evolution? Hint: consider the strong coupling.

5. The Landau pole signals an actual divergence of the strong coupling or it is an artefact of perturbation theory? Motivate your answer.