Soft Theorem from SCET Gravity

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Why SCET Gravity?

- Recent interest in a systematic construction
  [Beneke, Kirilin 1207.4926; Okui, Yunesi 1710.07685, + Chakraborty 1910.10738]
- So far no rigorous derivation of subleading Lagrangian
- Hints of a universal structure of soft physics
  - Low-Burnett-Kroll (LBK) and Soft Theorem:
    [Low 1958; Weinberg 1965; Burnett, Kroll 1968; Cachazo, Strominger 1404.4091]

$$\mathcal{A}^{\gamma}_{\rm rad} = \sum_i Q_i \left(\frac{\varepsilon_\mu p_i^\mu}{p_i \cdot k} + \frac{k_\nu \varepsilon_\mu J_i^{\mu\nu}}{p_i \cdot k} \right) \mathcal{A}_0$$

$$\mathcal{A}^{h}_{\rm rad} = -\frac{\kappa}{2} \sum_i \left(\frac{\varepsilon_{\mu\nu} p_i^\mu p_i^\nu}{p_i \cdot k} + \frac{k_\nu \varepsilon_{\mu\rho} p_i^\rho J_i^{\mu\nu}}{p_i \cdot k} + \frac{1}{2} \frac{k_\rho k_\sigma \varepsilon_{\mu\nu} J^{\mu\rho} J^{\nu\sigma}}{p_i \cdot k} \right) \mathcal{A}_0$$

  - Obscured in diagrammatic derivation
  - Understand universality and similarities within EFT treatment?
    [Larkoski, Neill, Stewart 1412.3108; Beneke, Garny, Szafron, Wang 1712.07462]

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Revision: scalar QCD

- Goal: obtain effective theory as power series in $\lambda \sim |\frac{p_\perp}{n_+p}|$ for collinear momenta $(n_+p, p_\perp, n_-p) \sim (1, \lambda, \lambda^2)$
- Use position-space formalism

  [Beneke, Chapovsky, Diehl, Feldmann hep-ph/0206152; Beneke, Feldmann hep-ph/211358]
- Full theory: split fields into soft and collinear modes.
  - Field content $A_c \sim (1, \lambda, \lambda^2), \phi_c \sim \lambda, A_s \sim \lambda^2, \phi_s \sim \lambda^2$
  - Duplicate gauge-symmetry, soft gauge field acts as background field

$$\hat{A}_c \to U_c \hat{A}_c U_c^\dagger + \frac{i}{g} U_c \left[D_s, U_c^\dagger \right] ,$$

- To control $n_+A_c \sim \lambda^0$, use collinear light-cone gauge $n_+\mathcal{A} = 0$
  - n_+A_c appears only in Wilson lines
  - Collinear fields appear manifestly gauge-invariant in sources

- Soft fields must be multipole expanded around $x_- = n_+ x \frac{n_-}{2}$
  - Equivalent to expanding small soft momenta in vertex
  - Gauge-symmetry must respect multipole expansion
  - New collinear fields necessary

$$\hat{A}_c \to U_c \hat{A}_c U_c^\dagger + \frac{i}{g} U_c \left[\hat{D}_s, U_c^\dagger \right] ,$$

  where $\hat{D}_s = \partial - ig \frac{n_+}{2} n_- A_s(x_-)$
  - This is equivalent to evaluating the soft field A_s in fixed-line gauge (light-cone analogue of Fock-Schwinger)

- Subleading interactions expressed via field-strength tensor, e.g.

$$x_\perp^\mu n_-^\nu \left[\partial_\mu A_\nu\right](x_-) n_+ J_c(x) \to x_\perp^\mu n_-^\nu F_{\mu\nu} n_+ J_c(x)$$

  - Compare to non-relativistic theory: $n_-^\mu \to \delta_0^\mu$

$$x_\perp^\mu F_{\mu -} J_+ \to x^i F^{i0} J^0 \sim \vec{x} \cdot \vec{E}\, J^0$$

  $\to$ Dipole interaction

Key Observation

By homogenising the gauge symmetry, we find a theory where all interactions due to multipole expansion are expressed via field-strength tensor. This gives a very transparent structure how gauge symmetry is organised in EFT.

Soft-collinear matter Lagrangian takes the simple structure

$$\mathcal{L}^{(0)} = \frac{1}{2} n_+ D_c \bar{\Phi} n_- D \Phi + \frac{1}{2} D_{c\perp} \bar{\Phi} D_{c\perp} \Phi + \text{h.c.}$$

$$\mathcal{L}^{(1)} = +\frac{1}{2} x_\perp^\mu n_-^\nu g F_{\mu\nu}^s n_+ J_c$$

$$\mathcal{L}^{(2)} = x_\perp^\mu g F_{\mu\nu}^s J_{c\perp}^\nu + \frac{1}{4} n_- x n_+^\mu n_-^\nu g F_{\mu\nu}^s n_+ J_c + \frac{1}{4} x_\perp^\mu x_\perp^\rho n_-^\nu g \left[D_\rho^s, F_{\mu\nu}^s \right] n_+ J_c$$

with $J_c^\mu = \bar{\Phi}_c \overleftarrow{D_c^\mu} \Phi_c + \bar{\Phi}_c \overrightarrow{D_c^\mu} \Phi_c$

What about Gravity?

- Can we bring Gravity into this transparent form?
- Based on soft theorem, we expect this to work
- Goal: formulate SCET Gravity in the framework of multipole expansion, to work out similarities and differences of both theories

QCD vs Gravity

	QCD	Gravity
Fundamental Degree of Freedom	$A_\mu \sim p_\mu$	$h_{\mu\nu} \sim \frac{p_\mu p_\nu}{\lambda}$
Field-strength / curvature	$F_{\mu\nu} \sim \partial A$	$R^\mu{}_{\nu\alpha\beta} \sim \partial^2 h$
Gauge Symmetry	$SU(3)$	Diff(M)
Coupling Dimensionful?	no	yes

Two Sources of Inhomogeneity

- In full theory: gauge charges P^μ and coupling κ are inhomogeneous in λ
  - Leads to relations for higher-order terms to form geometric objects
  - This is different from QCD – gauge charges have no scaling in λ

- From multipole expansion: evaluate soft fields at x_-.
  - Conceptually the same as in gauge theory
  - Deal with it in similar fashion

SCET Gravity Construction

- Minimally-coupled scalar field

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

- Perform κ expansion in collinear sector $g_{\mu\nu} = g_{s\mu\nu} + \kappa h_{\mu\nu}$
  - Collinear graviton $h_{\mu\nu}$ in presence of soft dynamical background $g_{s\mu\nu}$
  - Duplicate soft and collinear gauge symmetry, not yet homogeneous

- Field content $h_{\mu\nu}$, $g_{s\mu\nu} = \eta_{\mu\nu} + \kappa s_{\mu\nu}$, ϕ_c, ϕ_s

- Introduce collinear light-cone gauge $\mathfrak{h}_{+\mu} = 0$
  - $h_{+\mu}$ only appear in Wilson lines
  - Controls dangerous scaling $h_{++} \sim \lambda^{-1}$, $h_{+\perp} \sim \lambda^0$
    [Beneke, Kirilin 1207.4926; Okui, Yunesi 1710.07685, + Chakraborty 1910.10738]

Soft Multipole-expansion

- Homogeneous gauge symmetry: linear transformations in $(x - x_-)$

$$x^{\mu\prime} = x^\mu + \varepsilon^\mu(x_-) + \omega^\mu{}_\nu(x_-)(x - x_-)^\nu + \mathcal{O}(\varepsilon^2)$$

where $\omega_{\mu\nu} = \frac{1}{2}(\partial_\mu \varepsilon_\nu - \partial_\nu \varepsilon_\mu)$

- Specify a new soft background field $\hat{g}_{s\mu\nu}$
  - Light-cone generalisation of Riemann Normal Coordinates

- Should treat ε^μ and $\omega_{\mu\nu}$ as *independent* parameters

- Two fields appear in covariant derivative

- Systematically achieved by analogue of Wilson lines

Main Takeaway

"Homogeneous" symmetry in Gravity consists of local translations and local Lorentz transformations. This implies a covariant derivative that contains the momentum as well as the Lorentz generators. All other interaction terms are expressed via Riemann tensor and its derivative.

Schematically, the scalar-soft graviton Lagrangian takes the form

$$\mathcal{L}_{\phi\phi s} = \frac{1}{2} n_+ \partial\phi n_- D_s \phi + \frac{1}{2}\partial_\perp\phi\partial_\perp\phi - \frac{\kappa}{8} x_\perp^\alpha x_\perp^\beta R_{\alpha-\beta-} n_+\phi n_+\phi + \mathcal{O}(\lambda^3)\,,$$

where

$$n_- D_s = n_-\partial - \underbrace{\frac{\kappa}{2} s_{-\alpha}\partial^\alpha}_{\text{from vierbein}} - \underbrace{\frac{\kappa}{4}(\partial_\alpha s_{\beta-} - \partial_\beta s_{\alpha-})}_{\text{from spin-connection}} J^{\alpha\beta} + \mathcal{O}(s^2)$$

$$J^{\alpha\beta} = (x-x_-)^\alpha\partial^\beta - (x-x_-)^\beta\partial^\alpha$$

This is the transparent form we wanted, similar to QCD.

Towards the Soft Theorem

What we have learned:

- Gravity can be brought into the same multipole form as QCD
- Gauge symmetry consists of local translations and Lorentz transformations
- Covariant derivative consists of two fields, with charges P^μ and $J^{\mu\nu}$

Now we can tackle questions of soft radiation:

- How does the soft theorem arise?
- Why is soft emission universal?

Need to find N-jet operator basis

N-jet Amplitude and Operator Basis

- N-jet operator in SCET (hard scattering at $x = 0$)

$$J = \int \{dt_i\} C(\{t_i\}) J_s(0) \prod_i J_i(t_i n_{+i})$$

- Building blocks worked out for QCD, Gravity is similar
  [Beneke, Garny, Szafron, Wang 1712.04416, 1808.04742, 1907.05463]

- Operators J_s, J_i are individually collinear gauge-invariant, soft gauge-covariant

- Leading building blocks "$A0$-currents"

	collinear matter	collinear mediator	soft fields
QCD	$\Phi_i = W_i^\dagger \phi_i$	$\mathcal{A}_\perp = W_i^\dagger \left[iD_{\perp_i} W_i \right]$	$\phi_s(0), F^s_{\mu\nu}(0)$
	$\mathcal{O}(\lambda)$	$\mathcal{O}(\lambda)$	$\mathcal{O}(\lambda^3, \lambda^4)$
Gravity	$\Phi_i = W_i^{-1} \left[\phi_i \right]$	$\mathfrak{h}_{\perp\perp} = W_i^{-1} \left[h_{\mu\nu} \right]$	$\phi_s(0), R^s_{\mu\nu\alpha\beta}(0)$
	$\mathcal{O}(\lambda)$	$\mathcal{O}(\lambda)$	$\mathcal{O}(\lambda^3, \lambda^6)$

- $\mathcal{A}_-$, $\mathfrak{h}_{\perp -}$ and D_{s-} can be eliminated by equations of motion
- Subleading operators: act with $\partial_\perp$, or add building block of the same collinear sector
- In addition: time-ordered products with collinear Lagrangian $\mathcal{L}_i$

$$J^{T1} = i \int d^4x T\{J_i^{A0}, \mathcal{L}_i^{(1)}(x)\}$$

Observations

- The operator basis contains no soft building blocks up to
  - $F^s_{\mu\nu} \quad \sim \lambda^4 \sim k_s^2$ in QCD
  - $R^s_{\mu\nu\alpha\beta} \sim \lambda^6 \sim k_s^3$ in Gravity

- Any contribution to soft emission up to this order has to stem from the Lagrangian interactions, and is thus universal.

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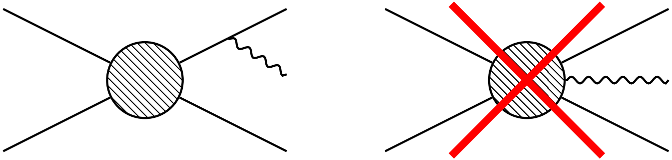

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- Any contribution to soft emission up to this order has to stem from the Lagrangian interactions, and is thus universal.
- This universality implies the soft theorem.
- The covariant derivative explains why there are three terms for Gravity.
  - First two are from $n_- D$
  - Third term is from $R_{\mu\nu\alpha\beta}$

Comparison of Soft Theorems

- Recall LBK and Soft Theorem

$$\mathcal{A}^{\gamma}_{\text{rad}} = -g \sum_i Q_i \left(\varepsilon_\mu \frac{p_i^\mu}{p_i \cdot k} + \frac{k_\nu \varepsilon_\mu J_i^{\mu\nu}}{p_i \cdot k} \right) \mathcal{A}_0$$

$$\mathcal{A}^{h}_{\text{rad}} = -\frac{\kappa}{2} \sum_i \left(\left(p_i^\nu \varepsilon_{\nu\mu} + J_i^{\nu\rho} k_\rho \varepsilon_{\nu\mu} \right) \frac{p_i^\mu}{p_i \cdot k} + \frac{1}{2} \frac{k_\rho k_\sigma \varepsilon_{\mu\nu} J^{\mu\rho} J^{\nu\sigma}}{p_i \cdot k} \right) \mathcal{A}_0$$

- First terms explained by covariant derivative

$$(n_- D_s)_{\mathsf{QCD}} = n_- \partial + i g n_- A_s^a Q^a$$

$$(n_- D_s)_{\mathsf{grav}} = n_- \partial - \frac{\kappa}{2} s_{-\nu} \partial^\nu - \frac{\kappa}{4} \omega_{\nu\rho} J^{\nu\rho}$$

- Soft decoupling in Gravity analogous to QCD
  [Bauer, Pirjol, Stewart hep-ph/0109045]

Soft Theorem from SCET Gravity

- Can evaluate in frame where $p_\perp = 0$, then we immediately see

$$\frac{p^\mu}{p \cdot k} \to \frac{n_+ p}{n_+ p \, n_- k} \frac{n_-^\mu}{2}$$

which we get from $\langle \phi_c(p) | n_+ \partial \phi_c(\tilde{p})$

- Relevant operators and terms

$$J^{A0} = \int dt \, C^{A0}(\{t_i\}) \Phi_1(t_1) \Phi_2(t_2) \ldots \Phi_n(t_n)$$

$$J^{A1} = \int dt \, C^{A1\mu}(\{t_i\}) \Phi_1(t_1) \ldots [i\partial_{\perp_i \mu} \Phi_i](t_i) \ldots \Phi_n(t_n)$$

$$J^{T2} = \int d^4x \, T\{J^{A0}, \mathcal{L}^{(2)}(x)\} + T\{J^{A1}, \mathcal{L}^{(1)}(x)\}$$

Derivation of Soft Theorem in Gravity

- Perform SCET Expansion: $\mathcal{A}_0 = \mathcal{A}_0^{(0)} + \mathcal{A}_0^{(1)} + \ldots$
- First term

$$-\frac{\kappa}{2}\frac{\varepsilon_{\mu\nu}p^\mu p^\nu}{p\cdot k}\mathcal{A}_0^{(0)} = -\frac{\kappa}{4}\varepsilon_{--}n_+p\frac{n_+p}{n_+p\,n_-k}\mathcal{A}_0^{(0)}$$

- In Lagrangian

$$\mathcal{L}_{\text{int}}^{(0)} = -\frac{\kappa}{8}s_{--}\partial_+\phi\partial_+\phi$$

- $\mathcal{A}_0^{(0)} = C^{A0}$ comes from leading non-radiative current

- Second term

$$-\frac{\kappa}{2}\frac{k_\nu \varepsilon_{\mu\rho} p^\rho J^{\mu\nu}}{p\cdot k} \to -\frac{\kappa}{8}\left(\varepsilon_{--}n_+k - \varepsilon_{+-}n_-k\right)\left(n_+p\frac{\partial}{\partial n_+p}\mathcal{A}_0^{(0)}\right)\frac{n_+p}{n_+pn_-k}$$
$$-\frac{\kappa}{4}\left(\varepsilon_{--}k^\alpha - \varepsilon_-^\alpha n_-k\right)\left(n_+p\frac{\partial}{\partial p_\perp^\alpha}\mathcal{A}_0^{(1)}\right)\frac{n_+p}{n_+pn_-k}$$

- In Lagrangian $x_\perp$ only contributes if $x_\perp^\alpha \partial_\perp^\beta = \eta_\perp^{\alpha\beta}$, $x_\perp^\alpha x_\perp^\beta = \frac{1}{2}\eta_\perp^{\alpha\beta}x_\perp^2$

$$\mathcal{L}^{(2)} = n_+\phi\Bigg(-\frac{\kappa}{16}\partial_\mu s_{\nu-}n_+^{[\mu}n_-^{\nu]}n_-x\partial_+\phi - \frac{\kappa}{4}\underbrace{\partial_{[\alpha}s_{\beta]-}x_\perp^\alpha\partial_\perp^\beta}_{=0 \text{ if } x_\perp^\alpha\partial_\perp^\beta=\eta_\perp^{\alpha\beta}}\phi$$
$$\underbrace{-\frac{\kappa}{8}x_\perp^\alpha x_\perp^\beta R_{\alpha-\beta-}\partial_+\phi}_{R^{\alpha_\perp}{}_{-\alpha_\perp-}=R_{--}=0}\Bigg) - \frac{\kappa}{8}\underbrace{s_{+-}\partial_\perp\phi\cdot\partial_\perp\phi}_{=0 \text{ if } p_\perp=0}$$

$\boxed{\text{First part comes from } T\{J^{A0},\mathcal{L}^{(2)}\}}$

- Second part

$$\frac{\kappa}{4}\left(\varepsilon_{--}k^{\alpha}-\varepsilon_{-}^{\alpha}n_{-}k\right)\left(n_{+}p\frac{\partial}{\partial p_{\perp}^{\alpha}}\mathcal{A}_0^{(1)}\right)\frac{n_{+}p}{n_{+}pn_{-}k}$$

- Must come from $T\{J^{A1},\mathcal{L}^{(1)}\}$,
- Matching coefficient completely fixed by RPI from non-radiative current

$$C^{A1\mu}\sim\frac{-2n_{j-}^{\mu}}{n_{i-}n_{j-}}\frac{\partial}{\partial n_{i+}p_i}C^{A0}$$

- In $\mathcal{L}^{(1)}$

$$\mathcal{L}^{(1)}=-\frac{\kappa}{8}\partial_{\mu}s_{\mu-}x_{\perp}^{[\mu}n_{-}^{\nu]}n_{+}\partial\phi n_{+}\partial\phi-\frac{\kappa}{4}\underbrace{s_{\mu_{\perp}-}\partial^{\mu_{\perp}}\phi n_{+}\partial\phi}_{=0\text{ if }p_{\perp}=0}$$

- $X_{\perp}$ acting on $p_{\perp}$ from $A1$ current generates $\frac{\partial}{\partial p_{\perp}}$

$$X_{\perp}^{\alpha}\tilde{p}_{\perp\beta}C^{A1\beta}(\tilde{p}_{+})\rightarrow\frac{\partial}{\partial p_{\perp\alpha}}\mathcal{A}_0^{(1)}$$

- Third term

$$-\frac{\kappa}{4}\frac{k_\rho k_\sigma \varepsilon_{\mu\nu} J^{\mu\rho} J^{\nu\sigma}}{p_i \cdot k} \to -\frac{\kappa}{8}\frac{p_+}{n_+ p n_- k} R_{\alpha-\beta-} p_+ \frac{\partial}{\partial p_\alpha}\frac{\partial}{\partial p_\beta}$$

- Generated from Riemann term

$$\mathcal{L}^{(2)} \supset -\frac{\kappa}{8} x_\perp^\alpha x_\perp^\beta R_{\alpha-\beta-} \partial_+\phi \partial_+\phi$$

$$\mathcal{L}^{(4)} \supset -\frac{\kappa}{32}(n_- x)^2 R_{+-+-} \partial_+\phi \partial_+\phi \,,$$

at $\mathcal{O}(\lambda^4)$ in e.g. $T\{J^{A0}, \mathcal{L}^{(4)}\}$ and $T\{J^{A2}, \mathcal{L}^{(2)}\}$

- Higher orders no longer universal, as we can add $R^s_{\mu\nu\alpha\beta}$ in the operator basis

$$\boxed{\text{process-dependent at } \mathcal{O}(\lambda^6) = \mathcal{O}(k_s^3)}$$

Conclusion

- Derived rigorously SCET for Gravity to subleading order
- Transparent structure of two-fold gauge symmetry of soft Gravity local translations and local Lorentz symmetry shows form of the soft theorem
- No soft graviton building blocks up to $\mathcal{O}(\lambda^6)$ in the operator basis implies universality of soft theorem
- Beautiful interpretation of the soft theorem based on gauge symmetry
- Similarity of LBK and soft theorem can be understood due to universal part of SCET – multipole expansion and homogeneous gauge symmetry

Auxiliary Slides

Full Lagrangian

$$
\begin{aligned}
\mathcal{L}^{(0)} &= \frac{1}{2}\sqrt{-\hat{g}_s}\hat{g}_s^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\,, \\
\mathcal{L}^{(1)} &= \frac{1}{2}\sqrt{-\hat{g}_s}\left(-\hat{g}_s^{\mu\alpha}\hat{g}_s^{\nu\beta}h_{\alpha\beta} + \frac{1}{2}\hat{g}_s^{\alpha\beta}h_{\alpha\beta}\hat{g}_s^{\mu\nu}\right)\partial_\mu\phi\partial_\nu\phi\,, \\
\mathcal{L}^{(2)} &= \frac{1}{2}\sqrt{-\hat{g}_s}\left(\hat{g}_s^{\mu\alpha}\hat{g}_s^{\nu\beta}\hat{g}_s^{\rho\sigma}h_{\alpha\rho}h_{\beta\sigma} - \frac{1}{2}\hat{g}_s^{\alpha\beta}h_{\alpha\beta}\hat{g}_s^{\mu\rho}\hat{g}_s^{\nu\sigma}h_{\rho\sigma} + \frac{1}{8}(\hat{g}_s^{\alpha\beta}h_{\alpha\beta})^2\right. \\
&\qquad\qquad \left. -\frac{1}{4}\hat{g}_s^{\mu\alpha}g_s^{\nu\beta}h_{\mu\nu}h_{\alpha\beta}\right)\partial_\mu\phi\partial_\nu\phi\,, \\
\mathcal{L}_R^{(2)} &= \frac{1}{2}\left[\partial_\alpha W^{-1}\phi\right]\left[\partial_\beta W^{-1}\phi\right]\Big(\det\left(R^\mu{}_\alpha\right)\left[R^{-1}\sqrt{-g_s}\right]R_\mu{}^\alpha R_\nu{}^\beta\left[R^{-1}g_s^{\mu\nu}(x)\right] \\
&\qquad -\sqrt{-\hat{g}_s}\hat{g}_s^{\alpha\beta}\Big)
\end{aligned}
$$

Metric in FLNC

- Fixed-line normal coordinates

$$
\begin{aligned}
x'^{\mu} = x^{\mu} + (E^{\mu}{}_{A} - \delta^{\mu}_{A})(x-x_{-})^{A} - \frac{1}{2}(x-x_{-})^{A}(x-x_{-})^{B}E^{\alpha}{}_{A}E^{\beta}{}_{B}\Gamma^{\mu}{}_{\alpha\beta} \\
+ \frac{1}{6}(x-x_{-})^{A}(x-x_{-})^{B}(x-x_{-})^{C}E^{\alpha}{}_{A}E^{\beta}{}_{B}E^{\nu}{}_{C}(2\Gamma^{\mu}{}_{\alpha\lambda}\Gamma^{\lambda}{}_{\beta\nu} - [\partial_{\nu}\Gamma^{\mu}{}_{\alpha\beta}]) \\
+ \mathcal{O}(x^{3}),
\end{aligned}
$$

- Dressed metric field

$$
\tilde{g}_{ab}(x) = \frac{\partial x'^{\mu}}{\partial x^{a}}\frac{\partial x'^{\nu}}{\partial x^{b}}\left(1 + \theta^{\alpha}\partial_{\alpha} + \frac{1}{2}\theta^{\alpha}\theta^{\beta}\partial_{\alpha}\partial_{\beta} + \dots\right)g_{\mu\nu}(x)
$$

- Background Metric field

$$
\begin{aligned}
\hat{g}_{ab}(x) &= \eta_{ab}\,, \\
\hat{g}_{a-}(x) &= e_{a-} - y^{A}\left[\omega_{-}\right]_{Aa}\,, \\
\hat{g}_{--}(x) &= (e_{-}{}^{A} - y^{R}\omega_{-R}{}^{A})(e_{-}{}^{B} - y^{S}\omega_{-S}{}^{B})\eta_{AB}\,.
\end{aligned}
$$

Simplified Lagrangian

$$\mathcal{L}^{(0)} = \frac{1}{2}\partial_+\phi D_-\phi + \frac{1}{2}\partial_{\alpha_\perp}\phi\partial^{\alpha_\perp}\phi$$

$$\mathcal{L}_h^{(1)} = -\frac{1}{2}\mathfrak{h}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{4}\mathfrak{h}^{\alpha_\perp}{}_{\alpha_\perp}\left(\partial_+\phi D_-\phi + \partial_{\alpha_\perp}\phi\partial^{\alpha_\perp}\phi\right)$$

$$\begin{aligned}\mathcal{L}^{(2)} &= -\frac{1}{8}x_\perp^\alpha x_\perp^\beta R_{\alpha-\beta-}(\partial_+\phi)^2 \\ &\quad + \frac{1}{8}s_{+-}\left(\partial_+\phi D_-\phi + \partial_{\alpha_\perp}\phi\partial^{\alpha_\perp}\phi\right)\end{aligned}$$

$$\begin{aligned}\mathcal{L}_h^{(2)} &= \frac{1}{2}\mathfrak{h}^{\mu\alpha}\mathfrak{h}_\alpha^\nu\partial_\mu\phi\partial_\nu\phi + \frac{1}{16}\left((\mathfrak{h}_{\alpha_\perp}^{\alpha_\perp})^2 - 2\mathfrak{h}^{\alpha\beta}\mathfrak{h}_{\alpha\beta}\right)\left(\partial_+\phi D_-\phi + \partial_{\mu_\perp}\phi\partial^{\mu_\perp}\phi\right) \\ &\quad + \frac{1}{4}\mathfrak{h}^{\mu\alpha}s_{\alpha-}\partial_+\phi\partial_\mu\phi\end{aligned}$$