Introduction to Parton Showers

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Goals of this lecture

Bird's eye overview

- Parton showers have been a topic of intense research over the past four decades, as they connect theory and experiment and need to make concessions to both
- There is renewed interest in understanding their interplay with analytic resummation, and in finding new and better algorithms that allow an extension to higher formal accuracy
- To some extent, the definition of accuracy itself is still being worked on

What to expect

- The background that allows you to understand what is being discussed in past and present parton shower literature, and why
- The tutorial as a chance for in-depth discussion of the basic concepts presented in the lecture (both parton showers and analytic resummation)

What not to expect

- All the latest and greatest plots, as well as a survey of all possible algorithms. This could fill the entire time of the school.

Suggested reading

1. R. K. Ellis, W. J. Stirling, B. R. Webber
   QCD and Collider Physics
   Cambridge University Press, 2003
2. R. D. Field
   Applications of Perturbative QCD
   Addison-Wesley, 1995
3. M. E. Peskin, D. V. Schroeder
   An Introduction to Quantum Field Theory
   Westview Press, 1995
4. L. Dixon, F. Petriello (Editors)
   Journeys Through the Precision Frontier
   Proceedings of TASI 2014, World Scientific, 2015
5. T. Sjöstrand, S. Mrenna, P. Z. Skands
   PYTHIA 6.4 Physics and Manual
   JHEP 05 (2006) 026

Additional references provided on the slides
Only if material not covered in these books

Hands on tutorials

- Resource for learning more about parton showers: Live "Hackathon"

  `http://cern.ch/shoeche/mcnet-cteq21/`
  `git clone https://gitlab.com/shoeche/tutorials.git`

Tutorial on MC event generators

Held by the MCnet collaboration at the CTEQ / MCnet School 2021.

Instructions

LL tutorial
PS coding tutorial
MC running tutorial

TASI Lectures

arXiv:1411.4085

Introduction to Parton Showers and Matching

Tutorial for summer schools

1 Introduction

In this tutorial we will discuss the construction of a parton shower, the implementation of on-the-fly uncertainty estimates, and of matrix-element corrections, and matching at next-to-leading order. At the end, you will be able to run your own parton shower for $e^+e^- \to$hadrons at LEP energies and compare its predictions to results from the event generator Sherpa (using a simplified setup). You will also have constructed your first MC@NLO and POWHEG generator.

2 Getting started

You can use any of the docker containers for the school to run this tutorial. Should you have problems with disk space, consider running `docker containers prune` and `docker system prune` first. To launch the docker container, use the following command

```
docker run -it -u $(id -u $USER) --rm -v $HOME:$HOME -w $PWD <container name>
```

You can also use your own PC (In this case you should have PyPy and Rivet installed). Download the tutorial and change to the relevant directory by running

Outline of lectures

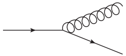
- Heuristic picture
- Technical ingredients
- Semi-classical picture
- Color coherence
- Higher-order effects
- Connection to resummation
- Forward vs. backward evolution

What is a parton shower?

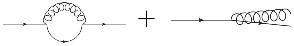
The heuristic view

Radiative corrections as a branching process

- Make two well motivated assumptions
  - Parton branching can occur in two ways



\- observed



\- unobserved

  - Evolution conserves probability
- The consequence is Poisson statistics
  - Let the decay probability be λ
  - Assume indistinguishable particles $\rightarrow$ naive probability for n emissions

$$P_{\text{naive}}(n,\lambda) = \frac{\lambda^n}{n!}$$

  - Probability conservation (i.e. unitarity) implies a no-emission probability

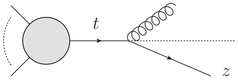
$$P(n,\lambda) = \frac{\lambda^n}{n!}\exp\{-\lambda\} \qquad \longrightarrow \qquad \sum_{n=0}^{\infty} P(n,\lambda) = 1$$

- In the context of parton showers $\Delta = \exp\{-\lambda\}$ is called a Sudakov factor

Radiative corrections as a branching process

- Decay probability for parton state in collinear limit

$$\lambda \to \frac{1}{\sigma_n}\int_t^{Q^2} \mathrm{d}\bar{t}\,\frac{\mathrm{d}\sigma_{n+1}}{\mathrm{d}\bar{t}} \approx \sum_{\text{jets}}\int_t^{Q^2}\frac{\mathrm{d}\bar{t}}{\bar{t}}\int \mathrm{d}z\,\frac{\alpha_s}{2\pi}P(z)$$



  Parameter t identified with evolution "time"

- Splitting function $P(z)$ spin & color dependent

$$P_{qq}(z) = C_F\left[\frac{2z}{1-z} + (1-z)\right] \qquad P_{gq}(z) = T_R\left[\,z^2 + (1-z)^2\,\right]$$

$$P_{gg}(z) = C_A\left[\frac{2z}{1-z} + z(1-z)\right] + (z \leftrightarrow 1-z)$$

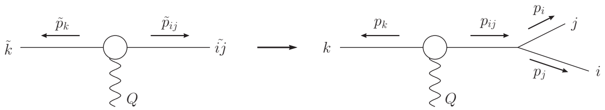
  Exercise: Why does the $2z/(1-z)$ term appear both in P_{qq} and P_{gg}?

- When adding partons
  - On-shell conditions must be maintained
  - Overall four-momentum must be conserved
  - Color must be conserved
- Later in this lecture we will derive part of these splitting functions and analyze their properties

How to deal with the phase space

Example momentum mapping

Final state momentum mapping



- Generate off-shell momentum by rescaling

$$p_{ij}^\mu = \tilde{p}_{ij}^\mu + \frac{p_{ij}^2}{2\tilde{p}_{ij}\tilde{p}_k}\,\tilde{p}_k^\mu\,, \qquad p_k^\mu = \left(1 - \frac{p_{ij}^2}{2\tilde{p}_{ij}\tilde{p}_k}\right)\tilde{p}_k^\mu$$

- Then branch into two on-shell momenta

$$p_i^\mu = \tilde{z}\,\tilde{p}_{ij}^\mu + (1-\tilde{z})\,\frac{p_{ij}^2}{2\tilde{p}_{ij}\tilde{p}_k}\tilde{p}_k^\mu + k_\perp^\mu$$

$$p_j^\mu = (1-\tilde{z})\,\tilde{p}_{ij}^\mu + \tilde{z}\,\frac{p_{ij}^2}{2\tilde{p}_{ij}\tilde{p}_k}\tilde{p}_k^\mu - k_\perp^\mu$$

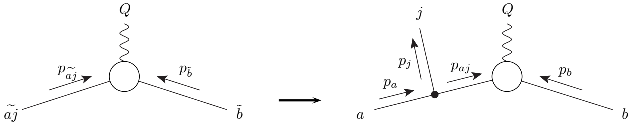
  Exercise: Is this momentum mapping collinear safe?

- On-shell conditions require that

$$\vec{k}_T^2 = p_{ij}^2\,\tilde{z}(1-\tilde{z}) \qquad \leftrightarrow \qquad \tilde{z}_\pm = \frac{1}{2}\left(1 \pm \sqrt{1 - 4\vec{k}_T^2/p_{ij}^2}\right)$$

  $\rightarrow$ for any finite $\vec{k}_T$ we have $0 < \tilde{z} < 1$

Initial state momentum mapping



- Rescale beam momentum to obtain new partonic cms energy

$$p_a^\mu = \frac{2\,p_a p_b}{2\,\tilde p_{aj}\tilde p_b}\,\tilde p_{aj}^\mu$$

- Compute final-state momentum and internal momentum

$$p_{aj}^\mu = \tilde z\,p_a^\mu + \frac{p_{aj}^2}{2p_b p_a}\,p_b^\mu + k_\perp^\mu$$

$$p_j^\mu = (1-\tilde z)\,p_a^\mu - \frac{p_{aj}^2}{2p_b p_a}\,p_b^\mu - k_\perp^\mu$$

- Recoil taken by complete final state via Lorentz transformation

$$p_i^\mu = p_{\tilde\imath}^\mu - \frac{2\,p_{\tilde\imath}(K+\tilde K)}{(K+\tilde K)^2}\,(K+\tilde K)^\mu + \frac{2\,p_{\tilde\imath}\tilde K}{\tilde K^2}\,K^\mu\ ,$$

where $K^\mu = p_a^\mu - p_j^\mu + p_b^\mu$ and $\tilde K^\mu = p_{\widetilde{aj}}^\mu + p_b^\mu$

How to color a shower

The improved large-N_c approximation

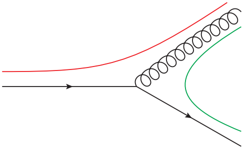
Color flow

- Write gluon propagator using completeness relations

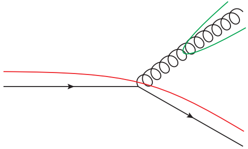
$$\underbrace{\delta^{ab}}_{\text{standard}} = 2\,\mathrm{Tr}(T^a T^b) = 2\,T^a_{ij} T^b_{ji} = T^a_{ij} \underbrace{2\,\delta_{ik}\delta_{jl}}_{\text{color flow}} T^b_{lk}$$

- Quark-gluon vertex

$$T^a_{ij} T^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\right)$$

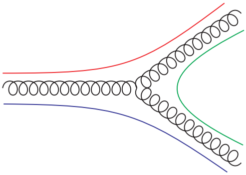


$$-\frac{1}{N_c}$$



- Gluon-gluon vertex

$$f^{abc} T^a_{ij} T^b_{kl} T^c_{mn} = \delta_{il}\delta_{kn}\delta_{mj} - \delta_{in}\delta_{ml}\delta_{kj}$$



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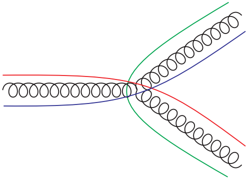
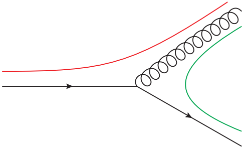


Exercise: Can you explain why there is no $1/N_c$ term here?

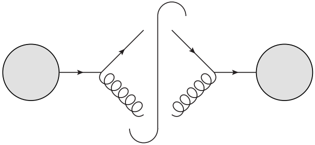
Color flow

- Typically, parton showers also make the leading-color approximation

$$T^a_{ij} T^a_{kl} \to \frac{1}{2} \delta_{il} \delta_{jk} \qquad \leftrightarrow$$



- If used naively, this would overestimate the color charge of the quark: Consider process $q \to qg$ attached to some larger diagram



$$\propto \qquad T^a_{ij} T^a_{jk} = C_F \delta_{ik}$$

  but now we have $\dfrac{1}{2}\, \delta_{il} \delta_{jm} \delta_{mj} \delta_{lk} = \dfrac{C_A}{2}\, \delta_{ik}$

- Color assignments in parton shower made at leading color
  but color charge of quarks actually kept at C_F
  Exercise: How should colors be assigned when a gluon splits into two gluons?

How to implement the algorithm

Monte-Carlo methods for parton showers

Monte-Carlo methods: Poisson distributions

- Assume decay process described by $g(t)$
- Decay can happen only if it has not happened already
  Must account for survival probability $\leftrightarrow$ Poisson distribution

$$\mathcal{G}(t) \;=\; g(t)\Delta(t,t_0) \qquad \text{where} \qquad \Delta(t,t_0) \;=\; \exp\left\{-\int_t^{t_0} \mathrm{d}t'\, g(t')\right\}$$

- If $G(t)$ is known, then we also know the integral of $\mathcal{G}(t)$

$$\int_t^{t_0} \mathrm{d}t' \mathcal{G}(t') \;=\; \int_t^{b} \mathrm{d}t' \;\frac{\mathrm{d}\Delta(t',t_0)}{\mathrm{d}t'} \;=\; 1-\Delta(t,t_0)$$

- Can generate events by requiring $1-\Delta(t,t_0)=1-R$

$$t = G^{-1}\Big[G(t_0)+\log R\Big]$$

You will use this formula in the tutorial

Monte-Carlo methods: Poisson distributions

- Importance sampling for Poisson distributions
  - Generate event according to $\mathcal{G}(t)$
  - Accept with $w(t) = f(t)/g(t)$
  - If rejected, continue starting from t
- Probability for immediate acceptance

$$\frac{f(t)}{g(t)}\, g(t) \exp\left\{-\int_t^{t_0} \mathrm{d}t'\, g(t')\right\}$$

- Probability for acceptance after one rejection

$$\frac{f(t)}{g(t)}\, g(t) \int_t^{t_0} \mathrm{d}t_1 \exp\left\{-\int_t^{t_1} \mathrm{d}t'\, g(t')\right\} \left(1 - \frac{f(t_1)}{g(t_1)}\right) g(t_1) \exp\left\{-\int_{t_1}^{t_0} \mathrm{d}t'\, g(t')\right\}$$

- For n intermediate rejections we obtain n nested integrals $\int_t^{t_0} \int_{t_1}^{t_0} \ldots \int_{t_{n-1}}^{t_0}$
- Disentangling yields $1/n!$ and summing over all possible rejections gives

$$f(t) \exp\left\{-\int_t^{t_0} \mathrm{d}t'\, g(t')\right\} \sum_{n=0}^{\infty} \frac{1}{n!} \left[\int_t^{t_0} \mathrm{d}t' \left[g(t') - f(t')\right]\right]^n = f(t) \exp\left\{-\int_t^{t_0} \mathrm{d}t'\, f(t')\right\}$$

Monte-Carlo method for parton showers

- Start with set of n partons at scale t', which evolve collectively
  Sudakovs factorize, schematically

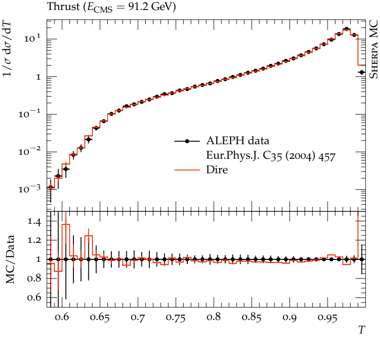
$$\Delta(t,t') = \prod_{i=1}^{n} \Delta_i(t,t')\ , \qquad \Delta_i(t,t') = \prod_{j=q,g} \Delta_{i\to j}(t,t')$$

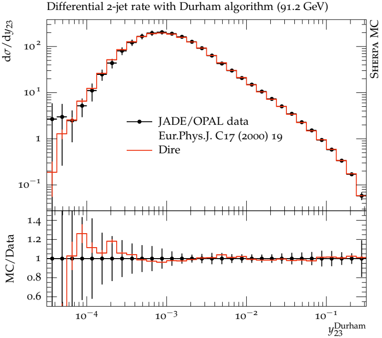
- Find new scale t where next branching occurs using veto algorithm
  - Generate t using overestimate $\alpha_s^{\max} P_{ab}^{\max}(z)$
  - Determine "winner" parton i and select new flavor j
  - Select splitting variable according to overestimate
  - Accept point with weight $\alpha_s(k_T^2) P_{ab}(z) / \alpha_s^{\max} P_{ab}^{\max}(z)$
- Construct splitting kinematics and update event record
- Continue until t falls below an IR cutoff

You will use this algorithm in the tutorial

Effects of the parton shower

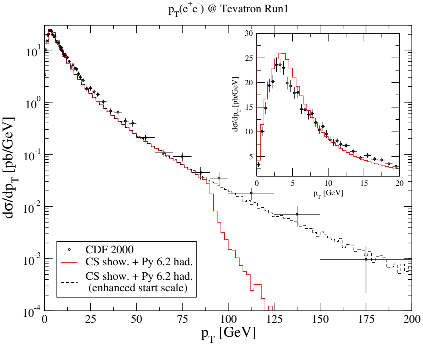
Effects of the parton shower





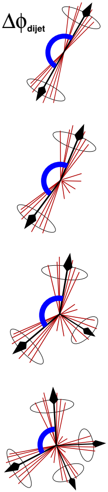
- Thrust and Durham $2 \to 3$-jet rate in $e^+e^- \to$hadrons
- Hadronization region to the right (left) in left (right) plot

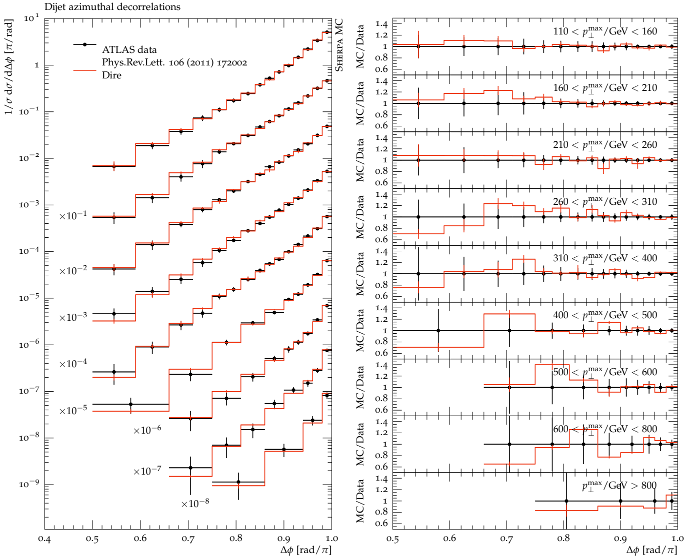
Effects of the parton shower



- Drell-Yan lepton pair production at Tevatron
- If hard cross section computed at leading order, then parton shower is only source of transverse momentum

Effects of the parton shower





What is a parton shower?

The semi-classical picture

Semi-classical source theory

- Classical point charge on trajectory $y^\mu(s) \to$ conserved current $j^\mu(\mathrm{x})$

$$j^\mu(x) = g \int \mathrm{d}t \, \frac{dy^\mu(t)}{dt} \, \delta^{(4)}(x - y(t)) \, , \qquad g = \sqrt{4\pi\alpha}$$

- Fourier transform to momentum space

$$j^\mu(k) = \int \mathrm{d}^4 x \, e^{ikx} \, j^\mu(x) = g \int \mathrm{d}t \, \frac{dy^\mu(t)}{dt} \, e^{iky(t)}$$

- Assume particle moves with momentum p_a if $t < 0$,
is 'kicked' at origin $y^\mu(0) = 0$, and moves with p_b if $t > 0$

$$y^\mu(t) = t \, \frac{p^\mu(t)}{p_0(t)} = \begin{cases} t \, p_a^\mu / p_{a,0} & \text{if} \quad t < 0 \\ t \, p_b^\mu / p_{b,0} & \text{if} \quad t > 0 \end{cases}$$

- Introduce a regulator and Fourier transform ...

$$j^\mu(k) = g \int_{-\infty}^{0} \mathrm{d}t \, \frac{p_a^\mu}{p_{a,0}} \, \exp\left\{ i \left(\frac{p_a k}{p_{a,0}} - i\varepsilon \right) t \right\} + g \int_{0}^{+\infty} \mathrm{d}t \, \frac{p_b^\mu}{p_{b,0}} \, \exp\left\{ i \left(\frac{p_b k}{p_{b,0}} + i\varepsilon \right) t \right\}$$

Semi-classical source theory

- Classical current

$$j^\mu(k) = \; ig\left(\frac{p_b^\mu}{p_b k + i\varepsilon} - \frac{p_a^\mu}{p_a k - i\varepsilon}\right)$$

  - Spin independent
  - Conserved

- Now add the quantum part $\rightarrow$ current can create gauge bosons
  Interaction Hamiltonian density

$$\mathcal{H}_{\mathrm{int}}(x) = j^\mu(x) A_\mu(x)$$

- Probability of no emission $\rightarrow$ vacuum persistence amplitude squared

$$|W_{a\to b}|^2 = |\langle 0| T\left[\exp\left\{i\int \mathrm{d}^4 x\, j^\mu(x) A_\mu(x)\right\}\right]|0\rangle|^2$$

- Can be expanded into power series

$$W_{a\to b} = \sum \frac{1}{n!} W_{a\to b}^{(n)}\,, \qquad\qquad W_{a\to b}^{(n)} \propto g^n$$

  - Zeroth order: $W_{a\to b}^{(0)} = 1$
  - First order: $\langle 0|A_\mu(x)|0\rangle = 0$

Semi-classical source theory

- Second order contribution

$$W^{(2)}_{a\to b} = -\int \mathrm{d}^4x \int \mathrm{d}^4y\, j^\mu(x) j^\nu(y) \langle 0|T\left[A_\mu(x)A_\nu(y)\right]|0\rangle$$
$$= -\int \mathrm{d}^4x \int \mathrm{d}^4y\, j^\mu(x) i\Delta_{F,\mu\nu}(x,y) j^\nu(y)$$

  - Emission of field quantum at x, propagation to y & absorption
  - Unobserved, i.e. a *virtual* correction

- Propagation described by time-ordered Green's function

$$i\Delta_F^{\mu\nu}(x,y) = \Theta(y_0 - x_0)\langle 0|A^\nu(y)A^\mu(x)|0\rangle + \Theta(x_0 - y_0)\langle 0|A^\mu(x)A^\nu(y)|0\rangle$$
$$= \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3\, 2E_k}\Big[\Theta(y_0 - x_0)e^{-ik(y-x)}$$
$$+\Theta(x_0 - y_0)e^{ik(y-x)}\Big]\sum_{\lambda=\pm}\varepsilon^\mu_\lambda(k,l)\varepsilon^{\nu\,*}_\lambda(k,l)$$
$$= -i\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{e^{-ik(y-x)}}{k^2 + i\varepsilon}\sum_{\lambda=\pm}\varepsilon^\mu_\lambda(k)\varepsilon^{\nu\,*}_\lambda(k)$$

Semi-classical source theory

- Insert into vacuum persistence amplitude

$$W^{(2)}_{a\to b} = -i\int \mathrm{d}^4x \int \mathrm{d}^4y \int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{e^{-ik(y-x)}}{k^2+i\varepsilon}\sum_{\lambda=\pm}\big(j(x)\varepsilon_\lambda(k)\big)\big(j(y)\varepsilon_\lambda(k)\big)^*$$

$$= -i\int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{1}{k^2+i\varepsilon}\sum_{\lambda=\pm}\big(j(k)\varepsilon_\lambda(k)\big)\big(j(k)\varepsilon_\lambda(k)\big)^*$$

- Use completeness relation for polarization vectors (e.g. axial gauge)

$$\sum_{\lambda=\pm}\varepsilon^\mu_\lambda(k,l)\,\varepsilon^{\nu\,*}_\lambda(k,l) = -g^{\mu\nu} + \frac{k^\mu l^\nu + k^\nu l^\mu}{kl}$$

- Complete second-order contribution $\big(p_a^2 = p_b^2 = 0$, dim.reg., $\overline{\mathrm{MS}}\big)$

$$W^{(2)}_{a\to b} = -i\,|g|^2\Big(\frac{\mu^2 e^{\gamma_E}}{4\pi}\Big)^\varepsilon \int \frac{\mathrm{d}^Dk}{(2\pi)^D}\,\frac{1}{k^2+i\varepsilon}\,\frac{2p_ap_b}{(p_ak)(p_bk)}$$

$$\xrightarrow{\text{IR only}} -\frac{\alpha}{\pi}\left(\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\log\frac{2p_ap_b}{\mu^2} + \frac{1}{2}\log^2\frac{2p_ap_b}{\mu^2} - \frac{\pi^2}{12} + \mathcal{O}(\varepsilon)\right)$$

Exercise: Compute matrix element in first line from eqns above using the Landau gauge $-g^{\mu\nu} + \frac{k^\mu k^\nu}{k^2}$

Semi-classical source theory

- Real-emission contribution

$$\mathrm{d}W^2_{a\to bc}(p_c) = \frac{\mathrm{d}^3\vec{p}_c}{(2\pi)^3\,2E_c}\left|\langle\vec{p}_c|T\left[\exp\left\{i\int \mathrm{d}^4x\, j^\mu(x)A_\mu(x)\right\}\right]|0\rangle\right|^2 .$$

- Can be expanded into power series

$$\mathrm{d}W_{a\to bc}(p_c) = \sum \frac{1}{n!}\mathrm{d}W^{(n)}_{a\to bc}(p_c)\,, \qquad \mathrm{d}W^{(n)}_{a\to bc}(p_c) \propto g^n$$

  - Zeroth order: $\langle\vec{p}_c|0\rangle = 0$

- First-order term ($p_a^2 = p_b^2 = 0$, dim.reg., $\overline{\mathrm{MS}}$)

$$\begin{aligned}
\int \mathrm{d}W^{2\,(1)}_{a\to bc}(p_c) &= \int \frac{\mathrm{d}^3\vec{p}_c}{(2\pi)^3\,2E_c}\left|\, i\int \mathrm{d}^4x\, j^\mu(x)\langle\vec{p}_c|A_\mu(x)|0\rangle\right|^2 \\
&= -\int \frac{\mathrm{d}^3\vec{p}_c}{(2\pi)^3\,2E_c}\sum_{\lambda=\pm}\big(j(p_c)\varepsilon_\lambda(p_c)\big)\big(j(p_c)\varepsilon_\lambda(p_c)\big)^* \\
&\to |g|^2\Big(\frac{\mu^2 e^{\gamma_E}}{4\pi}\Big)^\varepsilon \int \frac{\mathrm{d}^D\vec{p}_c}{(2\pi)^D}\,\frac{2p_ap_b}{(p_ap_c)(p_bp_c)}\,\delta(p_c^2) \\
&\approx +\frac{\alpha}{\pi}\left(\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\log\frac{2p_ap_b}{\mu^2} + \frac{1}{2}\log^2\frac{2p_ap_b}{\mu^2} - \frac{\pi^2}{12} + \mathcal{O}(\varepsilon)\right)
\end{aligned}$$

Semi-classical source theory

- So far we have

$$W^{(2)}_{a\to b} = -\frac{\alpha}{\pi}\left(\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\log\frac{2p_ap_b}{\mu^2} + \frac{1}{2}\log^2\frac{2p_ap_b}{\mu^2} - \frac{\pi^2}{12} + \mathcal{O}(\varepsilon)\right)$$

$$\int \mathrm{d}W^{2\,(1)}_{a\to bc}(p_c) = +\frac{\alpha}{\pi}\left(\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\log\frac{2p_ap_b}{\mu^2} + \frac{1}{2}\log^2\frac{2p_ap_b}{\mu^2} - \frac{\pi^2}{12} + \mathcal{O}(\varepsilon)\right)$$

  - Explicit form of unitarity condition (probability conservation)
  - Poles in ε cancel between virtual and real-emission correction
  - π^2 contributions due to D-dimensional phase space
- Double poles in ε only appear upon integration over loop momentum and full real-emission phase space $\to$ associated with unobserved region $\to$ can be removed explicitly (real-virtual cancelation)
- Remaining terms are double logarithms

$$W^{(2)}_{a\to b} \to -\frac{\alpha}{\pi}\left(\frac{1}{2}\log^2\frac{2p_ap_b}{\mu^2} - \frac{\pi^2}{12} + \mathcal{O}(\varepsilon)\right)$$

$$\int \mathrm{d}W^{2\,(1)}_{a\to bc}(p_c) \to +\frac{\alpha}{\pi}\left(\frac{1}{2}\log^2\frac{2p_ap_b}{\mu^2} - \frac{\pi^2}{12} + \mathcal{O}(\varepsilon)\right)$$

  - These terms survive if unitarity is broken by the measurement
    e.g. vetoed real radiation above a certain scale μ^2
    Exercise: Find more examples where real/virtual corrections are probed

Semi-classical source theory

- Order 2n contribution to vacuum persistence amplitude

$$W^{(2n)}_{a\to b} = \Big[\prod_{i=1}^{2n} i\int \mathrm{d}^4x_i j^{\mu_i}(x_i)\Big]\,\langle 0|T\Big[\prod_{i=1}^{2n} A_{\mu_i}(x_i)\Big]|0\rangle$$

- Decompose time-ordered product into Feynman propagators, use symmetry of integrand in currents

$$\begin{aligned}\frac{W^{(2n)}_{a\to b}}{(2n)!} &= \frac{(2n-1)(2n-3)\ldots 3\cdot 1}{(2n)!}\Big[\prod_{i=1}^{2n} i\int \mathrm{d}^4x_i j^{\mu_i}(x_i)\Big] \\ &\qquad\times\prod_{i=1}^{n}\langle 0|T\left[A_{\mu_{2i}}(x_{2i})A_{\mu_{2i+1}}(x_{2i+1})\right]|0\rangle \\ &= \frac{1}{2^n n!}\left(-\int \mathrm{d}^4x\int \mathrm{d}^4y\, j^{\mu}(x) i\Delta_{\mu\nu}(x,y)j^{\nu}(y)\right)^n = \frac{1}{n!}\left(\frac{W^{(2)}_{a\to b}}{2}\right)^n .\end{aligned}$$

- Sum all orders in $\alpha \to$ vacuum persistence amplitude squared

$$|W_{a\to b}|^2 = \left|\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{W^{(2)}_{a\to b}}{2}\right)^n\right|^2 = \exp\left\{W^{(2)}_{a\to b}\right\} .$$

Semi-classical source theory – Summary

- Sudakov factor from first principles

  $$\Delta = |W_{a\to b}|^2 = \exp\left\{ W^{(2)}_{a\to b} \right\}$$

  - Resummed virtual corrections at scale μ^2
  - Logarithmic structure same as real corrections

- For Abelian theories we can also use

  $$\Delta = \exp\left\{ -\int \mathrm{d}W^{2\,(1)}_{a\to bc} \right\}$$

  - Agrees with heuristics based on probability conservation
  - Sufficient for most use cases in non-Abelian theories, but not exact
    Exercise: What is different in QCD?

- Universal, semi-classical integrand (Eikonal)

  $$\frac{2p_a p_b}{(p_a p_c)(p_b p_c)}$$

  - Leads to double logarithm $1/2\log^2(2p_a p_b/\mu^2)$
  - Originates in gauge boson radiation off conserved charge

Dipole radiation pattern

Geometric properties of semi-classical result

Structure of semi-classical matrix element

[Marchesini,Webber] NPB310(1988)461

- Matrix element can be written in terms of energies and angles

  $$\frac{2p_ap_b}{(p_ap_c)(p_cp_b)} = \frac{W_{ab,c}}{E_c^2}$$

  Angular "radiator" function

  $$W_{ab,c} = \frac{1-\cos\theta_{ab}}{(1-\cos\theta_{ac})(1-\cos\theta_{bc})}$$

- Divergent as $\theta_{ac} \to 0$ and as $\theta_{bc} \to 0$
  $\to$ Expose individual singularities using $W_{ab,c} = \tilde{W}^a_{ab,c} + \tilde{W}^b_{ba,c}$

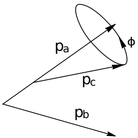
  $$\tilde{W}^a_{ab,c} = \frac{1}{2}\left[\frac{1-\cos\theta_{ab}}{(1-\cos\theta_{ac})(1-\cos\theta_{bc})} + \frac{1}{1-\cos\theta_{ac}} - \frac{1}{1-\cos\theta_{bc}}\right]$$

  - Divergent as $\theta_{ac} \to 0$, but regular as $\theta_{bc} \to 0$
  - Convenient properties upon integration over azimuthal angle

Structure of semi-classical matrix element

- Work in a frame where direction of $\vec{p}_a$ aligned with z-axis

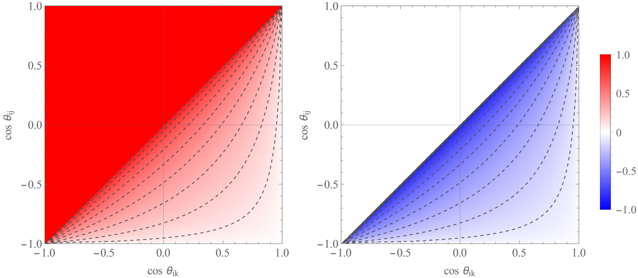
$$\cos\theta_{bc} = \cos\theta_b\cos\theta_c + \sin\theta_b\sin\theta_c\cos\phi_c$$



- Integration over ϕ_c yields

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi_c \tilde{W}^a_{ab,c} = \frac{1}{1-\cos\theta_c} \times \begin{cases} 1 & \text{if} \quad \theta_c < \theta_b \\ 0 & \text{else} \end{cases}$$

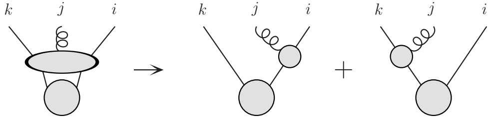
  - On average, no radiation outside cone defined by parent dipole
  - Differential radiation pattern more intricate:
    Positive & negative contributions outside cone sum to zero



Structure of semi-classical matrix element

- Alternative approach: partial fraction matrix element & match to collinear sectors [Ellis,Ross,Terrano] NPB178(1981)421, [Catani,Seymour] hep-ph/9605323

$$\frac{p_i p_k}{(p_i p_j)(p_j p_k)} \to \frac{1}{p_i p_j}\frac{p_i p_k}{(p_i+p_k)p_j} + \frac{1}{p_k p_j}\frac{p_i p_k}{(p_i+p_k)p_j}$$



  - Convenient, Lorentz invariant formulation
  - Easy to integrate and use in NLO IR subtraction
  - Captures matrix element both in angular ordered and unordered region

Seemingly ideal formulation of antenna radiation

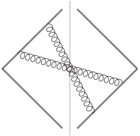
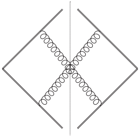
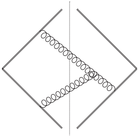
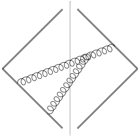
But theory still Abelian, so let's move on ...

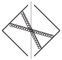
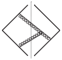
Approaching realistic QCD

Structure of non-Abelian result

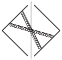
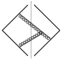
Explicit example – 2-gluon emission

- Semi-classical matrix element squared for $q(i)\bar{q}(j)g(1)g(2)$

 $+$  $+$  $+$  $+$ swaps

- Color factors
  -  $\propto \mathrm{Tr}\left[T^a T^b T^a T^b\right] = -C_F\left(\frac{C_A}{2} - C_F\right)$
  -  $\propto F^c_{ab}\mathrm{Tr}\left[T^a T^b T^c\right] = C_F\frac{C_A}{2}$

  Exercise: Where does the structure of the first term come from?

- Kinematical factors
  -  $\propto \frac{p_i p_j}{(p_i p_1)(p_1 p_j)}\frac{p_i p_j}{(p_i p_2)(p_2 p_j)}$
  -  $\propto \frac{p_i p_j}{(p_i p_1)(p_1 p_j)}\frac{p_i p_1}{(p_i p_2)(p_2 p_1)}$

  Exercise: Can you derive them?

Explicit example – 2-gluon emission

- Complete matrix element (Note: $s_{ij} = 2p_ip_j$)

$$C_F \frac{s_{ij}}{s_{i1}s_{j1}} \left(\frac{C_A}{2} \left(\frac{s_{i1}}{s_{i2}s_{12}} + \frac{s_{j1}}{s_{j2}s_{12}} \right) + \left(C_F - \frac{C_A}{2} \right) \frac{s_{ij}}{s_{i2}s_{j2}} \right)$$

  - Factorizes into first and second emission contribution
  - Non-Abelian color factors mix with Abelian kinematics
- Two important limits
  - $N_c \to \infty$, $C_A =$ const (large N_c limit):

$$\left(\frac{C_A}{2} \right)^2 \left(\frac{s_{ij}}{s_{i2}s_{12}s_{j1}} + \frac{s_{ij}}{s_{i1}s_{12}s_{j2}} \right)$$

  - $N_c \to 0$, $C_F =$ const (Abelian limit):

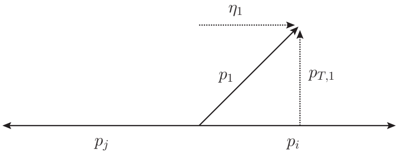
$$C_F^2 \frac{s_{ij}}{s_{i1}s_{j1}} \frac{s_{ij}}{s_{i2}s_{j2}}$$

Nice and simple formulae, but what have we learned?

Need a tool to visualize what's happening

Making sense of things – The Lund plane

- Compute everything in center-of-mass frame of quarks



- Write momenta in Sudakov decomposition

$$p_1 = p_1^+ + p_1^- + p_{T,1}$$

  - On-shell condition: $p_1^2 = 2(p_1^+ p_1^- - p_{T,1}^2)$
  - "$-$"-projection: $p_1^- = 2p_i p_1/\sqrt{2p_i p_j}$
  - "$+$"-projection: $p_1^+ = 2p_j p_1/\sqrt{2p_i p_j}$

- Simple expressions for transverse momentum and rapidity
  - $p_{T,1}^2 = \dfrac{2(p_i p_1)(p_j p_1)}{p_i p_j}$
  - $\eta_1 = \dfrac{1}{2} \ln \dfrac{p_i p_1}{p_j p_1}$
- Semi-classical abelian matrix element squared $\propto 1/p_T^2$

Making sense of things – The Lund plane

- Rewrite rapidity using transverse momentum

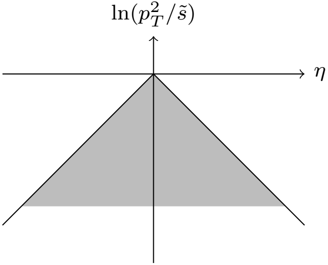
$$\eta_1 = \frac{1}{2}\ln\frac{p_i p_1}{p_j p_1} = \frac{1}{2}\ln\frac{s_{i1}^2}{p_{T,1}^2 s_{ij}} = \frac{1}{2}\ln\frac{p_{T,1}^2 s_{ij}}{s_{j1}^2}$$

- In momentum conserving parton branching $(\tilde{p}_i, \tilde{p}_j) \to (p_i, p_j, p_1)$

$$-\frac{1}{2}\ln\frac{\tilde{s}_{ij}}{p_{T,1}^2} \leq \eta_1 \leq \frac{1}{2}\ln\frac{\tilde{s}_{ij}}{p_{T,1}^2}$$

- Differential phase-space element $\propto \mathrm{d}p_T^2\,\mathrm{d}\eta$ (exercise)
- The Lund plane
  - $\eta, \ln(p_T^2/\tilde{s})$ plane
  - Phase space bounded by diagonals
  - Single-emission semi-classical radiation probability a constant

  Exercise: How do the double logarithms in the integrated matrix element emerge in the Lund plane?

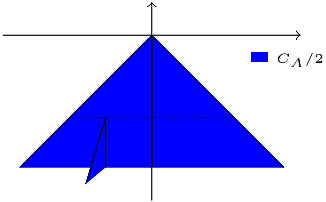


Explicit example – 2-gluon emission

- Limits of 2-gluon matrix element in Lund coordinates
  - $N_c \to \infty$, $C_A = \text{const}$ (large N_c limit):

    $$\frac{(C_A/2)^2}{p_{T,1}^{2\,(i,j)}\, p_{T,2}^{2\,(i,1)}} + \Big(i \leftrightarrow j\Big)$$

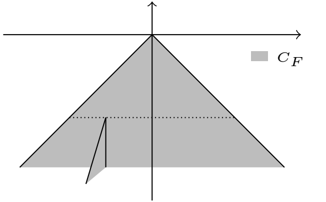
    - Gray area - C_F
    - Blue area - $C_A/2$



  - $N_c \to 0$, $C_F = \text{const}$ (Abelian limit):

    $$\frac{C_F^2}{p_{T,1}^{2\,(i,j)}\, p_{T,2}^{2\,(i,j)}}$$

    - Gray area - C_F



Explicit example – 2-gluon emission

- Full 2-guon matrix element

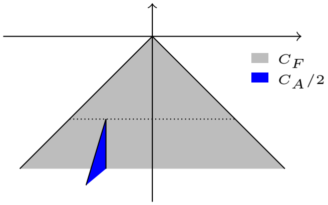
$$\frac{C_F}{p_{T,1}^2}\frac{1}{E_2^2}\left(\frac{C_A}{2}\left(\tilde{W}^i_{i1,2}+\tilde{W}^1_{i1,2}-\tilde{W}^i_{ij,2}\right)+C_F\tilde{W}^i_{ij,2}+\left(i\leftrightarrow j\right)\right)$$

- Rewrite using single-soft radiator $\bar{W}^{1,j}_{i,2}=\tilde{W}^i_{i1,2}-\tilde{W}^i_{ij,2}$

$$\frac{C_F}{p_{T,1}^2}\frac{1}{E_2^2}\left(\frac{C_A}{2}\left(\bar{W}^{1,j}_{i,2}+\tilde{W}^1_{i1,2}\right)+C_F\tilde{W}^i_{ij,2}+\left(i\leftrightarrow j\right)\right)$$

  - Azimuthally integrated $\bar{W}^{1,j}_{i,2}$ vanishes if $\theta_{i2}<\min(\theta_{i1},\theta_{ij})$
  - Azimuthally integrated $\tilde{W}^1_{i1,2}$ vanishes if $\theta_{12}>\theta_{i1}$

- For $\theta_{j1}\ll\theta_{ij}$ and $\theta_{12}>\theta_{i1}$, both $C_A/2$ terms vanish $\rightarrow$ Radiation from C_F term alone



The simplest manifestation of angular ordering in QCD

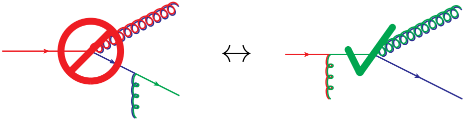
Color coherence and angular ordering

The heuristic picture

Color coherence and the dipole picture

[Marchesini,Webber] NPB310(1988)461
[Gustafsson,Pettersson] NPB306(1988)746

- Individual color charges inside a color dipole cannot be resolved if gluon wavelength larger than dipole size $\rightarrow$ emission off "mother"



- Net effect is destructive interference outside a cone with opening angle set by emitting color dipole
- Known in QED as the Chudakov effect

Let's have a look at the implementation

The phase-space integrals

Phase-space factorization

- Differential n-particle phase space element (massless partons)

$$\mathrm{d}\Phi_n(p_1,\ldots,p_n;P) = \left[\prod_{i=1}^{n}\frac{\mathrm{d}^4p_i}{(2\pi)^3}\delta(p_i^2)\right](2\pi)^4\delta^{(4)}\Big(P-\sum_i p_i\Big)$$

  - Obeys s-channel factorization formula [Byckling,Kajantie] NPB9(1969)568
- Use factorization to split off a $1\to 2$ decay

$$\mathrm{d}\Phi_n(p_1,\ldots,p_n;P) = \mathrm{d}\Phi_{n-1}(P_{12},p_3,\ldots,p_n;P)\,\frac{\mathrm{d}P_{12}^2}{2\pi}\,\mathrm{d}\Phi_2(p_1,p_2;P_{12})$$

- 2-body phase space in center-of-mass frame of light-like p_1 & p_2

$$\mathrm{d}\Phi_2(p_1,p_2;P) = \frac{1}{32\pi^2}\,\mathrm{d}\cos\theta\,\mathrm{d}\phi$$

- Rewrite in terms of light-cone momentum fraction $z=(1+\cos\theta)/2$

$$\mathrm{d}\Phi_n(p_1,\ldots,p_n;P) = \mathrm{d}\Phi_{n-1}(P_{12},p_3,\ldots,p_n;P)\,\frac{1}{16\pi^2}\,\mathrm{d}s_{12}\,\mathrm{d}z\,\frac{\mathrm{d}\phi}{2\pi}$$

  - Most parton showers evolve on-shell states into on-shell states
  - Must redefine $P_{12}\to\tilde{P}_{12}$, where $\tilde{P}_{12}^2=0$, while P^2=const

How the redefinition is achieved is to some extent arbitrary

This is referred to as the “recoil scheme”

Putting everything together

I – Angular ordered evolution

Angular ordered parton showers

- Matrix element
  $$|M|^2 = |g^2| \frac{2p_a p_b}{(p_a p_c)(p_b p_c)} + \text{spin dependent terms}$$
  - Define splitting function $2P_{ac} = 2(p_a p_c)\,|M|^2$
- Differential phase space
  $$\mathrm{d}\Phi_{+1} \approx \frac{1}{16\pi^2}\,\mathrm{d}s_{ac}\,\mathrm{d}z\,\frac{\mathrm{d}\phi}{2\pi}$$
  - Rewrite $z = \frac{1+\cos\theta_{ab}}{2} = \frac{p_a p_b}{(p_a + p_c)p_b}$
- Differential radiation probability
  $$\mathrm{d}\mathcal{P} = \mathrm{d}\Phi_{+1}|M|^2 \approx \frac{\mathrm{d}(p_a p_c)}{(p_a p_c)}\,\mathrm{d}z\,\frac{\alpha_s}{2\pi}\,P_{ac}(z) = \frac{\mathrm{d}\tilde{q}^2}{\tilde{q}^2}\,\mathrm{d}z\,\frac{\alpha_s}{2\pi}\,P_{ac}(z)$$
  - Semi-classical splitting function $P_{ac}(z) = 2C_a\,\frac{z}{1-z}$

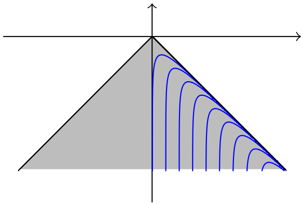
    Add spin-dependent terms for complete result in collinear limit
  - Ordering parameter $\tilde{q}^2 = \frac{2p_a p_c}{z(1-z)} \approx 4E_{ac}^2 \sin^2\frac{\theta_{ac}}{2}$

Angular ordered parton showers

- Differential radiation probability

$$\mathrm{d}\mathcal{P} = \mathrm{d}\Phi_{+1}|M|^2 = \frac{\mathrm{d}\tilde{q}^2}{\tilde{q}^2}\,\mathrm{d}z\,\frac{\alpha_s}{2\pi}\,P_{ac}(z)$$

  - Dipole radiation becomes monopole radiation
    $\rightarrow$ parton (not *dipole*) shower
  - Non-Abelian structure of QCD simplifies
    $\rightarrow$ radiation off mean charge C_F or C_A
- Lund plane filled from center to edges
  - Random walk in p_T^2
  - Color factors correct for observables insensitive to azimuthal correlations
  - Small dead zone at $\ln(p_T^2/\tilde{s}) \approx 0$



Putting everything together

II – Dipole evolution

Dipole showers

- Matrix element

$$|M|^2 = |g^2| \frac{2p_a p_b}{(p_a p_c)(p_b p_c)} + \text{spin dependent terms}$$

  - Define splitting function $P_{ac} = p_{T,c}^2 |M|^2$
- Differential phase space

$$\mathrm{d}\Phi_{+1} \approx \frac{1}{16\pi^2}\,\mathrm{d}s_{ac}\,\mathrm{d}z\,\frac{\mathrm{d}\phi}{2\pi}$$

  - Rewrite $z = 1 - \frac{s_{ac}}{\tilde{s} - s_{ac}}\,\mathrm{e}^{-2\eta_c}$
- Differential radiation probability for the dipole

$$\mathrm{d}\mathcal{P} = \mathrm{d}\Phi_{+1}|M|^2 \approx \frac{\mathrm{d}p_{T,c}^2}{p_{T,c}^2}\,\mathrm{d}\eta_c\,\frac{\alpha_s}{2\pi}\,\tilde{P}_{ac}(z)$$

  - Semi-classical splitting function $\tilde{P}_{ac}(z) = 2C_a$

    Add spin-dependent terms for complete result in collinear limit
  - Ordering parameter $p_{T,c}^2$

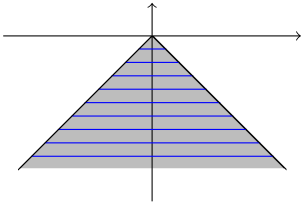
Dipole showers

- Differential radiation probability for the dipole

$$\mathrm{d}\mathcal{P} = \mathrm{d}\Phi_{+1}|M|^2 \approx \frac{\mathrm{d}p_{T,c}^2}{p_{T,c}^2}\,\mathrm{d}\eta_c\,\frac{\alpha_s}{2\pi}\,\tilde{P}_{ac}(z)$$

  - Semi-classical dipole radiation has constant probability
  - Due to ordering in $p_{T,c}^2$ no natural way to recover correct color factors ($\nearrow$ later)

- Lund plane filled from top to bottom
  - Random walk in η
  - Color factors in improved leading color approximation
  - Both ends of dipole evolve simultaneously
  - No dead zones



Putting everything together

III – Dipole-like evolution

Dipole-like showers

- Matrix element

$$|M|^2 = |g^2| \frac{2p_a p_b}{(p_a p_c)(p_b p_c)} + \text{spin dependent terms}$$

  - Partial fraction $|M|^2 = |g^2| \frac{1}{p_a p_c} \frac{2p_a p_b}{(p_a + p_b)p_c} + \Big(a \leftrightarrow b\Big)$
  - Define splitting function $2P_{ac} = 2|g^2| \frac{2p_a p_b}{(p_a + p_b)p_c}$

- Differential phase space

$$\mathrm{d}\Phi_{+1} \approx \frac{1}{16\pi^2} \, \mathrm{d}s_{ac} \, \mathrm{d}z \, \frac{\mathrm{d}\phi}{2\pi}$$

  - Rewrite $z = \frac{1 + \cos\theta_{ab}}{2} = \frac{p_a p_b}{(p_a + p_c)p_b}$

- Differential radiation probability

$$\mathrm{d}\mathcal{P} = \mathrm{d}\Phi_{+1}|M|^2 \approx \frac{\mathrm{d}p_{T,c}^2}{p_{T,c}^2} \, \mathrm{d}z \, \frac{\alpha_s}{2\pi} \, \bar{P}_{ac}(z)$$

  - Semi-classical splitting function $\bar{P}_{ac}(z) = 2C_a \left(\frac{1-z}{(1-z)^2 + p_{T,c}^2/\tilde{s}} - 1 \right)$

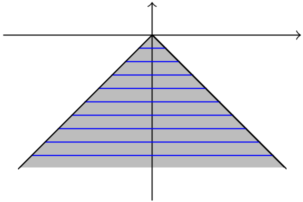
    Add spin-dependent terms for complete result in collinear limit
  - Ordering parameter $p_{T,c}^2$

Dipole-like showers

- Differential radiation probability

$$\mathrm{d}\mathcal{P} = \mathrm{d}\Phi_{+1}|M|^2 \approx \frac{\mathrm{d}p_{T,c}^2}{p_{T,c}^2}\,\mathrm{d}z\,\frac{\alpha_s}{2\pi}\,\bar{P}_{ac}(z)$$

  - Unified picture of parton and dipole evolution
  - Due to ordering in $p_{T,c}^2$ no natural way to recover correct color factors ($\nearrow$ later)
- Lund plane filled from top to bottom
  - Random walk in η
  - Color factors in improved leading color approximation
  - No dead zones

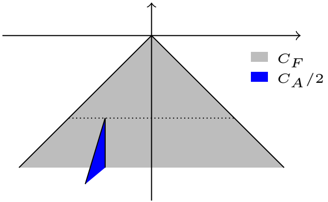


How to color the Lund plane

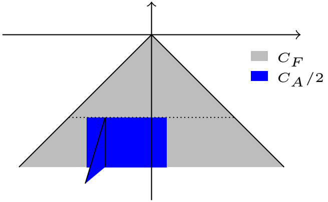
Multiple emission pattern of showers

Radiation pattern of angular ordered and dipole showers

- In angular ordered showers angles are measured in the event center-of-mass frame $\rightarrow$ coherence effects modeled by angular ordering variable agree on average with matrix element



- In dipole-like showers angles effectively measured in center-of-mass frame of emitting color dipole $\rightarrow$ angular coherence not reflected by setting average QCD charge

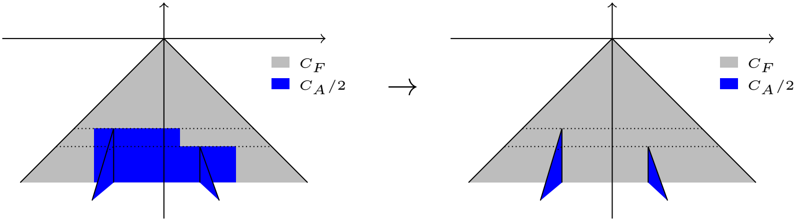


- Emission off "back plane" in Lund diagram should be associated with C_F, but is partly associated with $C_A/2$ in dipole showers
- All-orders problem that appears first in 2-gluon emission case

Correcting the radiation pattern of dipole showers

[Gustafsson] NPB392(1993)251

- Analyze rapidity of gluon emission in event center-of-mass frame
- Sectorize phase space and assign gluon to closest parton
  $\rightarrow$ choose corresponding color charge for evolution
- Same technology for higher number of emissions



- Starting with 4 emissions, there be "color monsters"
  - Quartic Casimir operators (easy)
  - Non-factorizable contributions (hard)

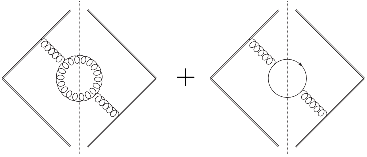
  Not captured in either angular ordered or corrected dipole evolution

Universal higher-order corrections

The CMW scheme

Soft-collinear enhanced terms at NLO

- Approximate soft-gluon emission times collinear decay in $q(i)\bar{q}(j)g(1)g(2)$ using semi-classical limit and gluon splitting function



$$= \sum_{b=q,g} j_{ij,\mu}(p_{12}) j_{ij,\nu}(p_{12}) \, \frac{P^{\mu\nu}_{gb}(z_1)}{s_{12}}$$

$$P^{\mu\nu}_{gq}(z) = T_R \left(-g^{\mu\nu} + 4\, z(1-z)\, \frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2} \right)$$

$$P^{\mu\nu}_{gg}(z) = C_A \left(-g^{\mu\nu} \left(\frac{z}{1-z} + \frac{1-z}{z} \right) - 2\,(1-\varepsilon) z(1-z)\, \frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2} \right)$$

- Combine with phase space for one parton emission in collinear limit $D = 4 - 2\varepsilon$, $y = s_{12}/Q^2$, see for example [Catani,Seymour] hep-ph/9605323

$$\mathrm{d}\Phi_{+1} = \frac{Q^{2-2\varepsilon}}{16\pi^2} \frac{(4\pi)^\varepsilon}{\Gamma(1-\varepsilon)} \, \mathrm{d}y \, \mathrm{d}z \, \big[y\, z(1-z) \big]^{-\varepsilon}$$

- Perform Laurent series expansion

$$\frac{1}{y^{1+\varepsilon}} = -\frac{\delta(y)}{\varepsilon} + \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \left(\frac{\ln^n y}{y} \right)_+$$

Soft-collinear enhanced terms at NLO

- $\mathcal{O}(\varepsilon^0)$ remainder terms proportional to

$$g \to q\bar{q}: \quad T_R \left[2z(1-z) + \big(1 - 2z(1-z)\big)\ln(z(1-z))\right]$$

$$g \to gg: \quad 2C_A \left[\frac{\ln z}{1-z} + \frac{\ln(1-z)}{z} + \big(-2 + z(1-z)\big)\ln(z(1-z))\right]$$

- Integration over z gives

$$\left(\frac{67}{18} - \frac{\pi^2}{3}\right) C_A - \frac{10}{9} T_R n_f$$

  - Some additional terms from semi-classical diagrams
  - Contribution from exact virtual correction (no unitarity!)
  - Only π^2 term changed (identical to $\mathcal{N} = 4$ SYM)

- Sums to two-loop cusp anomalous dimension

$$K = \left(\frac{67}{18} - \frac{\pi^2}{6}\right) C_A - \frac{10}{9} T_R n_f$$

  - Local K-factor for soft-gluon emission
  - Scheme dependent: originates in dim. reg. and $\overline{\mathrm{MS}}$

K can be absorbed into an effective coupling

This is called the CMW scheme [Catani,Marchesini,Webber] NPB349(1991)635

Connection to analytic resummation

Event shapes at NLL accuracy

How to assess formal precision?

- Angular ordered parton showers are proven to be NLL accurate for certain observables, provided that the CMW scheme is used
- But how do we quantify this for other showers?
  Can we establish a limit where parton showers should reproduce NLL exactly?
- Let's use a well-established result as an example
  - Observable: Thrust in $e^+e^- \to$hadrons
  - Method: Caesar [Banfi,Salam,Zanderighi] hep-ph/0407286

This discussion will be quite technical, so why have it at all?

Because the relevant limit is the $\alpha_s \to 0$ limit.

Sounds pretty unphysical, so it's definitely worth a closer look!

NLL resummation for simple additive observables

- Contribution of one emission with momentum k to "thrust" $v = 1 - T$

$$V(k) = \left(\frac{k_T}{Q}\right) e^{-\eta} \qquad \to \qquad V(\{p\},\{k\}) = \sum_i V(k_i)$$

  where k_T, $\eta = \log((1-z)Q/k_T) \to$ Lund coordinates of soft-gluon momentum
- Define a shower evolution variable $\xi = k_T^2/(1-z)$
- Integrated one-emission probability for $\xi > Q^2 v$

$$R_{\rm PS}(v) = 2\int_{Q^2 v}^{Q^2} \frac{d\xi}{\xi} \int_{z_{\min}}^{z_{\max}} dz\, \frac{\alpha_s\left(k_T^2\right)}{2\pi}\, C_F \left[\frac{2}{1-z} - (1+z)\right] \Theta(\eta)$$

  z-limits from momentum conservation, $\Theta(\eta)$ implements angular ordering
- Approximate to NLL accuracy

$$R_{\rm NLL}(v) = 2\int_{Q^2 v}^{Q^2} \frac{d\xi}{\xi} \left[\int_0^1 dz\, \frac{\alpha_s\left(k_T^2\right)}{2\pi} \frac{2\,C_F}{1-z} \Theta(\eta) - \frac{\alpha_s(\xi)}{\pi}\, C_F B_q\right]$$

  Exercise: Can you derive the value of B_q?

Origin of the $\alpha_s \to 0$ limit – The $\mathcal{F}$ function

- Define the cumulative cross section $\Sigma(v)$

$$\Sigma\left(v\right) = e^{-R(v)}\mathcal{F}\left(v\right)$$

- Obtained from the all-orders resummed result

$$\Sigma(v) = \int \mathrm{d}^3k_1|M(k_1)|^2 \exp\left\{-\int_{\varepsilon v_1} \mathrm{d}^3k|M(k)|^2\right\}$$
$$\times \sum_{m=0}^{\infty} \frac{1}{m!}\left(\prod_{i=2}^{m+1}\int_{\varepsilon v_1}^{v_1} \mathrm{d}^3k_i|M(k_i)|^2\right)\Theta\Big(v - V(\{p\}, k_1, \ldots, k_n)\Big)$$

by Taylor expansion of virtual corrections in ε

$$\exp\left\{-\int_{\varepsilon v_1} \mathrm{d}^3k|M(k)|^2\right\} = e^{-R(v)}\, e^{-R' \ln \frac{v}{\varepsilon v_1}}$$

- Definition of $\mathcal{F}(v)$

$$\mathcal{F}(v) = \int \mathrm{d}^3k_1|M(k_1)|^2\, e^{-R' \ln \frac{v}{\varepsilon v_1}} \sum_{m=0}^{\infty} \frac{1}{m!}\left(\prod_{i=2}^{m+1}\int_{\varepsilon v_1}^{v_1} \mathrm{d}^3k_i|M(k_i)|^2\right)$$
$$\times\, \Theta\Big(v - V(\{p\}, k_1, \ldots, k_n)\Big)$$

  - Purely NLL (no leading logarithms!)
  - Accounts for multiple-emission effects

Origin of the $\alpha_s \to 0$ limit – The $\mathcal{F}$ function

- In order to make this calculable, make the following approximations
  - Observable is recursively infrared and collinear safe
    $\to$ Can scale phase space $\int_{\varepsilon v_1}^{v_1} \to \int_{\varepsilon v}^{v}$
  - **Hold $\alpha_s(Q^2) \ln v$ fixed, while taking the limit $v \to 0$**
    $\to$ Can factorize integrals and neglect kinematic edge effects
- Reduces $\mathcal{F}$-function to convenient form

$$\mathcal{F}(v) = e^{R'(v) \ln \epsilon} \sum_{m=0}^{\infty} \frac{1}{m!} \left(\prod_{i=1}^{m} R'(v) \int_{\epsilon}^{1} \frac{d\zeta_i}{\zeta_i} \right) \Theta\Big(1 - \sum_{j=1}^{m} \zeta_j \Big)$$

  - For thrust and similar observales, $\mathcal{F}(v) = \dfrac{e^{-\gamma_E R'}}{\Gamma(1+R')}$

Remarkably simple and clean (no NNLL contamination)

Could only be achieved because of the limit $v \to 0 \;/\; \alpha_s \to 0$

$\alpha_s \to 0$ benchmark tests *exactly* NLL, nothing less or more

Differences between pure NLL and parton shower

[Reichelt,Siegert,SH] arXiv:1711.03497

- Schematic difference between analytic resummation and parton shower
  - $\Sigma_{\rm NLL}(v)$ determined at exactly NLL
  - $\Sigma_{\rm PS}(v)$ determined by unitarity
- One can find a unified NLL/PS expression for $R(v)$ and $\Sigma(v)$

$$\Sigma\,(v) = \exp\left\{ -\int_v \frac{d\xi}{\xi} R'_{>v}(\xi) - \int_{v_{\min}}^{v} \frac{d\xi}{\xi} R'_{<v}(\xi) \right\} \times \sum_{m=0}^{\infty} \frac{1}{m!} \left(\prod_{i=1}^{m} \int_{v_{\min}} \frac{d\xi_i}{\xi_i} R'_{<v}(\xi_i) \right) \Theta\left(v - \sum_{j=1}^{m} V(\xi_i) \right)$$

where

$$R'_{\lessgtr v}(\xi) = \frac{\alpha_s^{\lessgtr v,\mathrm{soft}}(\mu^2_{\lessgtr})}{\pi} \int_{z^{\min}}^{z^{\max}_{\lessgtr v,\mathrm{soft}}} dz\, \frac{C_{\rm F}}{1-z} - \frac{\alpha_s^{\lessgtr v,\mathrm{coll}}(\mu^2_{\lessgtr v})}{\pi} \int_{z^{\min}}^{z^{\max}_{\lessgtr v,\mathrm{coll}}} dz\, C_{\rm F} \frac{1+z}{2}$$

Differences between pure NLL and parton shower

- Isolated differences in terms of resolved/unresolved splitting probability:

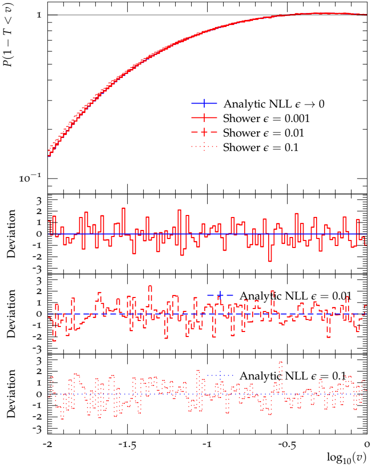
$$R'_{\lessgtr v}(\xi) = \frac{\alpha_s^{\lessgtr v,\mathrm{soft}}(\mu^2_{\lessgtr})}{\pi} \int_{z^{\min}}^{z^{\max}_{\lessgtr v,\mathrm{soft}}} dz\, \frac{C_\mathrm{F}}{1-z} - \frac{\alpha_s^{\lessgtr v,\mathrm{coll}}(\mu^2_{\lessgtr v})}{\pi} \int_{z^{\min}}^{z^{\max}_{\lessgtr v,\mathrm{coll}}} dz\, C_\mathrm{F}\, \frac{1+z}{2}$$

	NLL	Parton Shower
$z^{\max}_{>v,\mathrm{soft}}$	$1-(\xi/Q^2)^{\frac{a+b}{2a}}$	
$\mu^2_{>v,\mathrm{soft}}$	$\xi(1-z)^{\frac{2b}{a+b}}$	
$\alpha_s^{>v,\mathrm{soft}}$	2-loop CMW	
$z^{\max}_{<v,\mathrm{soft}}$	$1-v^{\frac{1}{a}}$	$1-(\xi/Q^2)^{\frac{a+b}{2a}}$
$\mu^2_{<v,\mathrm{soft}}$	$Q^2 v^{\frac{2}{a+b}}(1-z)^{\frac{2b}{a+b}}$	$\xi(1-z)^{\frac{2b}{a+b}}$
$\alpha_s^{<v,\mathrm{soft}}$	1-loop	2-loop CMW

	NLL	Parton Shower
$z^{\max}_{>v,\mathrm{coll}}$	1	$1-(\xi/Q^2)^{\frac{a+b}{2a}}$
$\mu^2_{>v,\mathrm{coll}}$	ξ	$\xi(1-z)^{\frac{2b}{a+b}}$
$\alpha_s^{>v,\mathrm{coll}}$	1-loop	2-loop CMW
$z^{\max}_{<v,\mathrm{coll}}$	0	$1-(\xi/Q^2)^{\frac{a+b}{2a}}$
$\mu^2_{<v,\mathrm{coll}}$	n.a.	$\xi(1-z)^{\frac{2b}{a+b}}$
$\alpha_s^{<v,\mathrm{coll}}$	n.a.	2-loop CMW

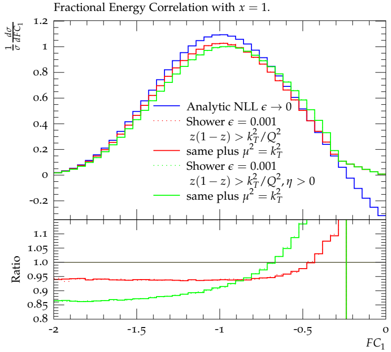
- Can cast pure NLL into PS language by using NLL expressions in PS
- Can study each effect in detail by reverting changes back to PS

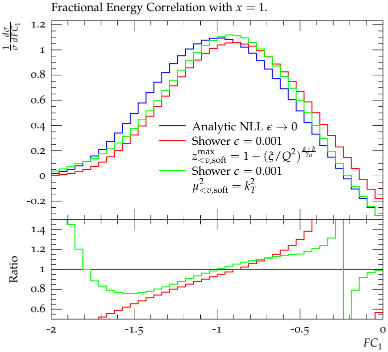
Implementing NLL resummation as a shower



- Modified parton shower exactly reproduces pure NLL result
- $E_{\rm cms}$=91.2 GeV, $\alpha_s(M_Z) = 0.118$ fixed flavor $n_f = 5$

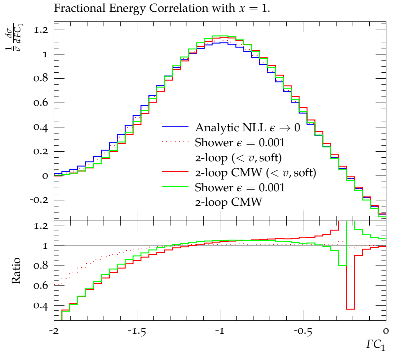
Local four momentum conservation and unitarity

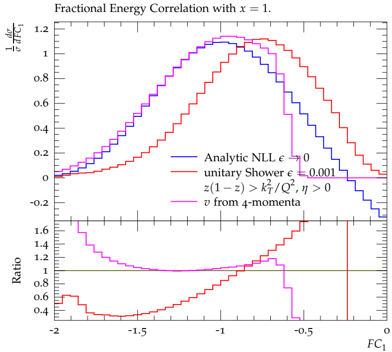




- NLL→PS in $z_{\mathrm{min/max}}$ (4-momentum conservation)
- NLL→PS in $z^{\max}_{>v,\mathrm{coll}}$ (phase-space sectorization)
- NLL→PS in $\mu^2_{>v,\mathrm{coll}}$ (conventional)
- NLL→PS in $z^{\max}_{<v,\mathrm{soft}}$ (from PS unitarity)
- NLL→PS in $\mu^2_{<v,\mathrm{soft}}$ (from PS unitarity)

Running coupling and global momentum conservation





- NLL→PS in 2-loop CMW $< v, \mathrm{soft}$ (from PS unitarity)
- NLL→PS in 2-loop CMW overall (conventional)

- NLL→PS in observable (use experimental definition)

Overall assessment

- Simplest process and simplest observable,
  still sizable differences away from $v \to 0$ limit
- Due to kinematic edge effects & unitarity
- At NLL, none of the methods is formally better
  $\to$ Difference is a true systematic uncertainty

The $\alpha_s \to 0$ limit is mandatory for exact comparison

Away from this limit there are important systematic effects

Problems with recoil

Correcting the momentum mapping

Momentum mapping in dipole-like showers

[Dasgupta,Dreyer,Hamilton,Monni,Salam] arXiv:1805.09327

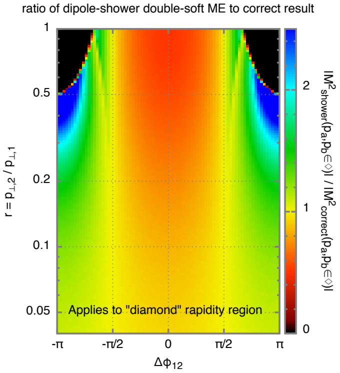
- Recently identified problem with standard dipole-like recoil

$$p_k^\mu = \left(1 - \frac{p_{ij}^2}{2\tilde{p}_{ij}\tilde{p}_k}\right)\tilde{p}_k^\mu$$

$$p_i^\mu = \tilde{z}\,\tilde{p}_{ij}^\mu + (1-\tilde{z})\frac{p_{ij}^2}{2\tilde{p}_{ij}\tilde{p}_k}\tilde{p}_k^\mu + k_\perp^\mu$$

$$p_j^\mu = (1-\tilde{z})\,\tilde{p}_{ij}^\mu + \tilde{z}\frac{p_{ij}^2}{2\tilde{p}_{ij}\tilde{p}_k}\tilde{p}_k^\mu - k_\perp^\mu$$

- Angular correlations across multiple emissions due to recoil on splitter in anti-collinear region
- Spoils $\alpha_s \to 0$ consistency check



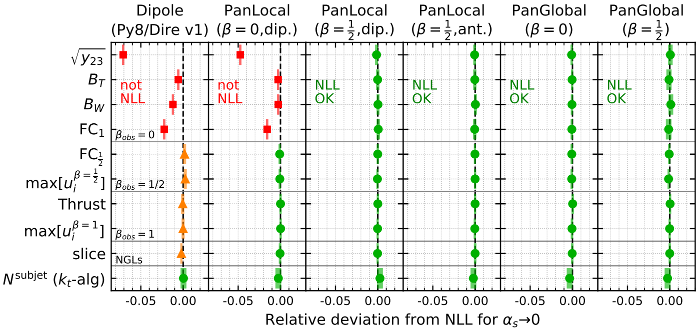
Momentum mapping in dipole-like showers

[Dasgupta,Dreyer,Hamilton,Monni,Salam,Soyez] arXiv:2002.11114

- Problem can be solved by partitioning of antenna radiation pattern and choosing a suitable evolution variable ($\beta \sim 1/2$)

$$k_T = \rho v e^{\beta|\bar{\eta}|} \qquad \rho = \left(\frac{s_i s_j}{Q^2 s_{ij}}\right)^{\beta/2}$$

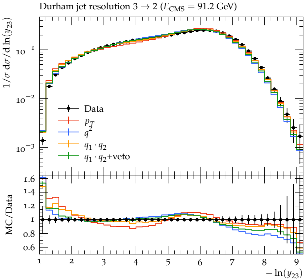
- Three different recoil schemes lead to NLL result if β chosen appropriately: Local dipole, local antenna, and global antenna
- NLL correct for global and non-global observables in $e^+e^- \to$hadrons

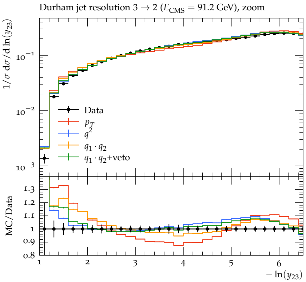


Momentum mapping in angular ordered showers

[Bewick,Ferrario-Ravasio,Richardson,Seymour] arXiv:1904.11866

- Recoil schemes affect logarithmic accuracy but impact also phase-space coverage
- In context of angular ordered Herwig 7 (NLL accurate for global observables)
  - q_T preserving scheme:
    Maintains logarithmic accuracy
    Overpopulates hard region
  - q^2 preserving scheme:
    Breaks logarithmic accuracy
    Good description of hard region
  - Dot product preserving scheme (new):
    Maintains logarithmic accuracy
    Good description of hard radiation





Analytic properties of branching equations

Forward vs. backward evolution

Properties of splitting kernels

- At any order of perturbation theory, splitting functions obey sum rules

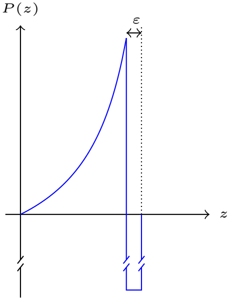
$$\int_0^1 \mathrm{d}\zeta\, \hat{P}_{qq}(\zeta) = 0 \qquad \rightarrow \qquad \text{flavor sum rule}$$

$$\sum_{c=q,g} \int_0^1 \mathrm{d}\zeta\, \zeta\, \hat{P}_{ac}(\zeta) = 0 \qquad \rightarrow \qquad \text{momentum sum rule}$$

$\rightarrow$ defines regularized splitting functions $\hat{P}_{ab}$ as

$$\hat{P}_{ab}(z) = \lim_{\varepsilon \to 0} \left[P_{ab}(z)\Theta(1-\varepsilon-z) - \delta_{ab} \frac{\Theta(z-1+\varepsilon)}{\varepsilon} \sum_{c=q,g} \int_0^{1-\varepsilon} d\zeta\, \zeta\, P_{ac}(\zeta) \right]$$

- What does that mean in physics terms?
  - Contribution $\propto \Theta(1-\varepsilon-z)$ corresponds to real-emission corrention
  - Contribution $\propto \Theta(z-1+\varepsilon)$ corresponds to virtual correction
  - Momentum sum rule is a unitarity constraint



Relation between parton shower and DGLAP evolution

- DGLAP equation for fragmentation functions

$$\frac{\mathrm{d}\, x D_a(x,t)}{\mathrm{d}\ln t} = \sum_{b=q,g} \int_0^1 \mathrm{d}\tau \int_0^1 \mathrm{d}z\, \frac{\alpha_s}{2\pi} \left[z P_{ab}(z) \right]_+ \tau D_b(\tau,t)\, \delta(x - \tau z)$$

- Refine plus prescription $\left[z P_{ab}(z) \right]_+ = \lim_{\varepsilon \to 0} z P_{ab}(z,\varepsilon)$

$$P_{ab}(z,\varepsilon) = P_{ab}(z)\, \Theta(1-\varepsilon-z) - \delta_{ab} \sum_{c\in\{q,g\}} \frac{\Theta(z-1+\varepsilon)}{\varepsilon} \int_0^{1-\varepsilon} \mathrm{d}\zeta\, \zeta\, P_{ac}(\zeta)$$

- Rewrite for finite ε

$$\frac{\mathrm{d}\ln D_a(x,t)}{\mathrm{d}\ln t} = - \sum_{c=q,g} \int_0^{1-\varepsilon} \mathrm{d}\zeta\, \zeta\, \frac{\alpha_s}{2\pi} P_{ac}(\zeta) + \sum_{b=q,g} \int_x^{1-\varepsilon} \frac{\mathrm{d}z}{z} \frac{\alpha_s}{2\pi} P_{ab}(z) \frac{D_b(x/z,t)}{D_a(x,t)}$$

- First term is derivative of Sudakov factor $\Delta = \exp\{-\lambda\}$

$$\Delta_a(t,Q^2) = \exp\left\{ - \int_t^{Q^2} \frac{\mathrm{d}\bar{t}}{\bar{t}} \sum_{c=q,g} \int_0^{1-\varepsilon} \mathrm{d}\zeta\, \zeta\, \frac{\alpha_s}{2\pi} P_{ac}(\zeta) \right\}$$

Relation between parton shower and DGLAP evolution

- Use generating function $\Pi_a(x,t,Q^2) = D_a(x,t)\Delta_a(t,Q^2)$ to write

$$\frac{\mathrm{d}\ln\Pi_a(x,t,Q^2)}{\mathrm{d}\ln t/Q^2} = \sum_{b=q,g}\int_x^{1-\varepsilon}\frac{\mathrm{d}z}{z}\,\frac{\alpha_s}{2\pi}\,P_{ab}(z)\,\frac{D_b(x/z,t)}{D_a(x,t)}\;.$$

- If hadron not resolved, obtain

$$\frac{\mathrm{d}}{\mathrm{d}\ln t/Q^2}\ln\left(\frac{\Pi_a(x,t,Q^2)}{D_a(x,t)}\right) = \frac{\mathrm{d}\Delta_a(t,Q^2)}{\mathrm{d}\ln t/Q^2} = \sum_{b=q,g}\int_0^{1-\varepsilon}\mathrm{d}z\,z\,\frac{\alpha_s}{2\pi}\,P_{ab}(z)$$

- Survival probabilities for one parton between scales t_1 and t_2:
  - $\dfrac{\Pi_a(x,t_2,Q^2)}{\Pi_a(x,t_1,Q^2)}$ Resolved hadron $\leftrightarrow$ constrained (backward) evolution
  - $\dfrac{\Delta_a(t_2,Q^2)}{\Delta_a(t_1,Q^2)}$ No resolved hadron $\leftrightarrow$ unconstrained (forward) evolution
- Parton-showers draw t_2-points starting from t_1 based on these probabilities

See heuristic introduction and tutorial for how to do this in practice

Summary of this lecture

- Parton showers are a topic of intense research, and they are expected to remain so as they provide the only effective means to simulate fully differential events with QCD radiation at both high and low scales
- The comparison with analytic resummation provides provides new, important constraints on old algorithms. Away from the $\alpha_s \to 0$ limit differences appear due to momentum and probability conservation
- The extension of parton showers to higher perturbative orders and to higher logarithmic accuracy as well as higher accuracy in the $1/N_c$ expansion will be an important step towards high-precision event simulation at the HL-LHC and future colliders