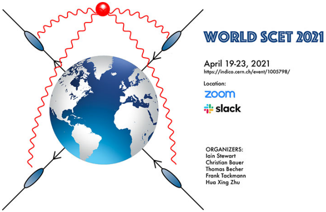
On soft-collinear factorization in the presence of endpoint divergences: *$e^-\mu^-$ backward-scattering as a template case*

Philipp Böer

*based on PB '18 (PhD thesis) and "in preparation" with G. Bell and T. Feldmann

22 April 2021



Technische Universität München

Introduction

- soft-collinear factorization well understood at leading power
  - $\rightarrow$ current status of precision: NNLL and N^{3}LL resummation
  - $\rightarrow$ NNLO or N^{3}LO matching to fixed-order calculations
- soft-collinear factorization at subleading power active and largely unexplored field:
  - $\rightarrow$ thrust spectrum for $H \rightarrow gg$ in pure glue QCD [Moult, Stewart, Vita, Zhu '18]
  - $\rightarrow$ threshold resummation for Drell-Yan and Higgs [Beneke et al. '18,19]
  - $\rightarrow$ soft quark Sudakov in $\mathcal{N} = 1$ [Moult, Stewart, Vita, Zhu '19]
  - $\rightarrow$ bottom-quark contribution in $h \rightarrow \gamma\gamma$ [Neubert et al. '19-21]
  - $\rightarrow$ threshold resummation for off-diagonal channels in DIS [Beneke et al. '20]
  - $\rightarrow$ ...

Endpoint-singular convolution integrals appear as a generic problem!

Heavy-to-light form factors

Factorization of $B \to \pi$ form factors at large recoil suffers from endpoint-singularities, e.g.

[Beneke, Feldmann '01]

$$\int_0^\infty d\omega \, \frac{\phi_+(\omega)}{\omega^2} \,, \qquad \int_0^1 du \, \frac{\phi_\pi(u)}{\bar{u}^2} \,, \qquad \ldots \qquad \text{log-divergent for } \omega, \bar{u} \to 0$$

Idea: Study process in a perturbative, non-relativistic set-up: $B_c \to \eta_c$, with $m_b \gg m_c \gg \Lambda_{\rm QCD}$

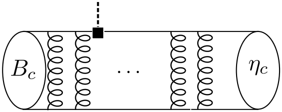
→ $\mu_s \sim \mu_c \sim m_c \to$ SCET$_{\rm II}$, requires additional rapidity regulator

Observations:

→ exponentiation of rapidity divergences (coll. anomaly) for some inverse moments, e.g.

$$\int_0^\infty d\omega \, \frac{\phi_+(\omega)}{\omega^2} \times \int_0^1 du \, \frac{1+\bar{u}}{\bar{u}^2} \phi_\pi(u) = r(\mu/m_c) \left(\frac{E}{m_c} \right)^{f(\mu/m_c)}$$

→ diagrammatically: "ladder-type" diagrams relevant for leading rapidity logarithms



Heavy-to-light form factors

Complicated (bare) factorization formula for non-factorizable "soft" form factor ξ_π!

Tree-level matching:

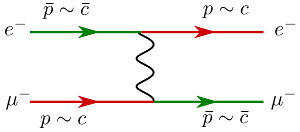
$$\xi_\pi(E_\pi) \sim C_F \int_0^\infty d\omega \int_0^1 du \left[\frac{\phi_B^-(\omega)}{\omega} \frac{1+\bar{u}}{\bar{u}^2} \phi_\pi(u) + \frac{\phi_B^+(\omega)}{\omega} \frac{u}{\bar{u}^2} \phi_\pi(u) \right.$$
$$\left. + \frac{\phi_B^+(\omega)}{\omega^2} \left(- \frac{m_q \bar{u} + 2 m_{\bar{q}}}{\bar{u}^2} \phi_\pi(u) + 3 \frac{\mu_\pi \phi_P(u)}{\bar{u}} + \frac{\tilde{\mu}_\pi}{6} \frac{\phi'_\sigma(u)}{\bar{u}} \right) \right]$$
$$- 2\,(C_F - C_A/2)\, \frac{f_{3\pi}}{f_\pi} \int_0^\infty d\omega \frac{\phi_B^+(\omega)}{\omega^2} \int \mathcal{D}\alpha \frac{\phi_{3\pi}(\{\alpha_i\})}{\alpha_g \alpha_{\bar{q}} (\alpha_g + \alpha_{\bar{q}})}$$
$$+ 2\,(C_F - C_A/2) \int_0^\infty d\omega \int_0^\infty d\xi \frac{\Psi_{A-V}(\omega,\xi)}{\omega\, \xi\, (\omega + \xi)} \int_0^1 du \frac{\phi_\pi(u)}{\bar{u}^2}\,.$$

Q: Is there a simpler perturbative process that shares all the relevant features we observed?

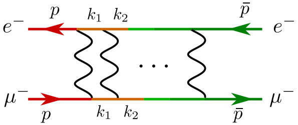
$e^- \mu^-$ backward scattering

$$e^- \mu^- \to e^- \mu^- \quad \text{at} \quad s \approx -t \gg m_\mu^2 \sim m_e^2 \quad \text{and} \quad \theta = \pi \; (u \approx 0)$$

- **counting**: $\log m_e^2/s \sim \log m_\mu^2/s \sim 1/\alpha_{\rm em}$ ($\to$ resummation), but $\log m_e/m_\mu \sim \mathcal{O}(1)$
  - $\to$ focus on resummation of double-log's: $m_e \approx m_\mu$ (or as hypothetical toy model)
  - $\to$ IR-finite, since $v_e = v_\mu$ (no Bremsstrahlung)



- double-log's from ladder diagrams in specific configuration:
  - $\to$ all photon-propagators become eikonal
  - $\to$ all lepton-propagators go on-shell and are ordered in rapidity



$$p_- > k_{1-} > k_{2-} > \dots$$

(note the unconventional momentum flow!)

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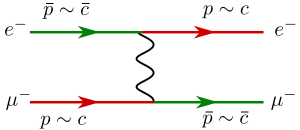

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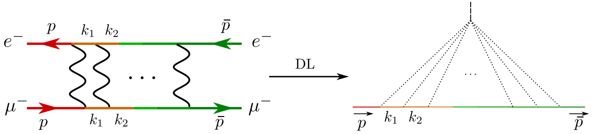
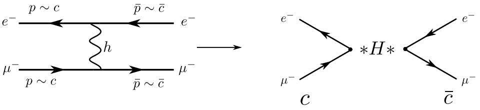
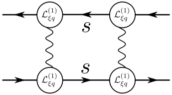


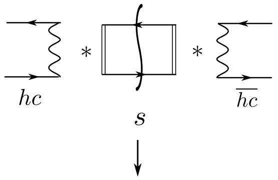
$\to$ large double log's (DL) map onto simple scalar integrals

i) 

$\sim \int_0^1 \frac{du}{u}\frac{dv}{v}\phi_c(u)\phi_{\bar{c}}(v)H(uvs)$

ii) 

$\mathcal{M} \sim \langle \mathcal{T}\left\{\left(\mathcal{L}^{(1)}_{\xi q}\right)^4\right\}\rangle$

$=$ 

$(\mathrm{SCET_I})$

$$\int_0^1 \frac{du}{u}\frac{dv}{v}\int_0^\infty \frac{dk_+}{k_+}\frac{dk_-}{k_-}\,\phi_c(u)J_{hc}(uk_+)S(k_+k_-)J_{\bar{hc}}(k_-v)\phi_{\bar{c}}(v) \qquad (\mathrm{SCET_{II}})$$

leading twist operators: (analog. for other chiralities)

$$\phi_c(u) = n_+p\int \mathrm{d}t\, e^{it(n_+p)u}\,\langle e^-_L(p)|\,\bar{\chi}^{(e)}(0)\frac{\not{n}_+}{2}\chi^{(\mu)}(tn_+)\,|\mu^-_L(p)\rangle = \delta(1-u) + \mathcal{O}(\alpha)$$

Bare factorization formula

$$\mathcal{M}(e\mu \to e\mu) \simeq \int_0^1 \frac{\mathrm{d}u}{u}\frac{\mathrm{d}v}{v}\phi_c(u)\phi_{\bar{c}}(v)\left\{H(uvs) + \int_0^\infty \frac{\mathrm{d}k_+}{k_+}\frac{\mathrm{d}k_-}{k_-}J_{hc}(uk_+)S(k_+k_-)J_{\bar{hc}}(k_-v)\right\}$$

Bare functions at DL-level/close to endpoint: [PB '18] $(\hat{\alpha} = \alpha/2\pi)$

$$H(x) = J_{hc}(x) = J_{\bar{hc}}(x) = 1 + \frac{\hat{\alpha}}{\varepsilon^2}\left(\frac{\mu^2}{x}\right)^{\varepsilon} + \frac{\hat{\alpha}^2}{4\varepsilon^4}\left(\frac{\mu^2}{x}\right)^{2\varepsilon} + \ldots$$

$$\phi_c(x) = \phi_{\bar{c}}(x) = \delta(1-x) + \theta(1-x)\left[\left(\frac{\mu^2}{m^2}\right)^{\varepsilon}\frac{\hat{\alpha}}{\varepsilon} + \hat{\alpha}^2\left(\frac{\mu^2}{m^2}\right)^{2\varepsilon}\frac{x^{\varepsilon} - 1 - 2\varepsilon\log x}{2\varepsilon^3} + \ldots\right]$$

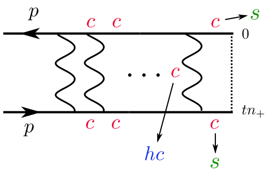
and [Bell,PB,Feldmann in progress]

$$S(x) = -\hat{\alpha}\left[\theta(x-m^2)S_a(x) + \theta(m^2-x)S_b(x)\right]$$

$$S_a(x \to \infty) = \left(\frac{\mu^2}{m^2}\right)^{\varepsilon}x^{-\varepsilon} + \hat{\alpha}\left(\frac{\mu^2}{m^2}\right)^{2\varepsilon}\frac{x^{-\varepsilon}}{\varepsilon^2} + \ldots \qquad S_b(x \to 0) = \hat{\alpha}\left(\frac{\mu^2}{m^2}\right)^{2\varepsilon}\frac{1}{\varepsilon^2} + \ldots$$

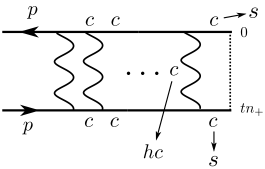
- endpoint singularities all over the place
  - $\to$ $u, v, k_+, k_- \to 0$
  - $\to$ $k_+, k_- \to \infty$ ("infinity-bin")
  - $\to$ requires rapidity regulator (sometimes)!

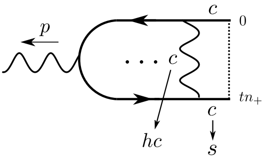
Endpoint re-factorization



	$e\mu$-scattering [PB '18; Bell,PB,Feldmann in progress]
re-factorization	$\phi_c(u \to 0) \simeq \int \frac{\mathrm{d}v}{v} \phi_c(v) \int \frac{\mathrm{d}k_+}{k_+} J(vk_+) S(k_+ u)$
subtraction	$\tilde{\phi}_c(u) = \int \frac{\mathrm{d}v}{v} \phi_c(v) \left\{ \delta(u-v) - \int \frac{\mathrm{d}k_+}{k_+} J(vk_+) S(k_+ u) \right\}$
soft function	$S(k_+ k_- \to 0) \simeq \int \frac{\mathrm{d}\ell_+}{\ell_+} \frac{\mathrm{d}\ell_-}{\ell_-} S(k_+ \ell_-) J(\ell_- \ell_+) S(\ell_+ k_-)$

$\rightarrow$ reflects the kinematic configuration of double log's in full theory (infinite number of modes?)





	$e\mu$-scattering [PB '18; Bell,PB,Feldmann in progress]	$h \to \gamma\gamma$ [Neubert et al. '19-21]
re-factorization	$\phi_c(u \to 0) \simeq \int \frac{dv}{v} \phi_c(v) \int \frac{dk_+}{k_+} J(vk_+) S(k_+ u)$	$\langle \mathcal{O}_2 \rangle \simeq \int \frac{dk_+}{k_+} J(-p_- k_+) S(k_+ u)$
subtraction	$\tilde{\phi}_c(u) = \int \frac{dv}{v} \phi_c(v) \left\{ \delta(u-v) - \int \frac{dk_+}{k_+} J(vk_+) S(k_+ u) \right\}$	$\langle \mathcal{O}_2 \rangle - [[\langle \mathcal{O}_2 \rangle]]$
soft function	$S(k_+ k_- \to 0) \simeq \int \frac{d\ell_+}{\ell_+} \frac{d\ell_-}{\ell_-} S(k_+ \ell_-) J(\ell_- \ell_+) S(\ell_+ k_-)$	$S(k_+ k_- \to 0) \to 0$

$\langle \mathcal{O}_2 \rangle \sim \phi_\gamma(u)$: photon-LCDA

- $\rightarrow$ reflects the kinematic configuration of double log's in full theory (infinite number of modes?)
- $\rightarrow$ in $h \to \gamma\gamma$: S and ϕ_γ m_b-suppressed $\to$ cannot appear twice!
- $\rightarrow$ in SCET$_\mathrm{I}$ problems get new mode with lower virtuality ($\rightarrow$ see next talk by S. Jaskiewicz)

Resummation from consistency relations

Double log's completely fixed by re-factorization and pole cancellation!

1. choose analytic rapidity regulator $(\nu/k_-)^\delta \quad \Rightarrow \quad$ Second term scaleless

$$\mathcal{M}(e^-\mu^- \to e^-\mu^-) \sim \int_0^1 \frac{\mathrm{d}u}{u}\frac{\mathrm{d}v}{v}\,\phi_c(u;\mu/m;\nu/\sqrt{s})\,\phi_{\bar{c}}(v;\mu/m,\nu\sqrt{s}/m^2)\,H(\mu^2/uvs)$$

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2. use fixed-order expansion of hard function: $H = \sum_n \frac{\hat{\alpha}^n}{\varepsilon^{2n}}\left(\frac{\mu^2}{uvs}\right)^{n\varepsilon} h_n$

$$\frac{\mathcal{M}}{\mathcal{M}^0} \sim \langle u^{-1}\rangle\langle v^{-1}\rangle + \frac{\hat{\alpha}}{\varepsilon^2}h_1\left(\frac{\mu^2}{s}\right)^{\varepsilon}\langle u^{-1-\varepsilon}\rangle\langle v^{-1-\varepsilon}\rangle + \frac{\hat{\alpha}^2}{\varepsilon^4}h_2\left(\frac{\mu^2}{s}\right)^{2\varepsilon}\langle u^{-1-2\varepsilon}\rangle\langle v^{-1-2\varepsilon}\rangle + \ldots$$

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$$= r_0(\mu/m)\left(\frac{s}{m^2}\right)^{f_0(\mu/m)} + \frac{\hat{\alpha}}{\varepsilon^2}h_1\left(\frac{\mu^2}{s}\right)^{\varepsilon} r_1(\mu/m)\left(\frac{s}{m^2}\right)^{f_1(\mu/m)} + \frac{\hat{\alpha}^2}{\varepsilon^4}h_2\left(\frac{\mu^2}{s}\right)^{2\varepsilon} r_2(\mu/m)\left(\frac{s}{m^2}\right)^{f_2(\mu/m)} + \dots$$

3. each order in matching: large rapidity log's exponentiate
   - $\rightarrow$ infinite number of remainder functions $r_i = 1 + \mathcal{O}(\alpha)$
   - $\rightarrow$ and infinite number of coll. anomalies $f_i \sim \mathcal{O}(\alpha^{i+1})$ (e.g. f_1 contributes first at 3-loop!)
   - $\rightarrow$ rapidity divergences under control, but complicated cross-talk of $1/\varepsilon$ poles

Resummation from consistency relations

Double log's completely fixed by re-factorization and pole cancellation!

$$\frac{\mathcal{M}}{\mathcal{M}^0} \sim r_0(\mu/m)\left(\frac{s}{m^2}\right)^{f_0(\mu/m)} + \frac{\alpha}{\varepsilon^2} h_1 \left(\frac{\mu^2}{s}\right)^{\varepsilon} r_1(\mu/m)\left(\frac{s}{m^2}\right)^{f_1(\mu/m)} + \frac{\alpha^2}{\varepsilon^4} h_2 \left(\frac{\mu^2}{s}\right)^{2\varepsilon} r_2(\mu/m)\left(\frac{s}{m^2}\right)^{f_2(\mu/m)} + \dots$$

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4. Plug in double-log structure

$$r_i = \sum_{n=0}^{\infty} \frac{\hat{\alpha}^n}{\varepsilon^{2n}} \left(\frac{\mu^2}{m^2}\right)^{n\varepsilon} r_i^{(n)} \qquad\qquad f_i = \sum_{n=i+1}^{\infty} \frac{\hat{\alpha}^n}{\varepsilon^{2n-1}} \left(\frac{\mu^2}{m^2}\right)^{n\varepsilon} f_i^{(n)}$$

$\rightarrow$ minimal set-up: all coefficients h_i, $r_i^{(n)}$, $f_i^{(n)}$ fixed by demanding $\mathcal{M} = \mathcal{O}(\varepsilon^0)$

Resummation from consistency relations

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$$\boxed{\frac{\mathcal{M}}{\mathcal{M}^0} = 1 + \frac{\hat{\alpha}}{2}h_1 L^2 + \frac{\hat{\alpha}^2}{12}(h_1)^2 L^4 + \dots = \sum_{n=0}^{\infty}\frac{\hat{\alpha}^n}{n!(n+1)!}L^{2n} = \frac{I_1\left(2\sqrt{\hat{\alpha}h_1 L^2}\right)}{\sqrt{\hat{\alpha}h_1 L^2}}}$$

(I_1 :**mod. Bessel-function**)

$L = \log m^2/s$

Resummation from consistency relations

Double log's completely fixed by re-factorization and pole cancellation!

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(I_1 :**mod. Bessel-function**)

5. one-loop gives $h_1 = 1$.

known since 1967, but highly non-trivial endpoint dynamics in SCET!

(see e.g. Landau/Lifschitz "QED")

Conclusion

- renormalized factorization theorem for $e\mu$ backward-scattering currently unknown
- but template case to study endpoint singularities in $\mathrm{SCET}_{\mathrm{II}}$

$$\mathcal{M} \sim \int_0^1 \frac{du}{u}\frac{dv}{v}\,\phi_c(u)\phi_{\bar{c}}(v)H(uv) \qquad \text{for regulator } (\nu/k_-)^\delta$$

$\rightarrow$ perturbative, simple factorization formula
$\rightarrow$ each DL from endpoint configuration (consistency constraints, simple scalar integrals)

- self-similar endpoint structure

$\rightarrow$ exponentiation of rapidity poles in inverse moments (as in heavy-to-light form factors)
$\rightarrow$ infinite number of coll. anomaly exponents
$\rightarrow$ more complicated than $h \rightarrow \gamma\gamma$ (single rapidity divergence to all orders)
$\rightarrow$ single additive re-arrangement does not cure the problem

For discussion part: problem of dim.-reg./analytic regularization?

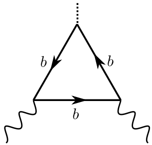
$\rightarrow$ factorization with rapidity cutoff's (w.i.p.; see backup slides)

Backup-Slides

Comparison $e\mu$-scattering and $h \to \gamma\gamma$

probably oversimplified!

$h \to \gamma\gamma$

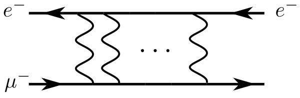


- $\to$ single endpoint-divergence $\otimes$ off-shell Sudakov form-factor?
- $\to$ double log's:

$$ {}_2F_2 \sim \int^{M_h} \underbrace{J\,S\,J}_{\text{exponentiation}} $$

- $\to$ ren. factorization theorem with cutoff

$e\mu$-scattering



- $\to$ all DL's from endpoint configuration

$$ F \sim I_1 \sim 1 + \hat{\alpha} L^2 \int F * F $$

- $\to$ renormalized factorization theorem unknown

Factorization with rapidity cutoff's

Idea: Decompose phase-space of DL integrals in full theory ($L = \log \lambda^2 = \log m^2/s$)

$$\mathcal{M} = \sum_{n=0}^{\infty} \hat{\alpha}^n \int_{\lambda^2}^{1} \frac{du_1}{u_1} \cdots \int_{\lambda^2}^{u_{n-1}} \frac{du_n}{u_n} \int_{\lambda^2/u_n}^{1} \frac{dv_n}{v_n} \cdots \int_{\lambda^2/u_1}^{v_2} \frac{dv_1}{v_1} = \frac{I_1(2\sqrt{\hat{\alpha}L^2})}{\sqrt{\hat{\alpha}L^2}}$$

into momentum regions defined by longitudinal cutoffs ξ^*, η^*: $\left(\xi = \frac{\log u}{\log \lambda^2}, \eta = \frac{\log v}{\log \lambda^2}\right)$

hard: $\xi < \xi^*, \eta < \eta^*$ collinear: $\xi < \xi^*, \eta > \eta^*$ soft: $\xi > \xi^*, \eta > \eta^*$

$\rightarrow$ recover factorization formula ...

$$\mathcal{M} = \int_0^{\xi^*} \mathrm{d}\xi \int_0^{\eta^*} \mathrm{d}\eta\, \phi_c(\xi,\eta^*) H(\xi^* - \xi, \eta^* - \eta) \phi_{\bar{c}}(\eta,\xi^*) \quad + \quad \text{2nd term} \sim S$$

...with longitudinal cutoff's! (endpoint $u(v) \rightarrow 0$ maps onto $\xi(\eta) \rightarrow \infty$)

$\Rightarrow$ no endpoint divergence!

Factorization with rapidity cutoff's

$$\mathcal{M} = \int_0^{\xi^*} \mathrm{d}\xi \int_0^{\eta^*} \mathrm{d}\eta\, \phi_c(\xi,\eta^*) H(\xi^* - \xi, \eta^* - \eta) \phi_{\bar{c}}(\eta, \xi^*) \quad + \quad \text{2nd term} \sim S$$

RG evolution for hard function

$$\frac{d^2 H(\xi,\eta)}{d\xi d\eta} = \hat{\alpha} L^2 H(\xi,\eta)$$

$$H(\xi,\eta) = I_0(2\sqrt{\hat{\alpha} L^2 \xi\eta}) = \int_0^{\xi} d\xi'\, H(\xi',\eta_H) U_H(\xi - \xi', \eta - \eta_H)$$

Wave function obeys similar differential equation

$$\frac{d^2 \phi_c(\xi,\eta^*)}{d\xi d\eta^*} = \hat{\alpha} L^2 H(\xi,\eta^*)$$

$$\phi_c(\xi,\eta^*) = \delta(\xi) + \theta(\xi)(\eta^* - \xi)\hat{\alpha} L^2 \frac{I_1(2\sqrt{\hat{\alpha} L^2 \xi\eta^*})}{\sqrt{\hat{\alpha} L^2 \xi\eta^*}} = \int_0^{\xi} d\xi'\, \phi_c(\xi',\eta_c) U_c(\xi - \xi', \eta^* - \eta_c)$$

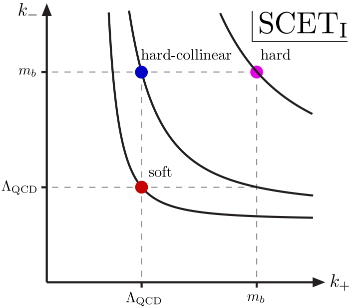
In dim-reg. get:

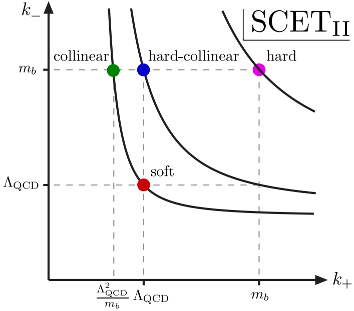
not allowed to expand in ε!

$$H(uvs) = I_0\left(2\sqrt{\frac{\hat{\alpha}}{\varepsilon^2}\left(\frac{\mu^2}{uvs}\right)^{\varepsilon}}\right) \quad \text{and} \quad \phi_c(u) = \delta(1-u) + \theta(1-u)\frac{\hat{\alpha}}{\varepsilon}\frac{I_1(2\sqrt{-\hat{\alpha}/\varepsilon \ln u})}{\sqrt{-\hat{\alpha}/\varepsilon \ln u}}$$

Modes in SCET_{I} and SCET_{II}

$\rightarrow$ The scaling shown in the fig's below is relevant for heavy-to-light form factors at large recoil.





$$
\begin{array}{|r|}
\hline
h:\ (1,1,1) \\
hc:\ (1,\lambda,\lambda^2) \\
c:\ (1,\lambda^2,\lambda^4) \\
s:\ (\lambda^2,\lambda^2,\lambda^2) \\
\hline
\end{array}
\qquad (\lambda^2 \equiv \Lambda_{\text{QCD}}/m_b)
$$