UK Research and Innovation

Pushing the periodic boundaries

Ken Wilson Award Acceptance

Maxwell T. Hansen

July 28th, 2021

THE UNIVERSITY of EDINBURGH

2013 2014 2015 2016 2017 2018 2019 2020

I'm humbled and honored to receive this award… Thank you!

I find it a great privilege to be a member of the vibrant and fascinating scientific community surrounding lattice field theory!

"For pushing the boundary of our finite-volume quantum field theories"

Special thanks to…
- The Selection Committee
- The Conference Organizers/IAC
- Robert Edwards *(for the nomination)*

- My predecessors, collaborators and competitors
- My colleagues and mentors in Seattle, Mainz, CERN and Edinburgh
- The full lattice community

I want to acknowledge the countless, brilliant and lasting contributions to lattice field theory that do not fit the parameters of this award!

Thanks for being an outstanding mentor

Thanks for being a great collaborator (and friend)

Community has been crucial to keep going when the work gets tough

Seattle… *INT and particle group, all my peers and friends, 5249*

Mainz… *all students, postdocs, everyone in Muschenheim and MZM*

CERN… *TH, lattice group, Geneva runners*

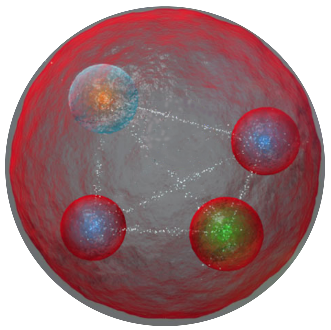
Edinburgh… *to new beginnings!*

Mom, Dad, Simon and Esther

now physics!…

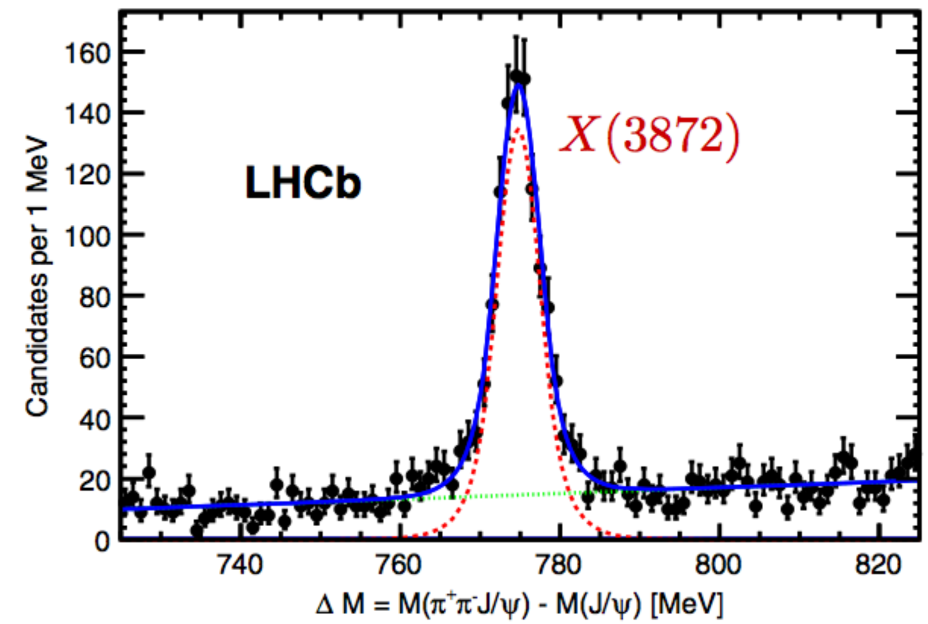
Multi-hadron observables

- Exotics, XYZs, tetra- and penta-quarks, *H* dibaryon



e.g. X(3872)
$$\sim |D^0 \overline{D}^{*0} + \overline{D}^0 D^{*0}\rangle\, ?$$

LHCb (PRD92, 2015)



- Eletroweak, CP violation, resonant enhancement

CP violation in strange $K \to \pi\pi$ See… C. Kelly - Tues Plenary

CP violation in charm $D \to \pi\pi, K\overline{K}$

$f_0(1710)$ could enhance ΔA_{CP}
Soni (2017)

$$\Delta A_{CP} = (-15.4 \pm 2.9) \times 10^{-4}$$
LHCb (PRL, 2019)

Resonant B decays $B \to K^* \ell\ell \to K\pi\, \ell\ell$

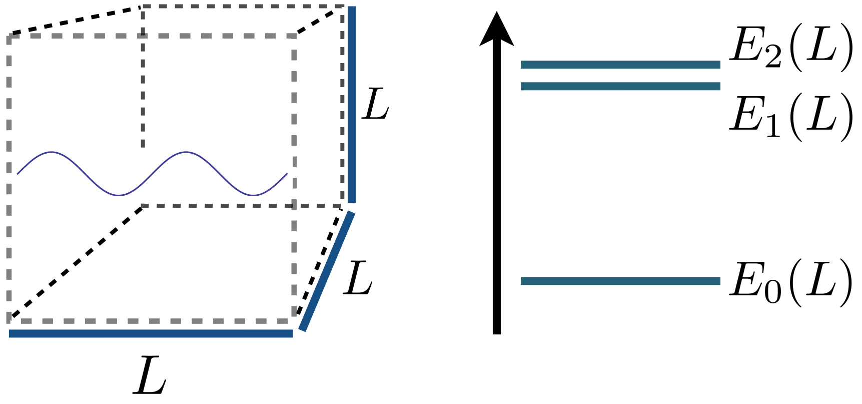
$|X\rangle, |\rho\rangle\,, |K^*\rangle\,, |f_0\rangle \quad \notin$ **QCD Fock space**

Importance of the finite volume

$|X\rangle, |\rho\rangle\,, |K^*\rangle\,, |f_0\rangle \quad \notin$ **QCD Fock**

$|\pi\pi, \text{out}\rangle, \quad |K\pi, \text{out}\rangle, \quad \cdots \in$ **QCD Fock space (continuum of states)**

Relation is (highly) non-trivial

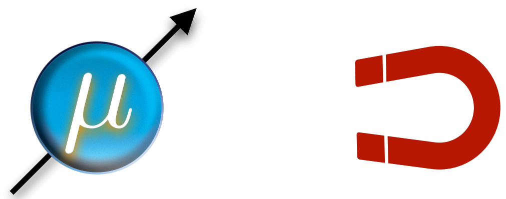


$\in$ **Discrete set of finite-volume states**

Finite-volume landscape

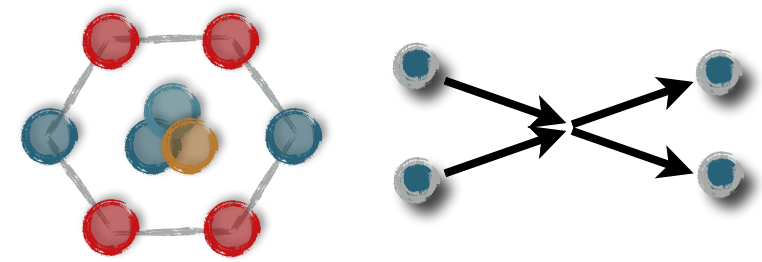
(Important) $e^{-\mu L}$ effects

$(g-2)_\mu$, PDFs, deuteron, $\cdots$



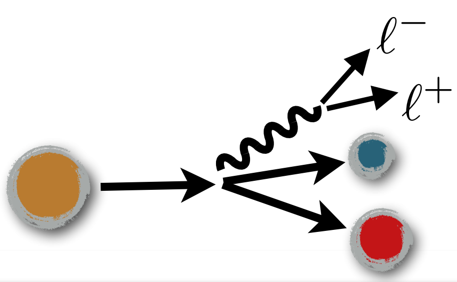
2 → 2 scattering

$\rho(770)$, $\sigma/f_0(500)$, $K^\star$, κ, a_0, Δ, $\cdots$



(1+)$\mathcal{J}$ → 2 transitions

$K \to \pi\pi$, $\gamma^\star \to \pi\pi$, $\cdots$



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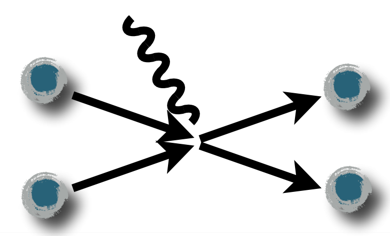

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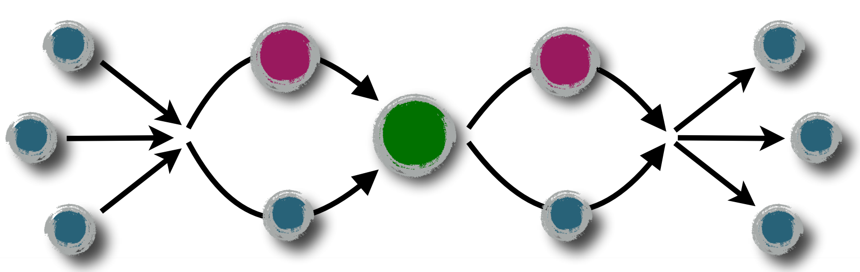
$N\pi\pi \to N(1420) \to N\pi\pi$, $\cdots$



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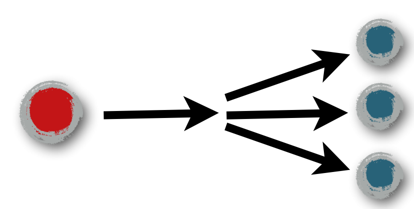

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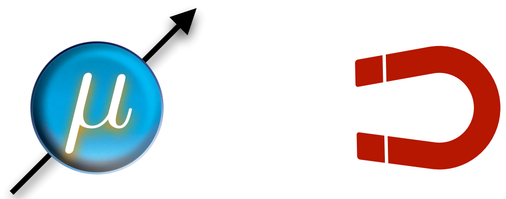

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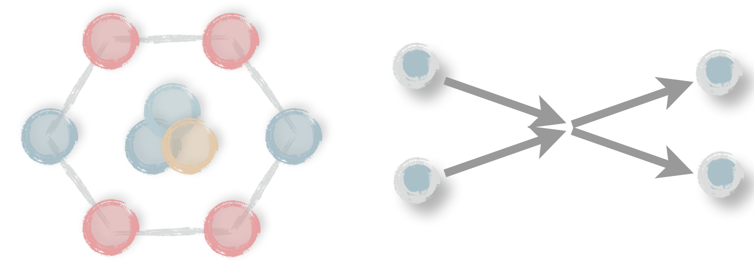

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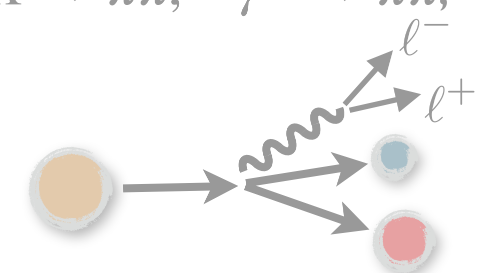

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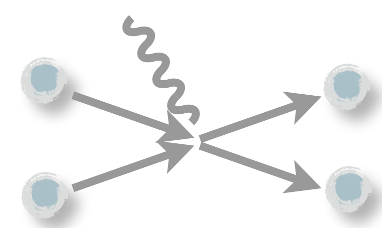

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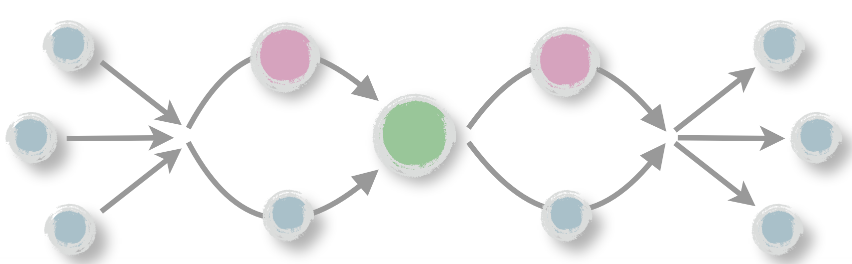
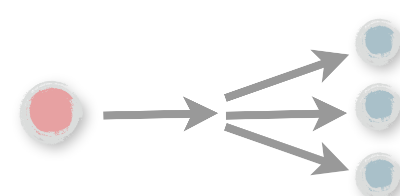


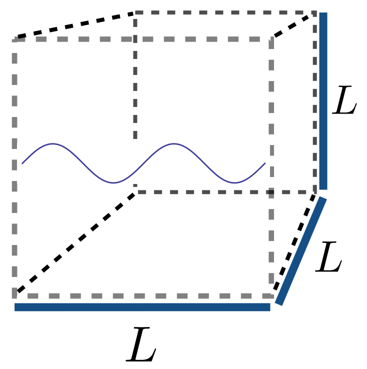
$(1+)\mathcal{J} \to 3$ transitions

$K \to \pi\pi\pi$, $\gamma^\star \to \pi\pi\pi$, $\cdots$



LO HVP (hadronic vacuum polarization)

- $T \times L^3$ periodic, Euclidean signature



$$a_\mu^{\mathrm{LO,HVP}}(L,T) \equiv \int_0^{T/2} dx_0\, \mathcal{K}(x_0) \int_{L^3} d^3\boldsymbol{x}\, \langle j_k^\dagger(x_0,\boldsymbol{x}) j_k(0) \rangle_{L,T}$$

- Continuous kernel function: $\mathcal{K}(x_0) \overset{x_0\to\infty}{\propto} x_0^2$
- For asymptotically large *T, L* …

$$\begin{aligned} a_\mu^{\mathrm{LO,HVP}}(L,T) - a_\mu^{\mathrm{LO,HVP}} = {} & \mathcal{O}(e^{-M_\pi L}) + \mathcal{O}(e^{-\sqrt{2}M_\pi L}) + \mathcal{O}(e^{-\sqrt{3}M_\pi L}) + \mathcal{O}(e^{-1.9 M_\pi L}) + \cdots \\ & + \mathcal{O}(e^{-M_\pi T}) + \mathcal{O}(e^{-\frac{3}{2}M_\pi T}) + \cdots \\ & + \mathcal{O}(e^{-M_\pi \sqrt{T^2+L^2}}) + \cdots \\ & + \mathcal{O}(e^{-M_K L}) + \cdots \end{aligned}$$

$\sqrt{2+\sqrt{3}} \approx 1.92$

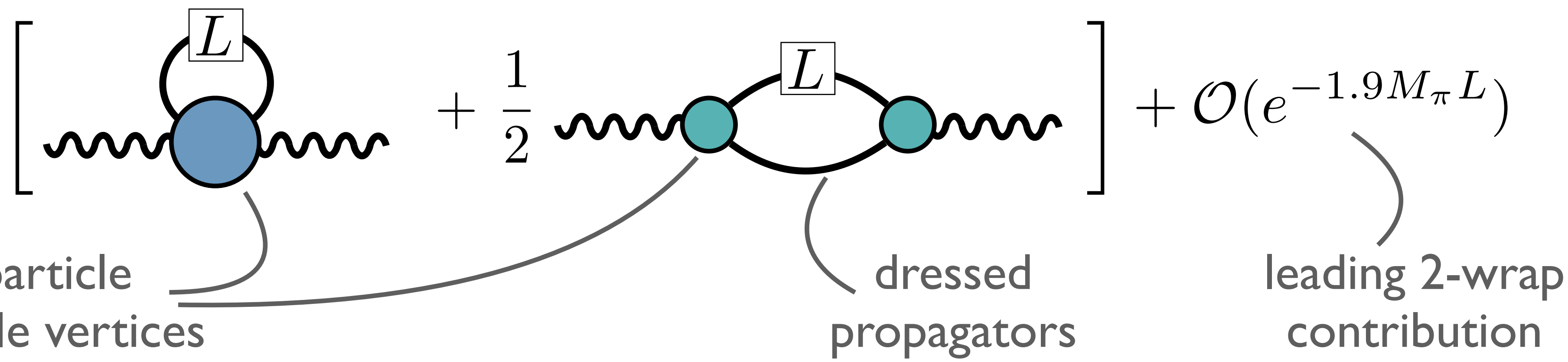
1904.10010 (pink) — 2004.03935 (blue) — for now (green)

MTH and A. Patella • *strong inspiration from…* Lüscher 1986

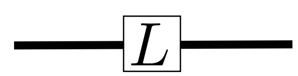
Leading terms

- Leading contributions: *only one pion wraps the torus*

$$\Delta a_\mu(L) = \int_0^\infty dx_0\, \mathcal{K}(x_0) \Big[\quad + \frac{1}{2} \quad \Big] + \mathcal{O}(e^{-1.9 M_\pi L})$$

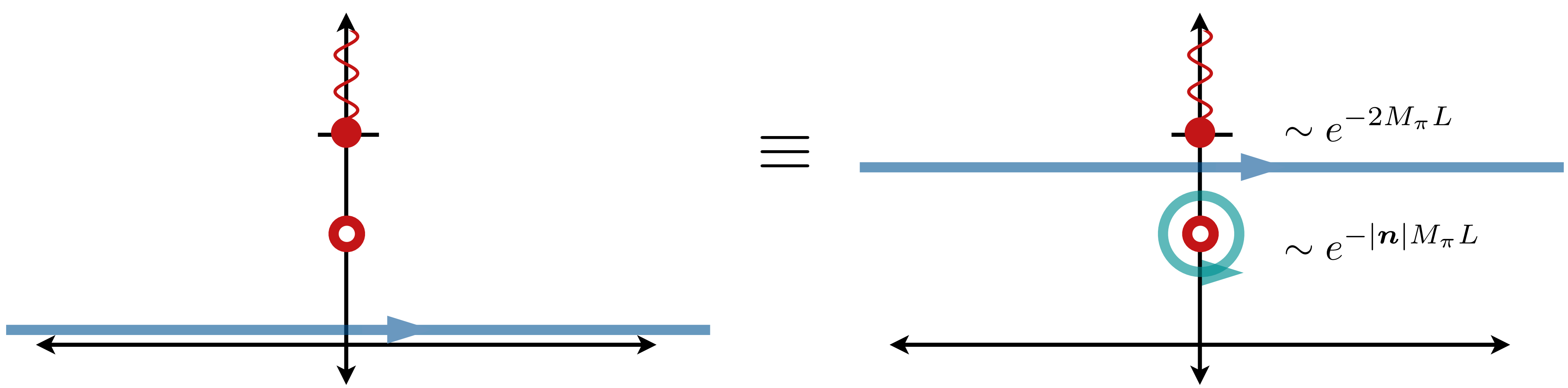


- All *L* dependence inside…



$$\equiv \sum_{\boldsymbol{n}} \frac{e^{iL\boldsymbol{n}\cdot\mathbf{p}}}{p^2 + M_\pi^2 + \Sigma(p^2)}$$

- Exponential decay?



pole contributions → on-shell pions → ***physical Compton amplitude***

Result

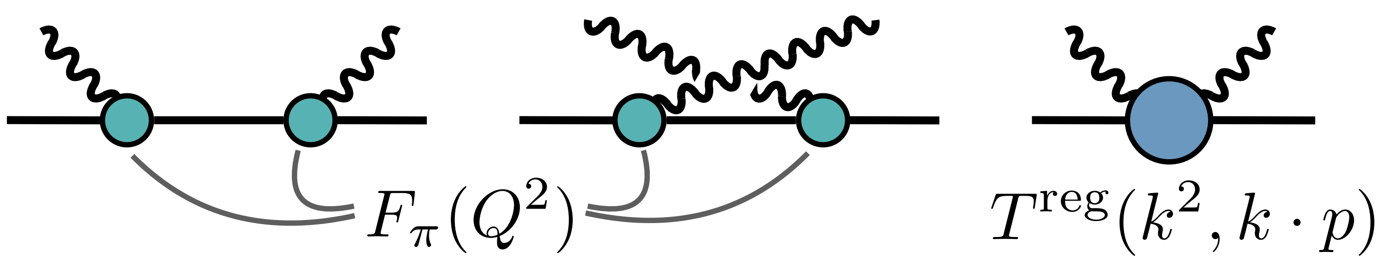
- Sum of ***all single-winding terms*** gives

$$\Delta a_\mu(L) = -\frac{\alpha^2}{m_\mu^2} \sum_{\boldsymbol{n}\neq\boldsymbol{0}} \int \frac{dp_3}{2\pi} \frac{e^{-|\boldsymbol{n}|L\sqrt{M_\pi^2+p_3^2}}}{2\pi L|\boldsymbol{n}|} \int_0^\infty dx_0\, \widehat{\mathcal{K}}(m_\mu x_0) \int \frac{dk_3}{2\pi} \cos(x_0 k_3)\, \mathrm{Re}\, T(-k_3^2, -k_3 p_3)$$

volume dependence — kernel — Compton amplitude

reference to EFT has disappeared — requires only spacelike photons

$$T(k^2, k\cdot p) \equiv i \lim_{\boldsymbol{p}'\to\boldsymbol{p}} \sum_{q=0,\pm1} \int d^4x\, e^{ikx} \langle \boldsymbol{p}', q|\mathrm{T}\mathcal{J}_\rho(x)\mathcal{J}^\rho(0)|\boldsymbol{p}, q\rangle$$

$=$ 

Numerical estimates

$M_\pi = 137$ MeV $\qquad m_\mu = 106$ MeV $\qquad F_\pi(Q^2) = [1 + Q^2/M^2]^{-1} \qquad M = 727$ MeV

$$-100 \times \Delta a_\mu(L)/(700 \times 10^{-10})$$

$M_\pi L$	$\lvert\mathbf{n}\rvert = 1$	$\sqrt{2}$	$\sqrt{3}$	2	$\sqrt{5}$	$\sqrt{6}$	$2\sqrt{2}$	3	$\sum_{\mathbf{n}}$
4	1.26	1.16	0.317	0.104	0.193	0.0944	0.0128	0.0174	3.17
5	0.852	0.428	0.0813	0.0199	0.0287	0.0112	0.00102	0.00117	1.42
6	0.461	0.141	0.0189	0.00349	0.00394	0.00124	0.0000764	0.0000735	0.630
7	0.226	0.0433	0.00417	0.000582	0.000515	0.000130	5.46×10^{-6}	4.41×10^{-6}	0.274
8	0.104	0.0128	0.000883	0.0000936	0.0000652	0.0000132	3.79×10^{-7}	2.57×10^{-7}	0.118

- Slow convergence of $e^{-\alpha_n M_\pi L}$ series expected if large x_0 dominates (Meyer)

- 3 paths to f.v. effects for g-2

correct f.v. matrix elements — this work — fixed-order ChPT

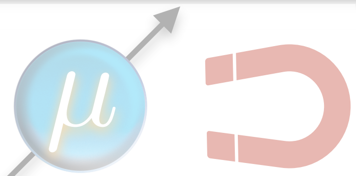
timelike ← $F_\pi(Q^2)$ → spacelike

- Combined information → finite-volume = sub-dominant uncertainty

See… F. M. Stokes - Mon 13:00… Plenaries - Fri 9:00… M. San José Pérez - Tue 7:30

Finite-volume landscape

(Important) $e^{-\mu L}$ effects



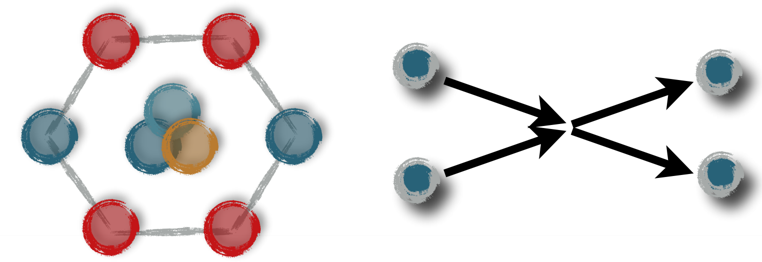
Dictated by analyt. cont. of observables

Relevant for precision & when $1/\mu$ is enhanced

Convergence of $e^{-\alpha_n \mu L}$ series can be delicate

2 → 2 scattering

$\rho(770),\quad \sigma/f_0(500),\quad K^\star,\quad \kappa,\quad a_0,\quad \Delta,\quad \cdots$



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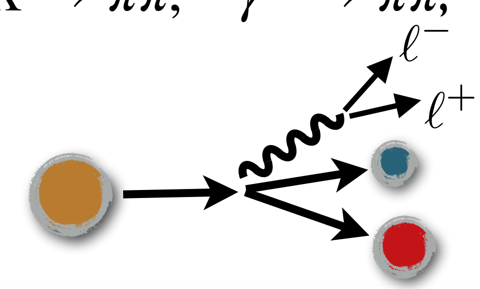

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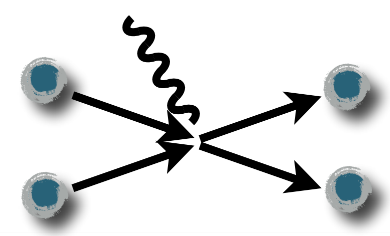

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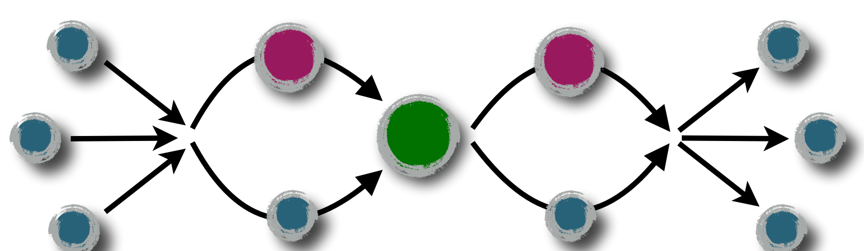

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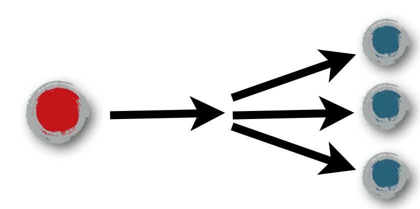

Finite-volume landscape

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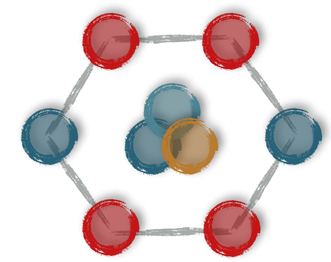

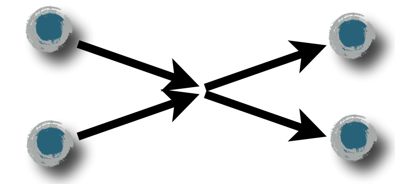

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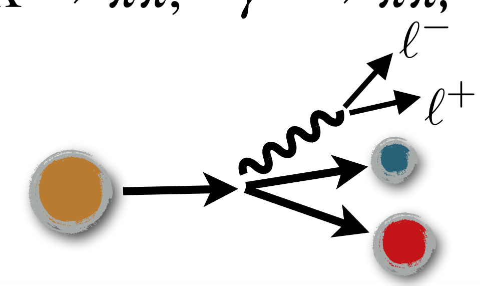

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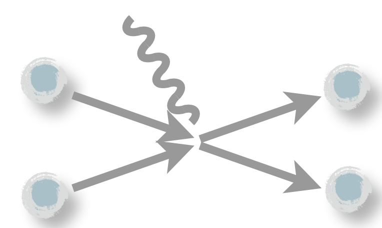

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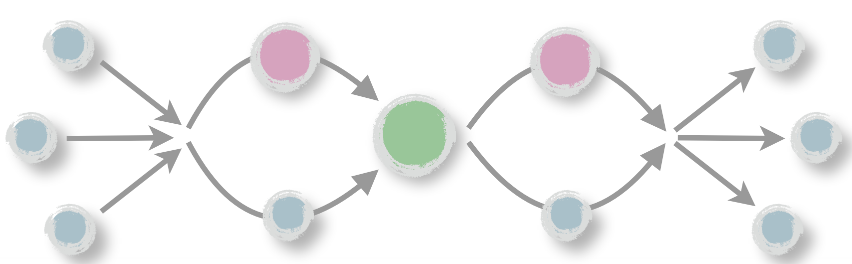

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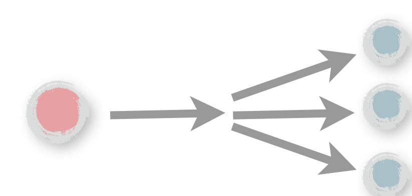
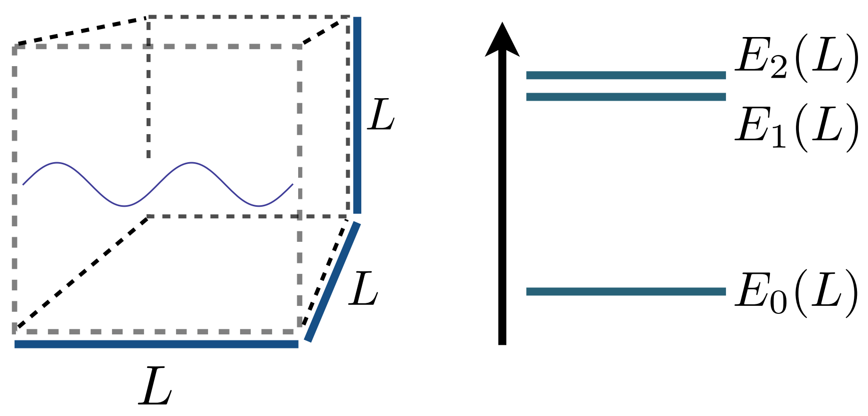


The finite-volume as a tool

- Finite-volume set-up

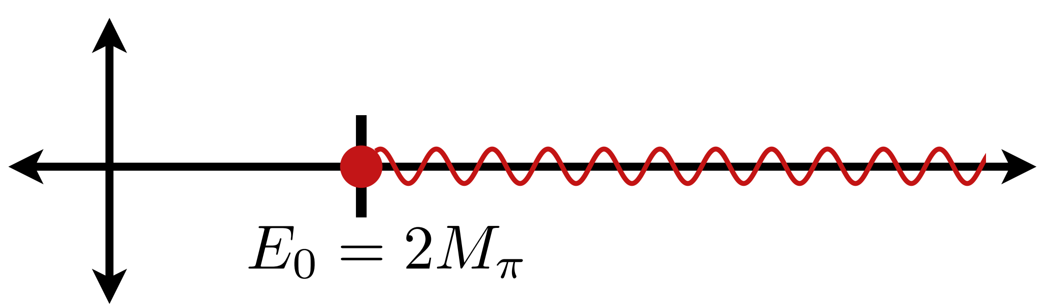


- cubic, spatial volume (extent L)
- periodic

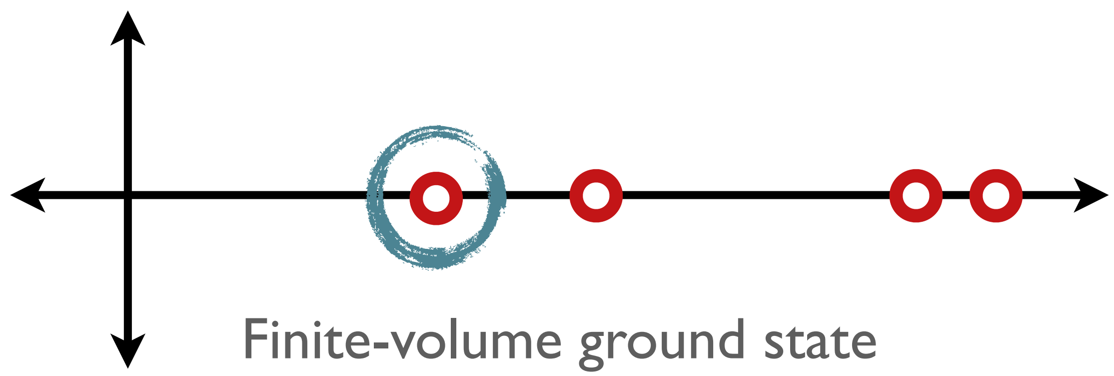
$$\vec{p} = \frac{2\pi}{L}\vec{n}\,, \qquad \vec{n} \in \mathbb{Z}^3$$

- L is large enough to neglect $e^{-M_\pi L}$
- T and lattice also neglected

- Scattering leaves an ***imprint*** on finite-volume quantities



Infinite-volume threshold



Finite-volume ground state

scattering length

$$\mathcal{M}_{\ell=0}(2M_\pi) = -32\pi M_\pi a$$

$$E_0(L) = 2M_\pi + \frac{4\pi a}{M_\pi L^3} + \mathcal{O}(1/L^4)$$

Huang, Yang (1958)

Derivation

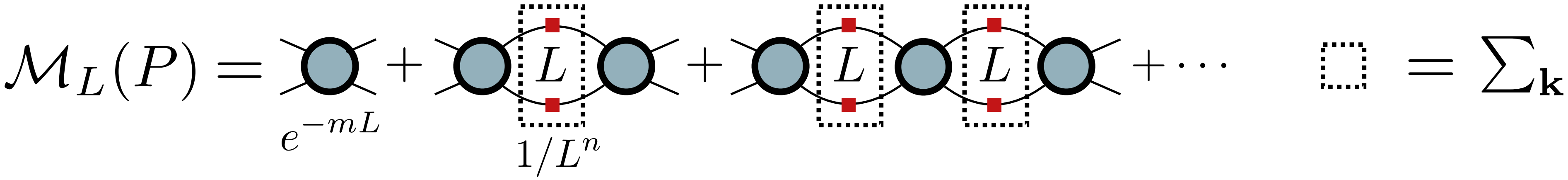
- Consider the finite-volume correlator:

$\mathcal{M}(s)$

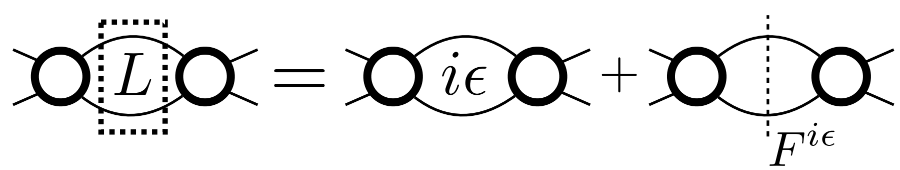
probability amplitude

$\mathcal{M}_L(P)$

poles give f.v. spectrum

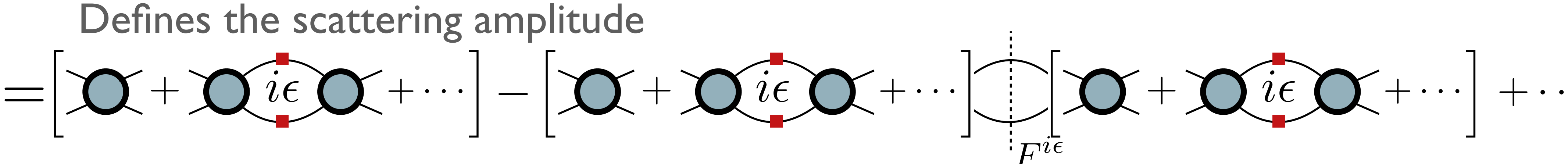


For two-particle energies $(2m)^2 < s < (4m)^2$, what is the L dependence?

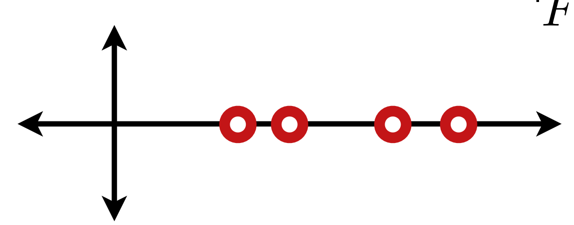


Cut projects loop to ***on-shell energies***
$F^{i\epsilon}$= matrix of known geometric functions

Defines the scattering amplitude



$$= \frac{1}{\mathcal{M}(s)^{-1} + F^{i\epsilon}(P,L)}$$



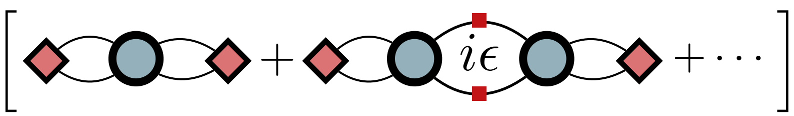
Lüscher, Rummukainen, Gottlieb, Kim, Sachrajda, Sharpe, Christ, Yamazaki, Lin, Martinelli, Testa, Savage, Beane, Meyer, Agadjanov, Bernard, Rusetsky, Hoja, Meißner, Davoudi, Gockeler (et al.)…

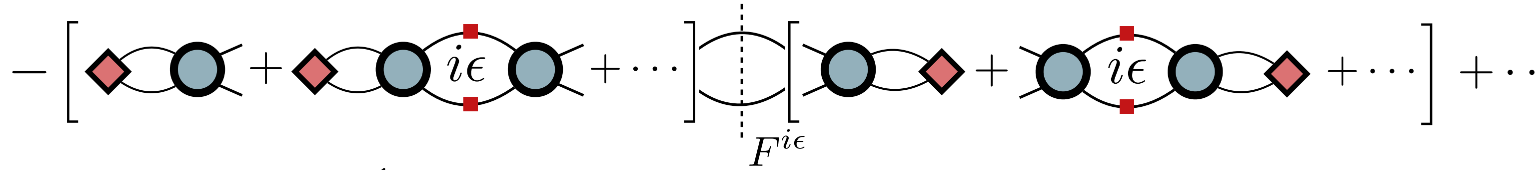
$1 + \mathscr{J} \to 2$

$$C_L(P) = \langle \pi | \mathcal{J}\mathcal{J} | \pi \rangle_L$$

Instead of this…

$$\mathcal{M}_L(P) = \mathcal{M}(s) - \mathcal{M}(s) \frac{1}{\mathcal{M}(s) + F^{i\epsilon}(P,L)^{-1}} \mathcal{M}(s)$$





$$= C_\infty^{i\epsilon}(s) - A_{\text{in}}^{i\epsilon}(s) \frac{1}{\mathcal{M}(s) + F^{i\epsilon}(P,L)^{-1}} A_{\text{out}}^{i\epsilon}(s)$$

Useful, since…

$$\lim_{E \to E_n(L)} [E - E_n(L)] C_L(P) \;\propto\; |\langle n, L | \mathcal{J} | \pi, L \rangle|^2$$

Crucial information = residue at the pole

$$\mathcal{R}(P,L) = - \lim_{E \to E_n(L)} \frac{E - E_n(L)}{\mathcal{M}(s) + F^{i\epsilon}(P,L)^{-1}} = \frac{1}{\mu'(E)} \mathbf{v}^T \mathbf{v}$$

MTH, Sharpe (2012) • Briceño, MTH, Walker-Loud (2015) • Briceño, MTH (2016) •

Lüscher, Rummukainen, Gottlieb, Kim, Sachrajda, Sharpe, Christ, Yamazaki, Lin, Martinelli, Testa, Savage, Beane, Meyer, Agadjanov, Bernard, Rusetsky, Hoja, Meißner, Davoudi, Gockeler (et al.)…

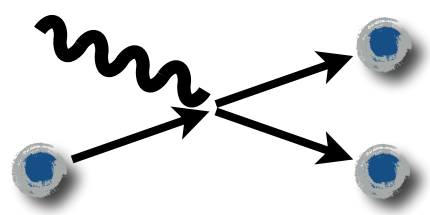
Transition amplitudes

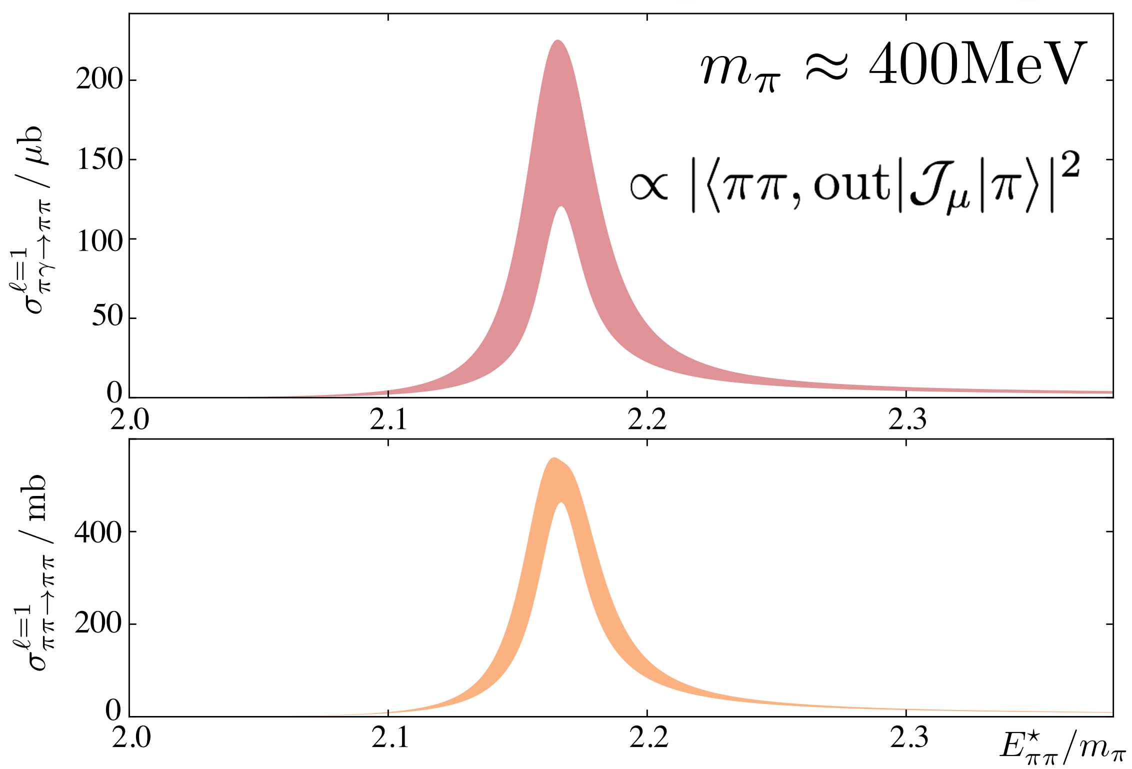
- An effective relation on states

$$\langle n, L|\mathcal{J}|\pi, L\rangle \propto \sum_\alpha \mathbf{v}_\alpha \langle \alpha, \text{out}|\mathcal{J}|\pi\rangle$$

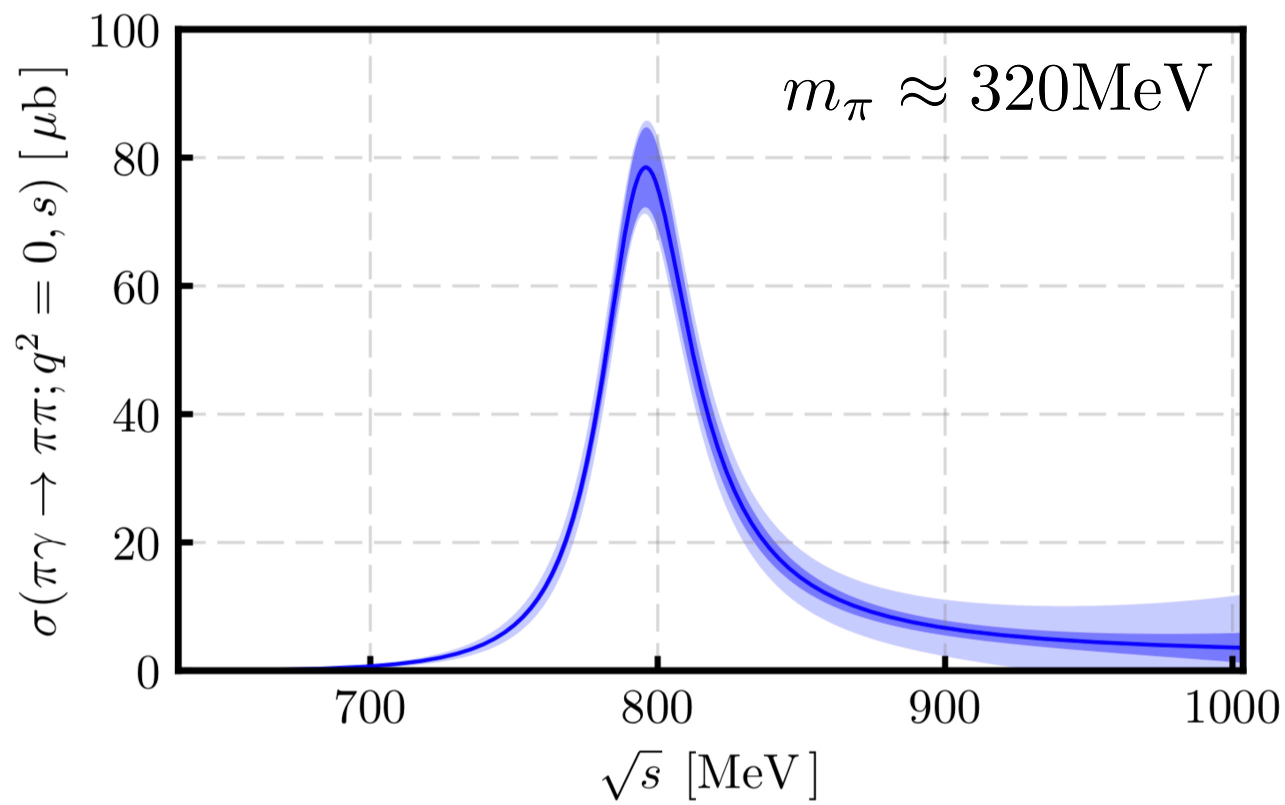
$$\mathcal{R}(P,L) = -\lim_{E\to E_n(L)} \frac{E - E_n(L)}{\mathcal{M}(s) + F^{i\epsilon}(P,L)^{-1}} = \frac{1}{\mu'(E)}\mathbf{v}^T\mathbf{v}$$

$$\langle \pi\pi, \text{out}|\mathcal{J}_\mu|\pi\rangle \equiv$$





Briceño et. al., Phys. Rev. D93, 114508 (2016)



Alexandrou et. al., Phys. Rev. D98, 074502 (2018)

Finite-volume landscape

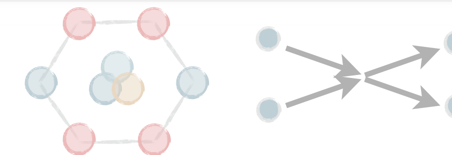
(Important) $e^{-\mu L}$ effects

Dictated by analyt. cont. of observables

Relevant for precision & when $1/\mu$ is enhanced

Convergence of $e^{-\alpha_n \mu L}$ series can be delicate

2 → 2 scattering

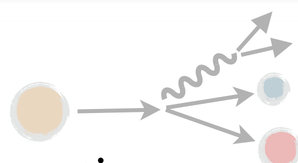


Sets the standard of production-ready formalism

$E_n(L)$ has power-like dependence on all open channels

Shows the utility of the skeleton expansion framework

(1+)$\mathscr{J}$ → 2 transitions



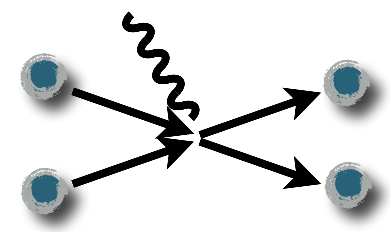
Eigenvectors of q.c. matrix = useful information

Looks like a relation on states:

$$|n, L\rangle = a(L)\,|\pi\pi, \text{out}\rangle + b(L)\,|K\bar{K}, \text{out}\rangle + \cdots$$

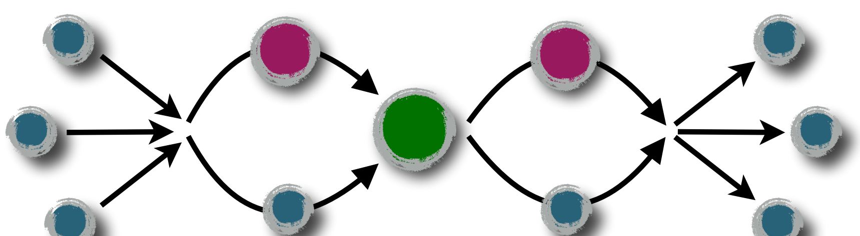
2+$\mathscr{J}$ → 2 scattering

$\rho\gamma \to \rho, \quad \pi\pi \to \gamma\pi\pi, \quad \cdots$



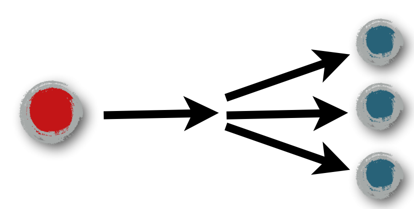
2 → 3, 3 → 3 scattering

$N\pi\pi \to N(1420) \to N\pi\pi, \quad \cdots$



(1+)$\mathscr{J}$ → 3 transitions

$K \to \pi\pi\pi, \quad \gamma^\star \to \pi\pi\pi, \quad \cdots$



Finite-volume landscape

(Important) $e^{-\mu L}$ effects

Dictated by analyt. cont. of observables

Relevant for precision & when $1/\mu$ is enhanced

Convergence of $e^{-\alpha_n \mu L}$ series can be delicate

$2 \to 2$ scattering

Sets the standard of production-ready formalism

$E_n(L)$ has power-like dependence on all open channels

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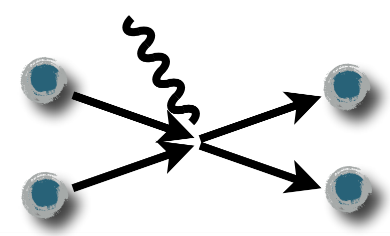
$$|n, L\rangle = a(L)\,|\pi\pi, \text{out}\rangle + b(L)\,|K\bar{K}, \text{out}\rangle + \cdots$$

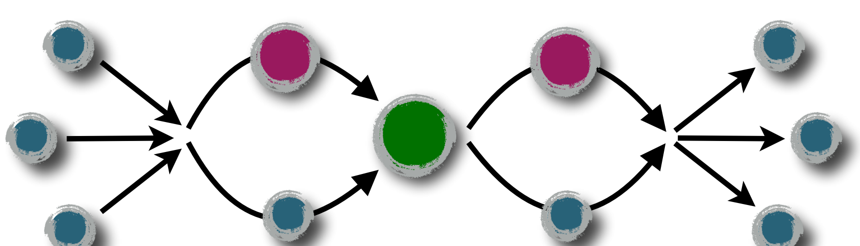
$2{+}\mathscr{J} \to 2$ scattering

$$\rho\gamma \to \rho, \quad \pi\pi \to \gamma\pi\pi, \quad \cdots$$



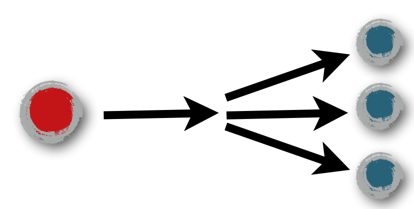
$2 \to 3,\ 3 \to 3$ scattering

$$N\pi\pi \to N(1420) \to N\pi\pi, \quad \cdots$$



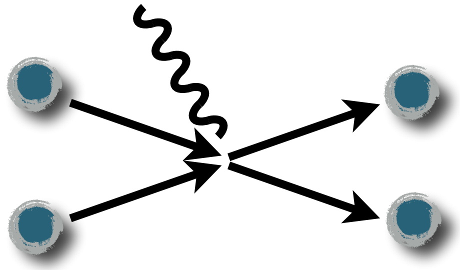
$(1+)\mathscr{J} \to 3$ transitions

$$K \to \pi\pi\pi, \quad \gamma^\star \to \pi\pi\pi, \quad \cdots$$

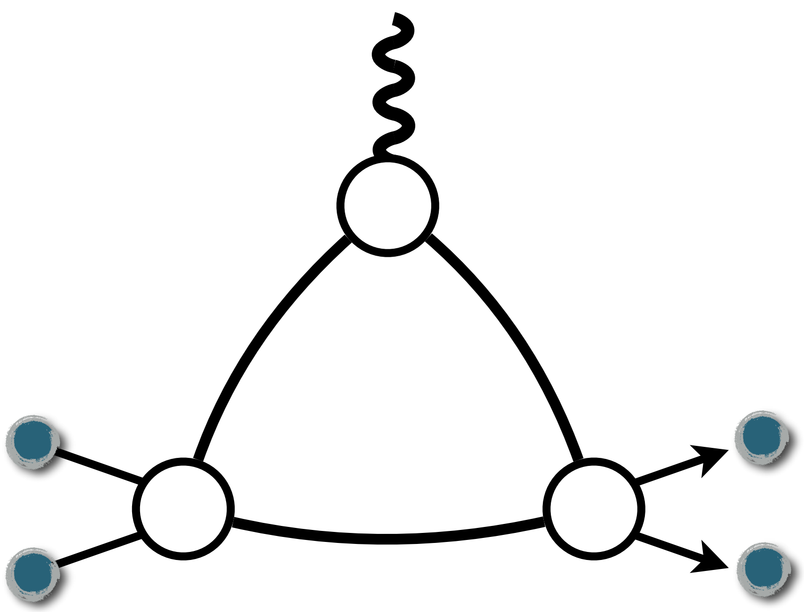


$2 + \mathscr{J} \rightarrow 2$

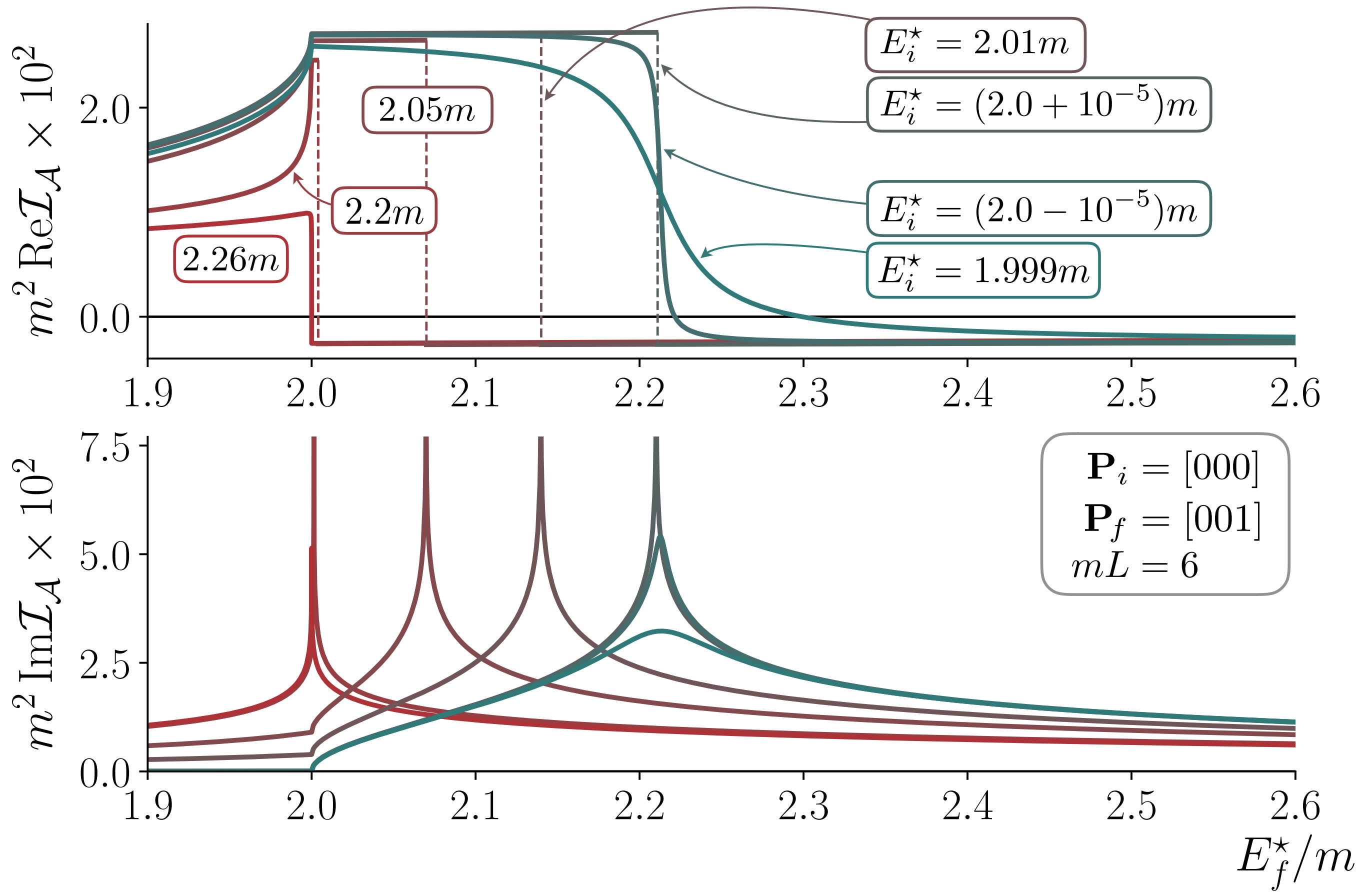
- Fully developed formalism for multi-hadron form factors

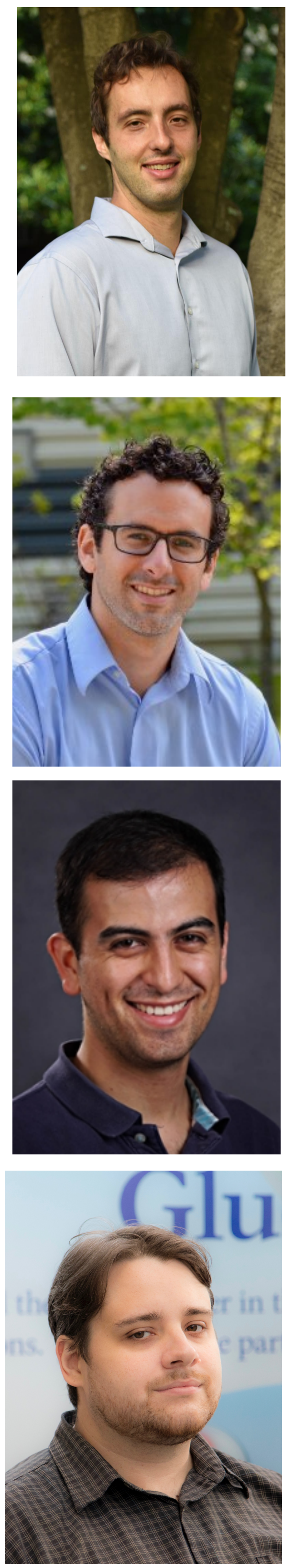
$$\langle \pi\pi, \text{out} | \mathcal{J}_\mu | \pi\pi, \text{in} \rangle \equiv$$ 

- Continuation to the pole → ***resonance form factors***
- Must carefully treat ***triangle*** singularities



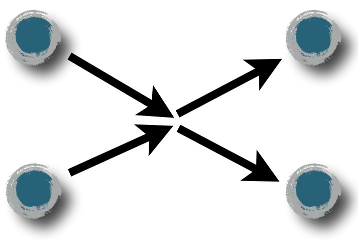
$\sim$





Briceño, MTH (2017) • Baroni, Briceño, MTH, Ortega-Gama (2018) • Briceño, MTH, Jackura (2020)

Complication: degrees of freedom

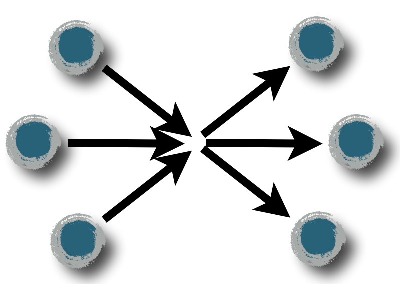


12 momentum components

-10 Poincaré generators

2 degrees of freedom

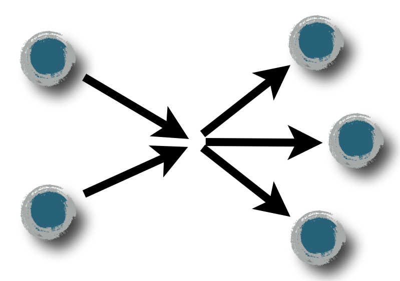
$$\vec{p}_1 + \vec{p}_2 \;\to\; \vec{p}_3 + \vec{p}_4 \;\longrightarrow\; \text{Mandelstam } s, t$$



18 momentum components

-10 Poincaré generators

8 degrees of freedom



15 momentum components

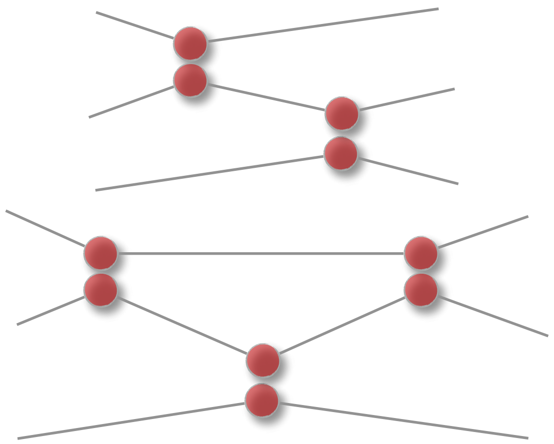
-10 Poincaré generators

5 degrees of freedom

Complication: on-shell states

- Classical pairwise scattering

for $m_1 = m_2 = m_3$ up to ***3*** binary collisions are possible



Dispersion Relations for Three-Particle Scattering Amplitudes. I*

MORTON RUBIN
Physics Department, University of Wisconsin, Madison, Wisconsin

AND

ROBERT SUGAR
Physics Department, Columbia University, New York, New York

AND

GEORGE TIKTOPOULOS
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey

(Received 31 January 1966)

$$b = \frac{(m_1+m_3)(m_2+m_3)}{m_1 m_2}$$

It follows that if

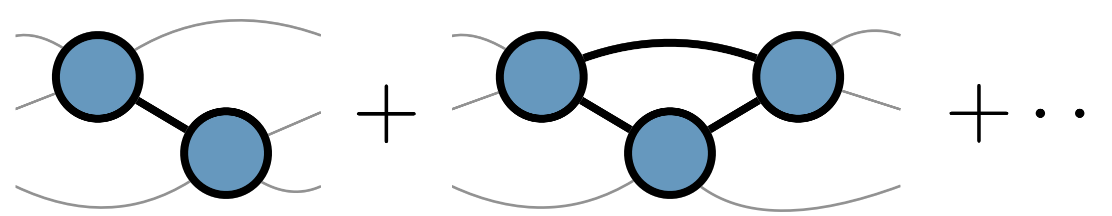
$$b^{n-2}(b-1) > 1\,, \qquad \text{(IV.18)}$$

then $2n+1$ successive binary collisions are kinematically impossible.

$m_1 = m_2 = m_3 - \varepsilon$:
4 collisions possible
$\pi\pi K$

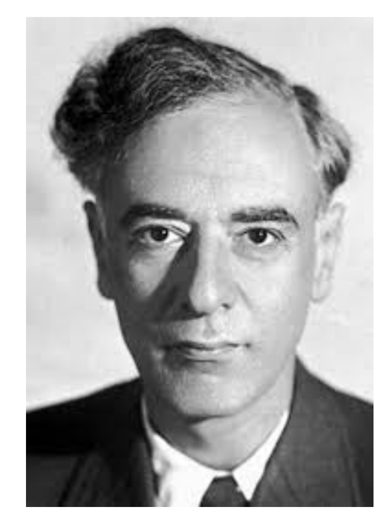
b < 2
5 collisions possible
$\pi K K$

- Correspond to Landau singularities

$i\mathcal{M}_{3\to 3} \equiv$ fully connected correlator $=$ 

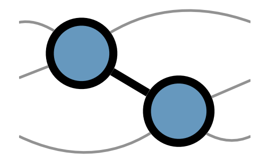
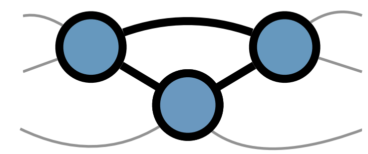
complicate analyticity & unitarity

difficult to disentangle kinematic singularities from resonance poles



Two key observations

- Intermediate $K_{\mathrm{df},3}$ removes singularities

$\mathcal{K}_{\mathrm{df},3} \equiv$ fully connected diagrams w/ *PV pole prescription* $-$  $+$  $+ \cdots$

same degrees of freedom as M_3

smooth real function

relation to M_3 = known

- $K_{\mathrm{df},3}$ has a systematic low-energy expansion

$$\mathcal{K}_{\mathrm{df},3}(p_3, p_2, p_1; k_3, k_2, k_1) = \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},0} + \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},1}\,\Delta + \cdots \qquad \Delta = \frac{s-(3m)^2}{(3m)^2}$$

smooth real function

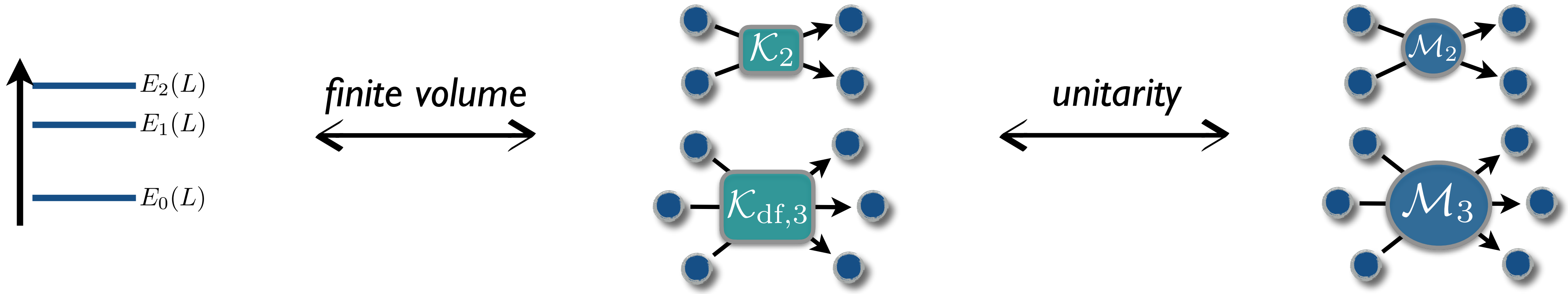
analogous to effective range expansion

$$p \cot\delta = -\frac{1}{a} + \frac{1}{2}\, r\, p^2 + \mathcal{O}(p^4)$$

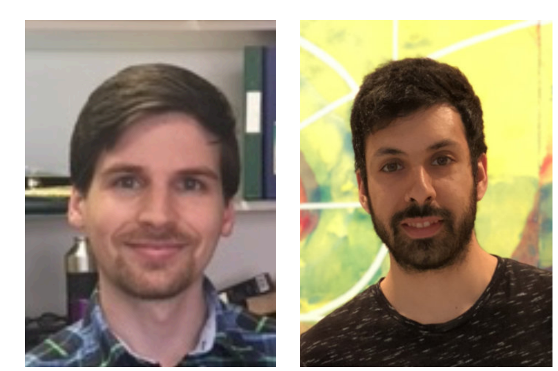
gives handle on many *degrees of freedom*

(DOFs enter order by order)

Status…



- ☑ Identical spin-zero, no 2-to-3, no K2 poles • MTH, Sharpe (2014, 2015) •
- ☑ as above… but including 2-to-3 • Briceño, MTH, Sharpe (2017) •
- ☑ as above… but including K2 poles • Briceño, MTH, Sharpe (2018) •
- ☑ Non-identical, non-degenerate spin-zero $\pi\pi\pi \to \rho\pi \to \omega \to \rho\pi \to \pi\pi\pi$
  • MTH, Romero-López, Sharpe (2020) • Blanton, Sharpe (2020, 2021)
- ☐ Multiple three-particle channels… Spin!



Related work

- Finite-volume unitarity method
  Döring, Mai (2016,2017)

  Gives connection to unitarity relations

- Non-relativistic EFT method
  Hammer, Pang, Rusetsky (2017)

  Simplified derivation + integral equations

  Do not yet include non-degenerate, 2-to-3

- All methods

  Rely on intermediate, scheme-dependent quantity

  Hold up to e^{-mL} and for $E_3^\star < 5m_\pi$

  Equivalent where comparable

- Review articles

  MTH and Sharpe, 1901.00483 • Rusetsky, 1911.01253 • Mai, Döring, Rusetsky, 2103.00577

Not covered here

- Activity extracting and fitting three-hadron energies

• Hörz, Hanlon (2019) • Blanton, Romero-López, Sharpe (2019) •
• Alexandru, Brett, Culver, Döring, Guo, Lee, Mai (2019) •
• Fischer, Kostrzewa, Liu, Romero-López, Ueding, Urbach (2020) •
• Mai, Alexandru, Brett, Culver, Döring, Lee, Sadasivan (2021) •
• Blanton, Hanlon, Hörz, Morningstar, Romero-López, Sharpe (2021) •

- Activity connecting and extending formalisms

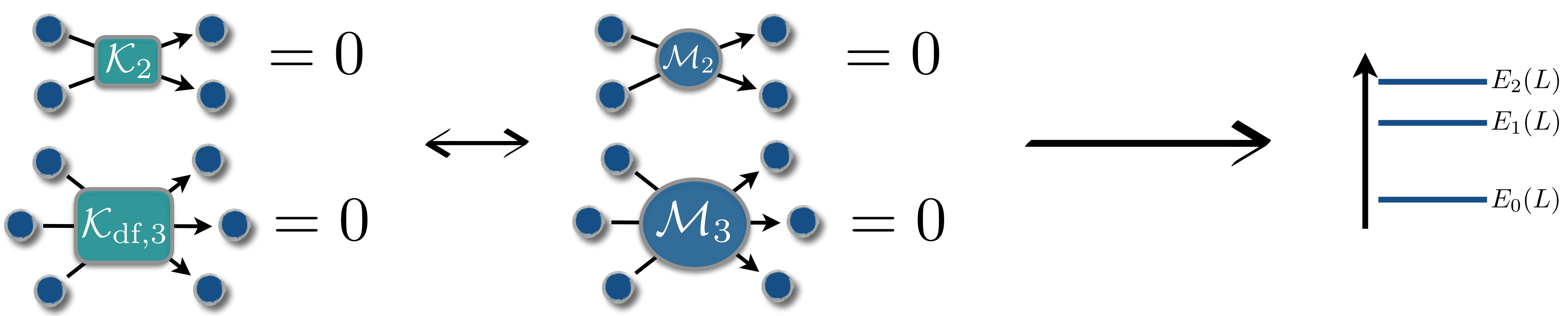
Relating infinite-volume equations • Jackura *et al.* (2019) •

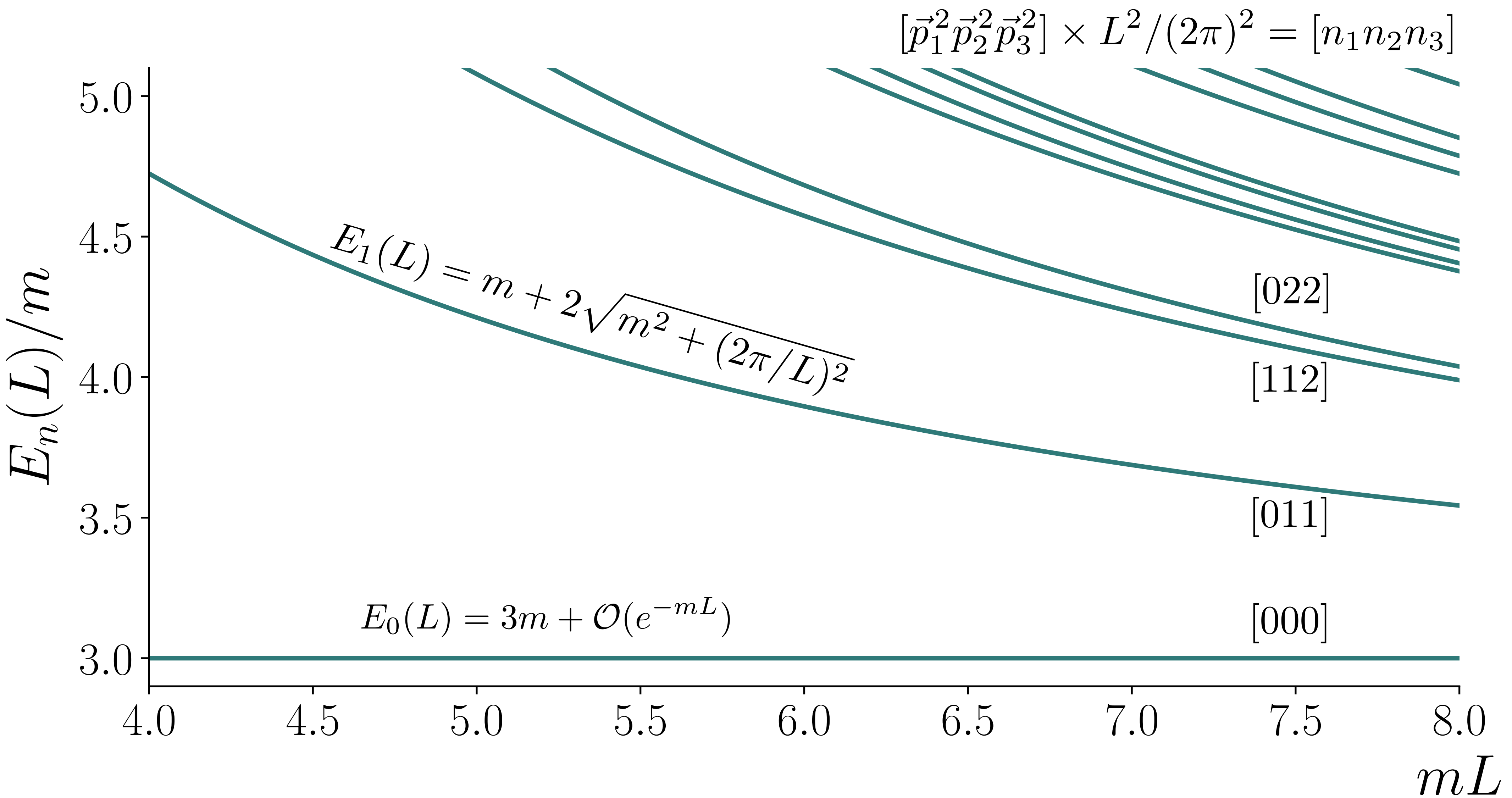
Alternative derivations • Blanton, Sharpe (2020) •

Equivalence of formalisms (where comparable) • Blanton, Sharpe (2020) •

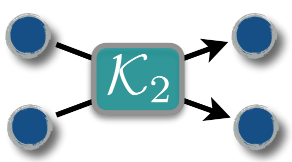
See… B. Hörz - Fri 1:00… Grabowska, Sharpe, Hanlon, Mai, Brett, Blanton, Dawid - Thurs 13:00

Non-interacting energies

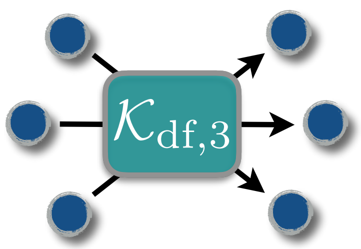




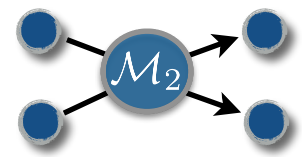
Two-particle interactions



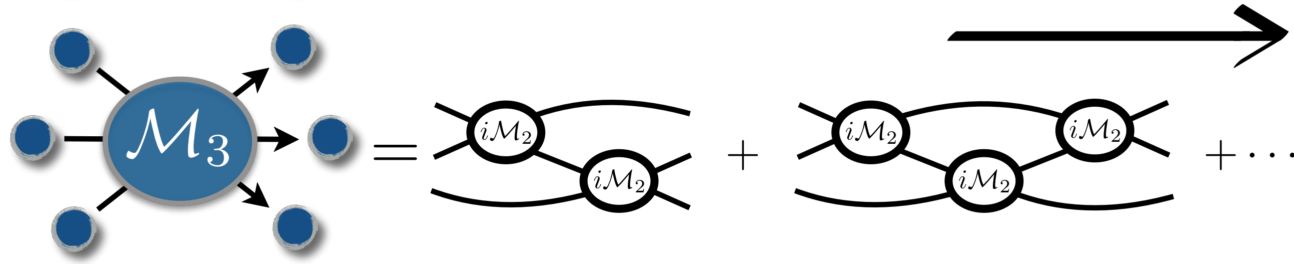
$= -16\pi\sqrt{s}\,a$

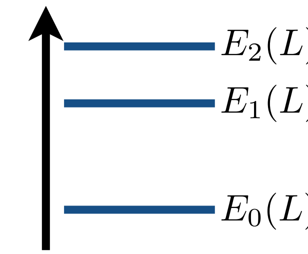


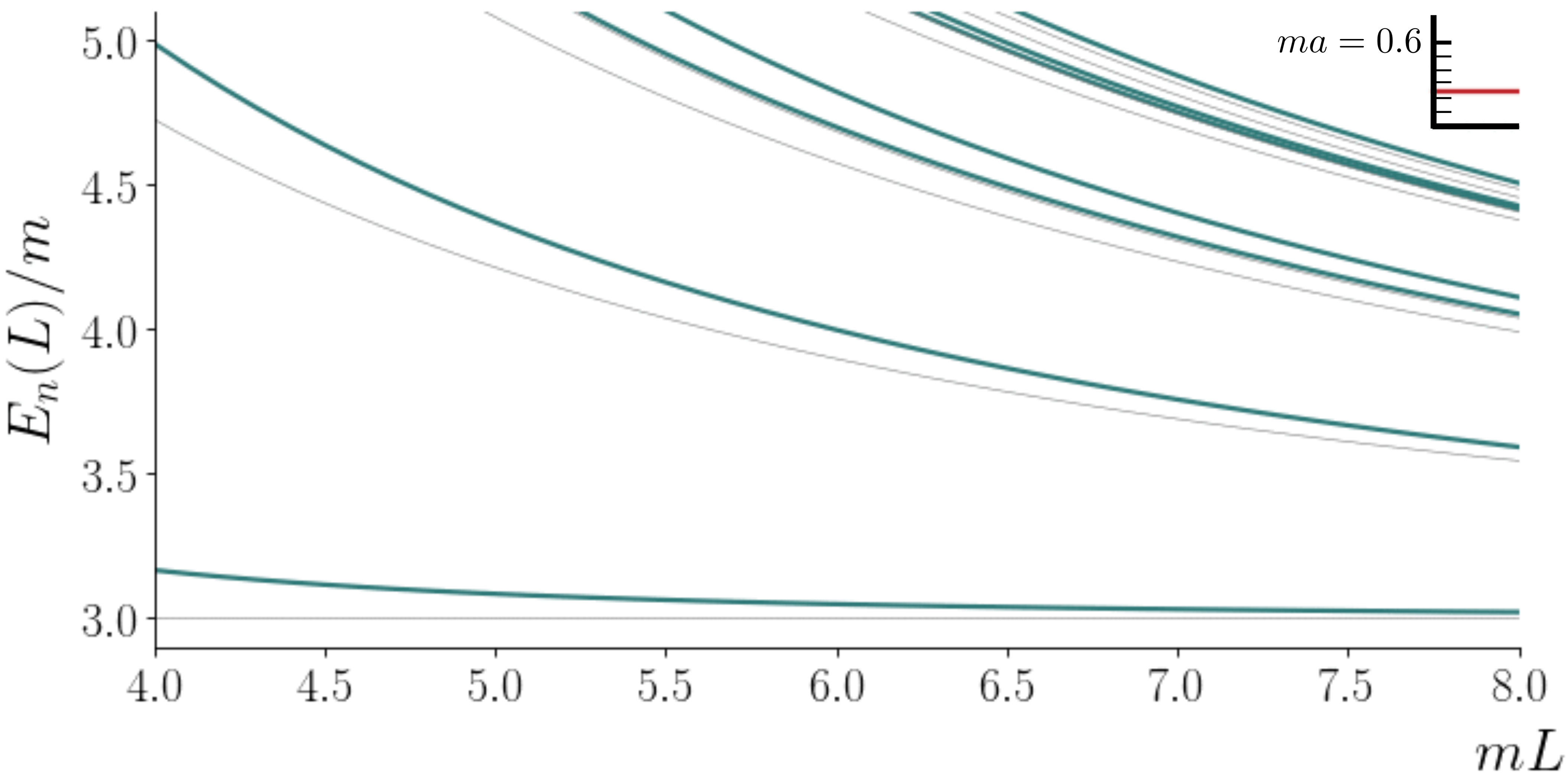
$= 0$



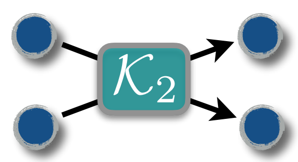
$$= \frac{16\pi\sqrt{s}}{-1/a - ip}$$



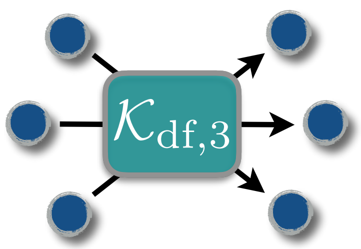




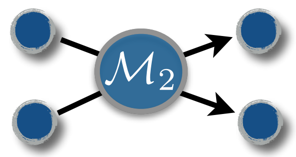
Two-particle interactions



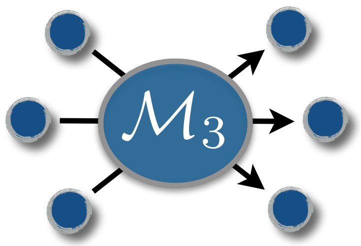
$= -16\pi\sqrt{s}\,a$



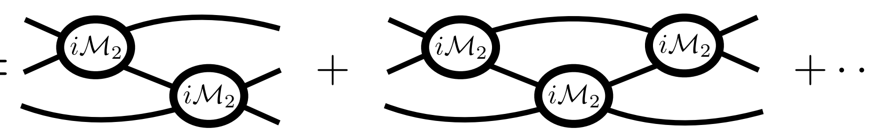
$= 0$

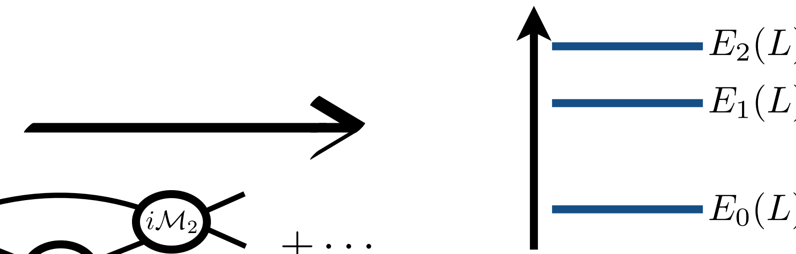


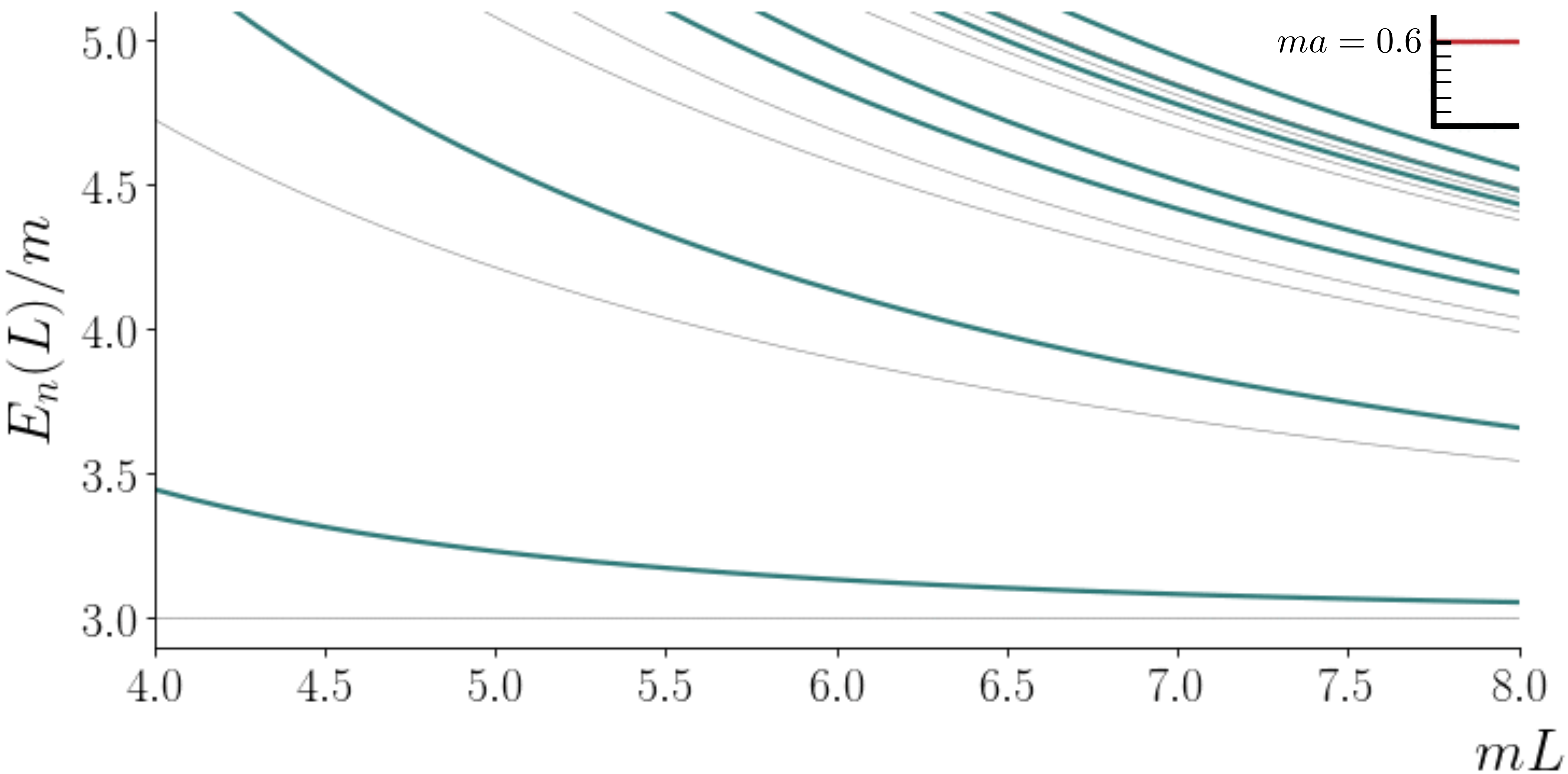
$$= \frac{16\pi\sqrt{s}}{-1/a - ip}$$



$=$

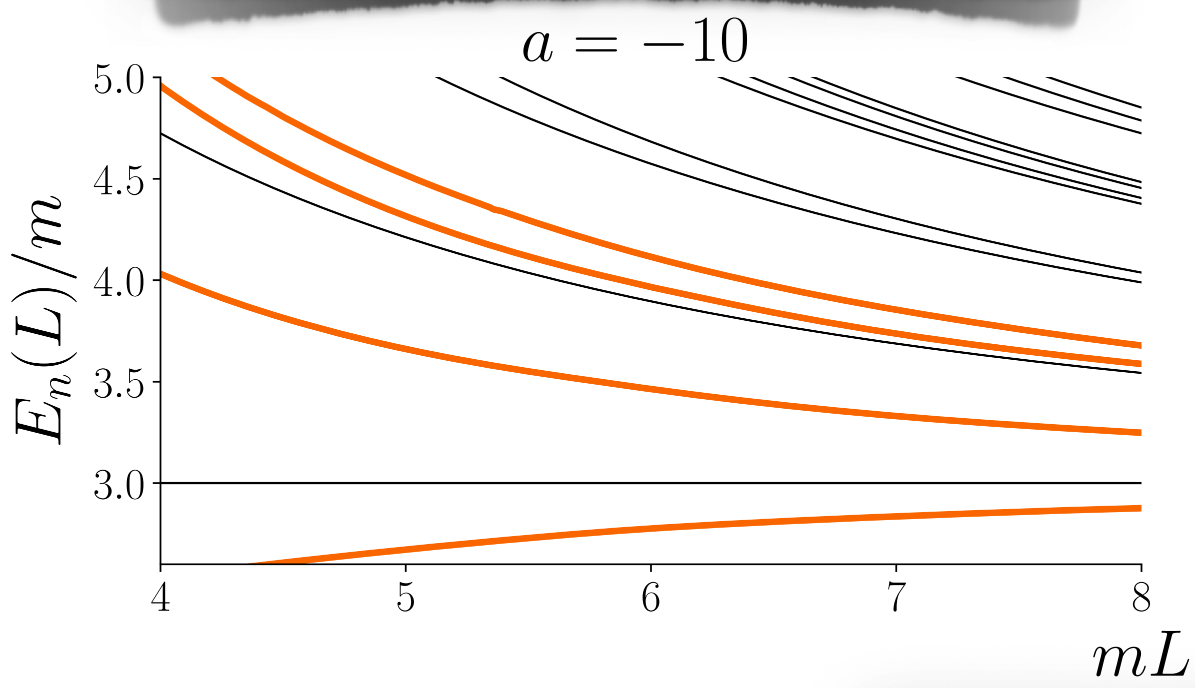




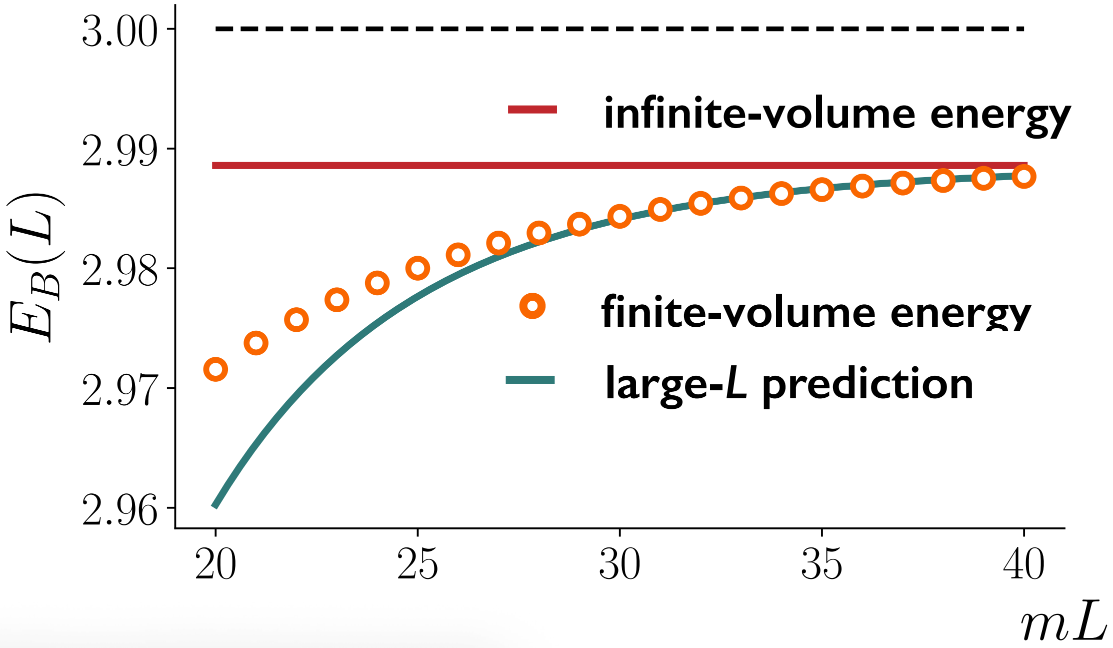


Many toy results

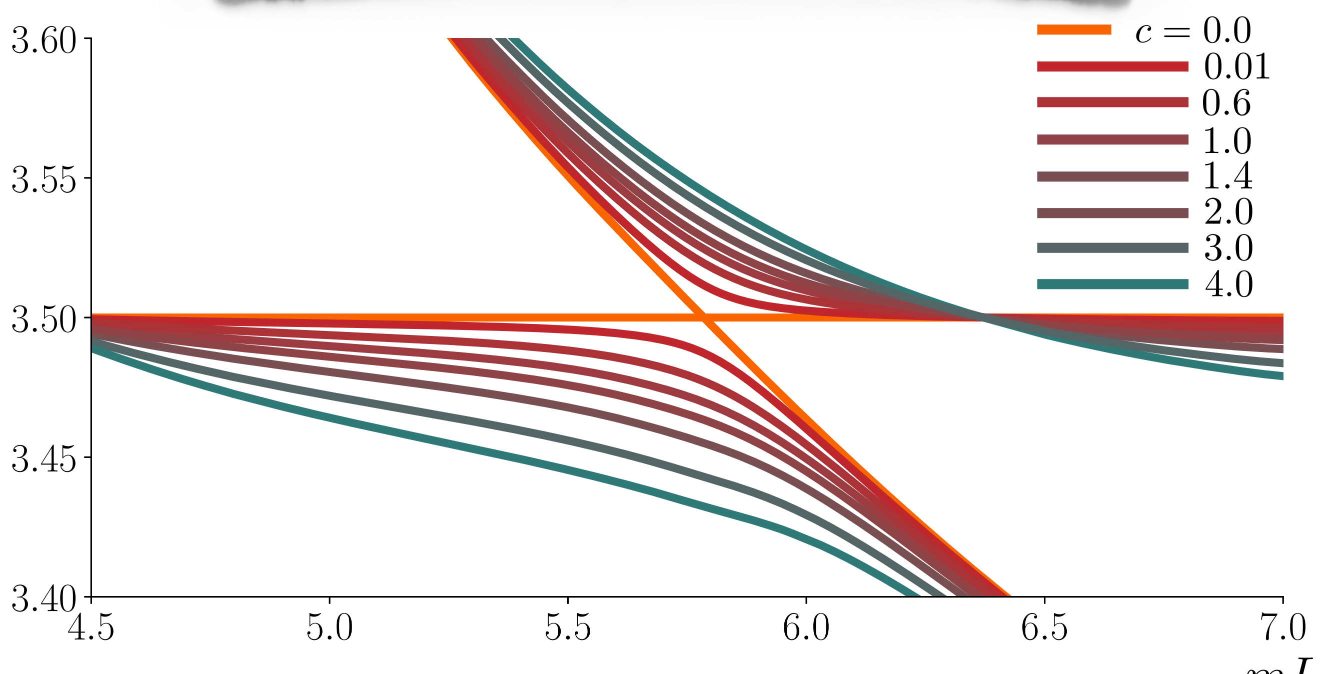
Spectrum with no 3-particle interaction



Finite-volume effects on a 3-particle bound state



Model of a 3-particle resonance

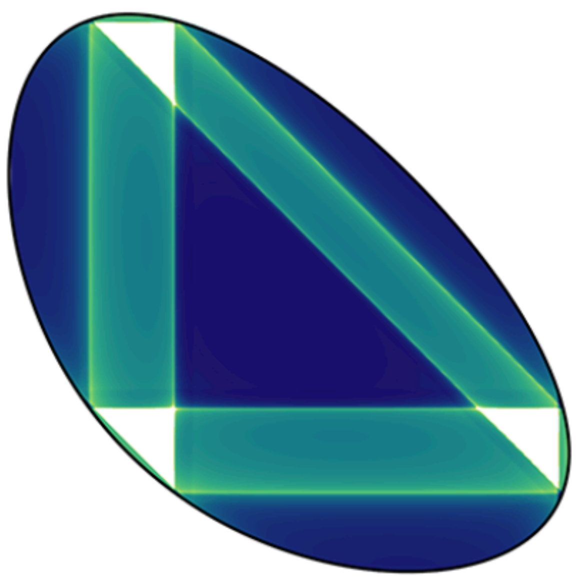


• Briceño, MTH, Sharpe (2018) •

Energy-Dependent $\pi^+\pi^+\pi^+$ Scattering Amplitude from QCD

Maxwell T. Hansen,[1,2,*] Raul A. Briceño,[3,4,†] Robert G. Edwards,[3,‡]
Christopher E. Thomas,[5,§] and David J. Wilson[5,‖]

(for the Hadron Spectrum Collaboration)



EDITORS' SUGGESTION

Energy-Dependent $\pi^+ \pi^+ \pi^+$ Scattering Amplitude from QCD

A three-hadron scattering amplitude is computed using lattice QCD for the first time.

Maxwell T. Hansen *et al.*

Phys. Rev. Lett. **126**, 012001 (2021)

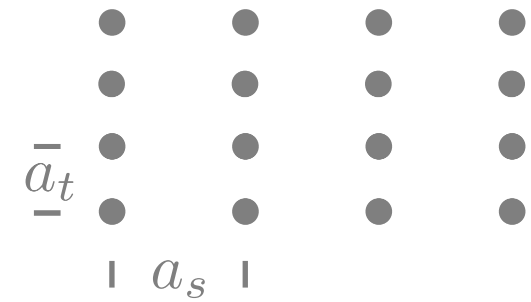
$\pi^+\pi^+\pi^+ \to \pi^+\pi^+\pi^+$

lattice details

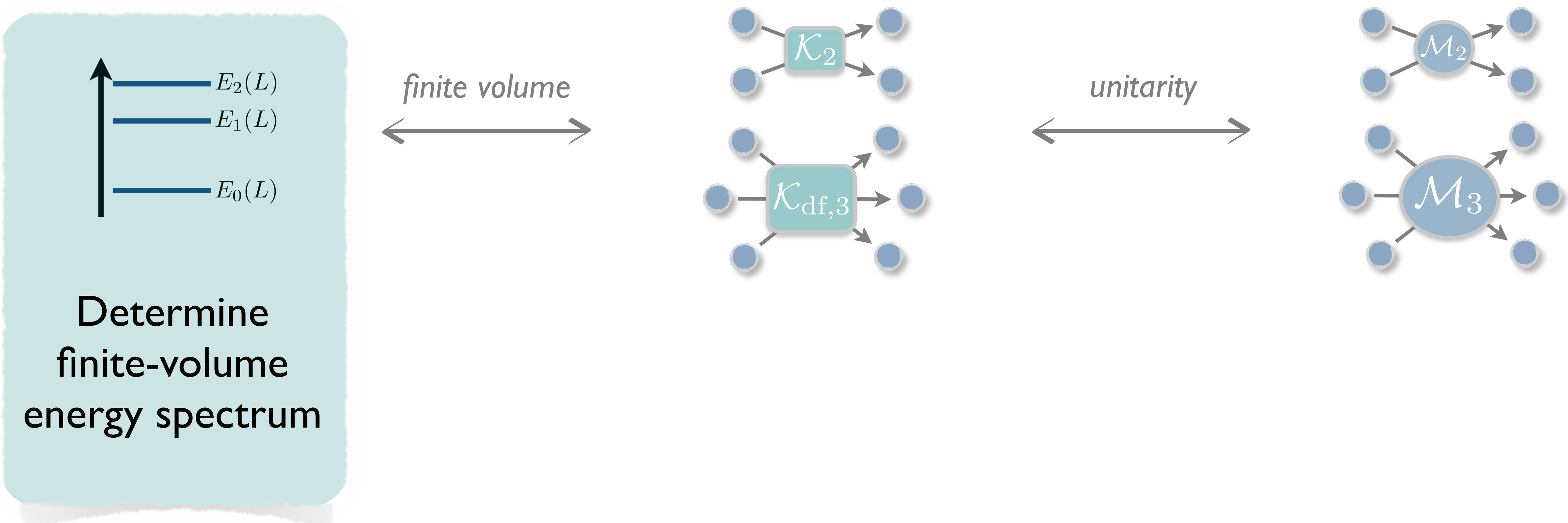
$N_f = 2+1 \qquad a_s/a_t = 3.444(6)$

$m_\pi \approx 400\text{MeV} \qquad a_s \approx 0.12\text{fm}$

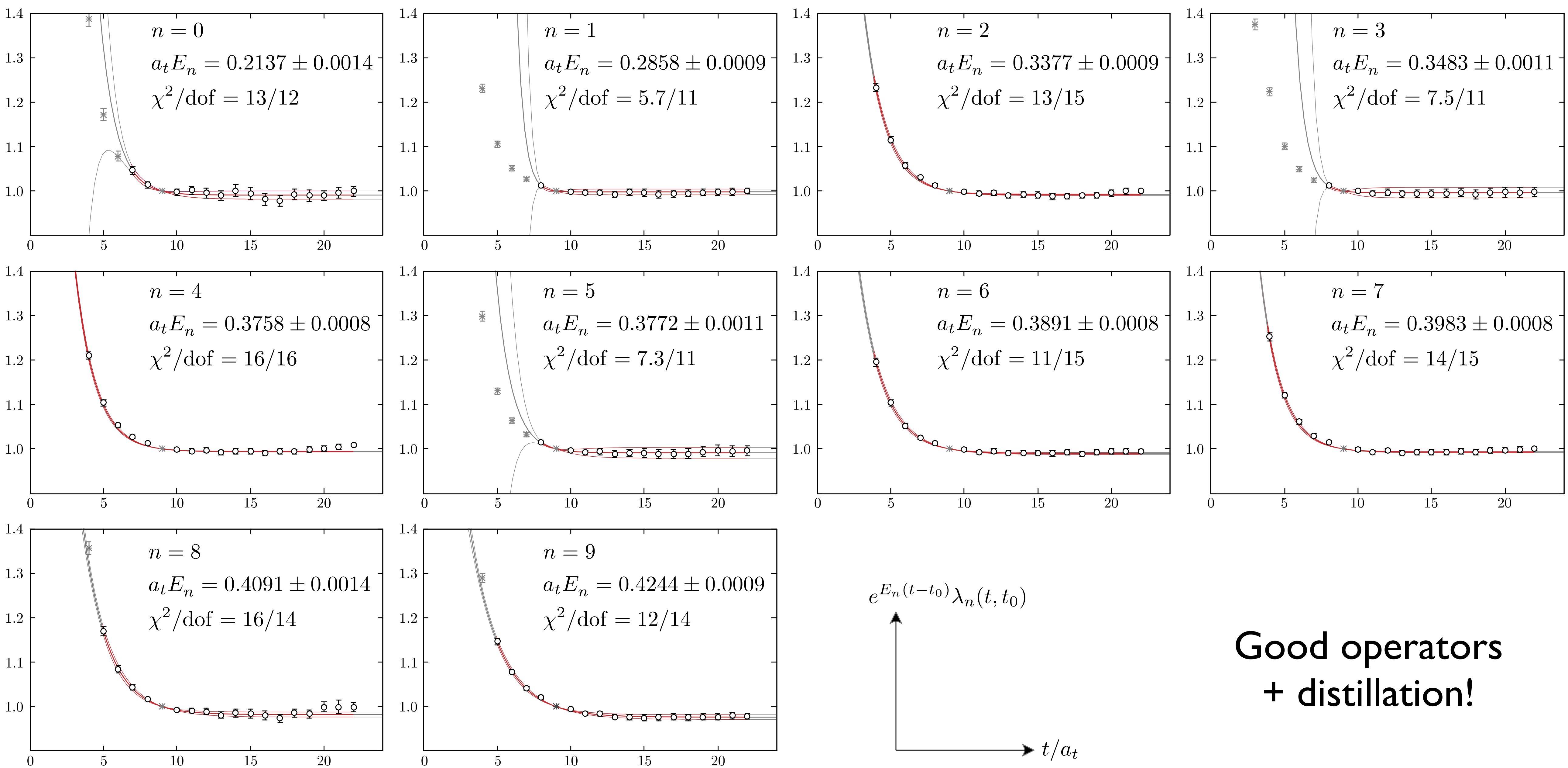
$L_s/a_s = 20, 24$



❑ Workflow outline



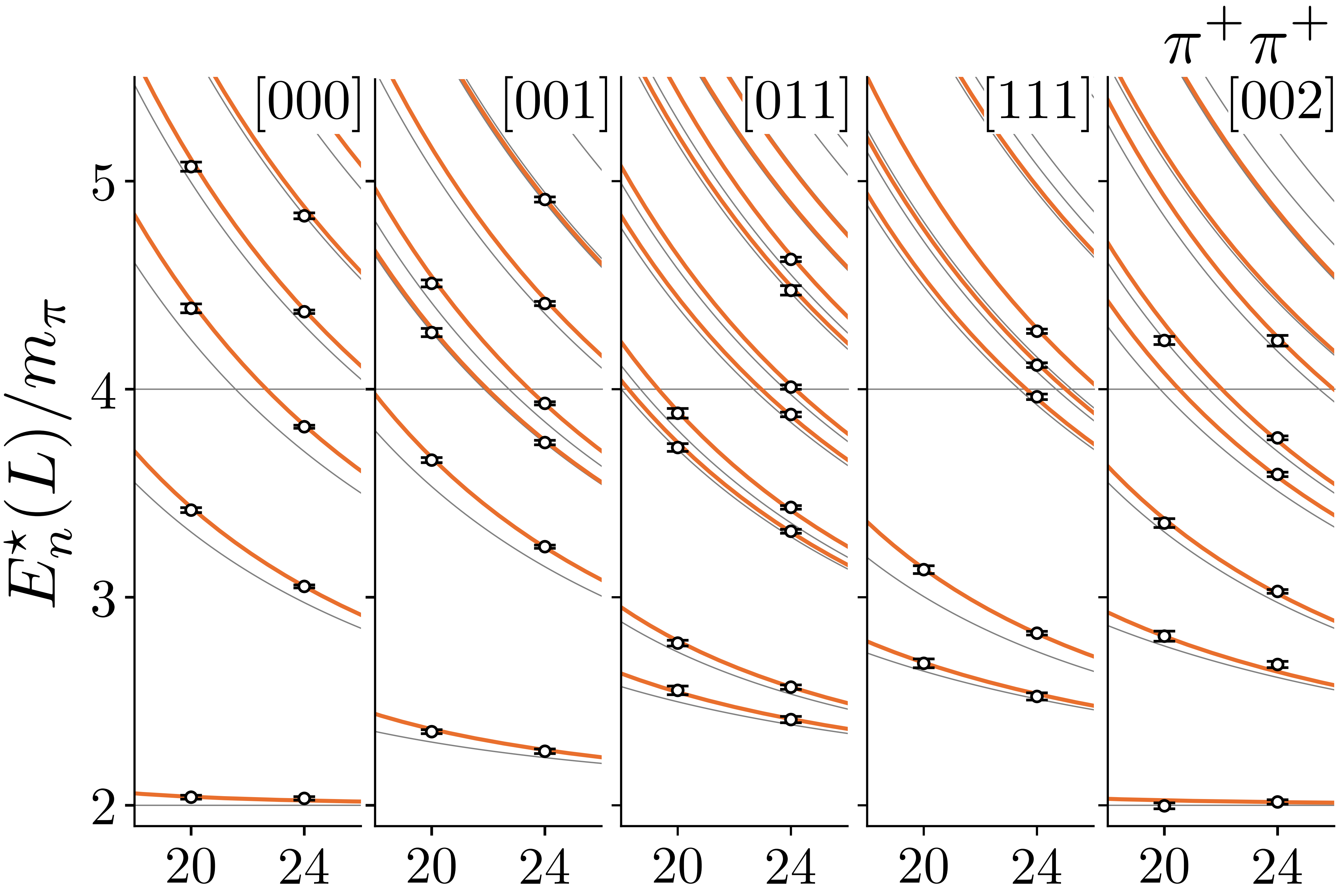
$$I = 3\,(\pi^+\pi^+\pi^+),\ \boldsymbol{P} = [000],\ \Lambda = A_1^-,\ L/a_s = 24$$



Good operators
+ distillation!

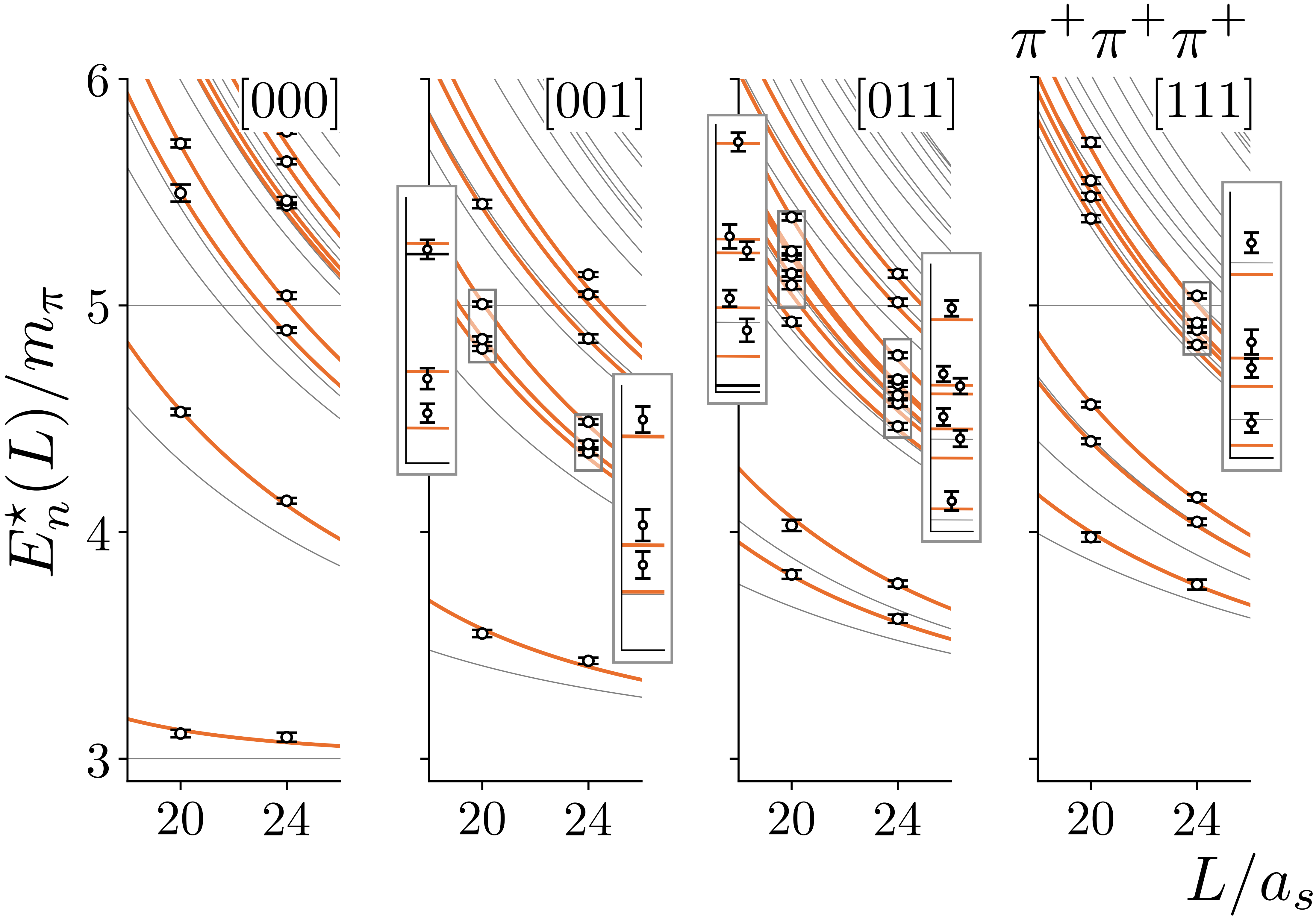
MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

$\pi^+\pi^+$ energies



MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

$\pi^+\pi^+\pi^+$ energies



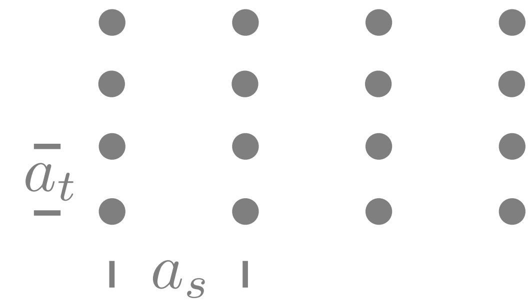
MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

$$\pi^+\pi^+\pi^+ \to \pi^+\pi^+\pi^+$$

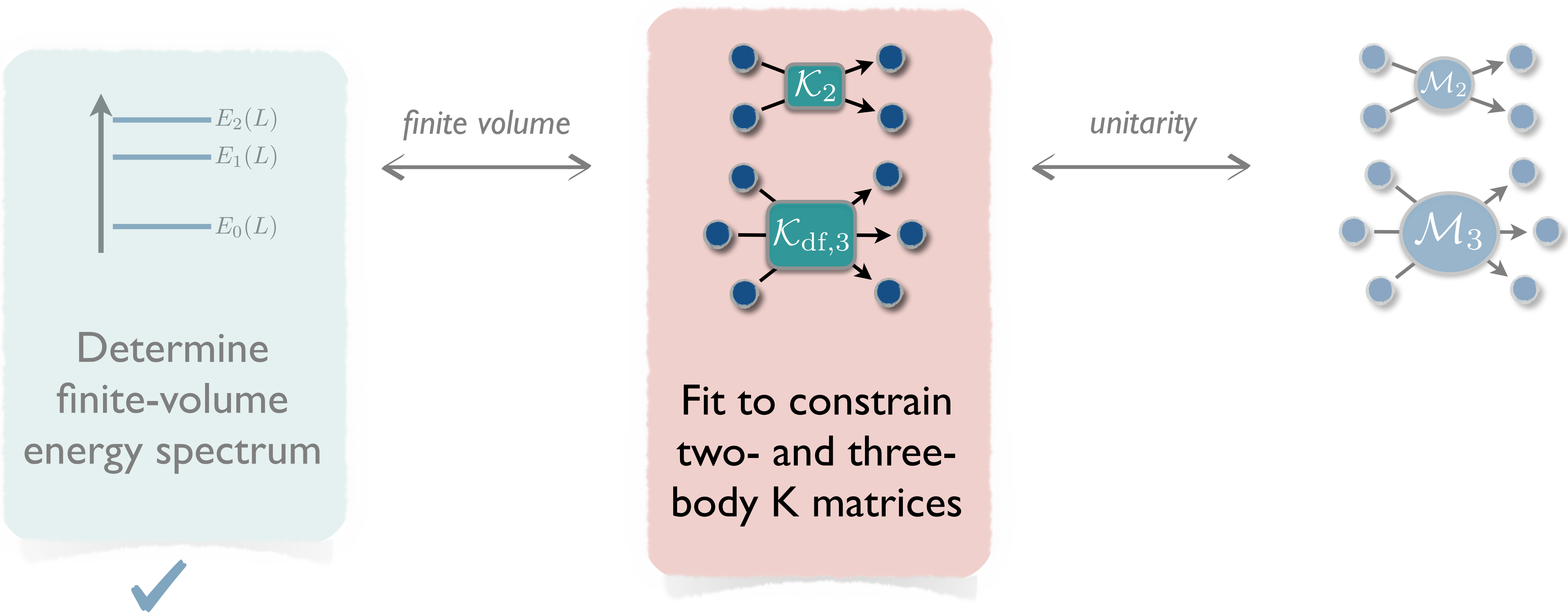
lattice details

$N_f = 2+1$ $\quad a_s/a_t = 3.444(6)$ $\quad L_s/a_s = 20, 24$

$m_\pi \approx 400\text{MeV}$ $\quad a_s \approx 0.12\text{fm}$

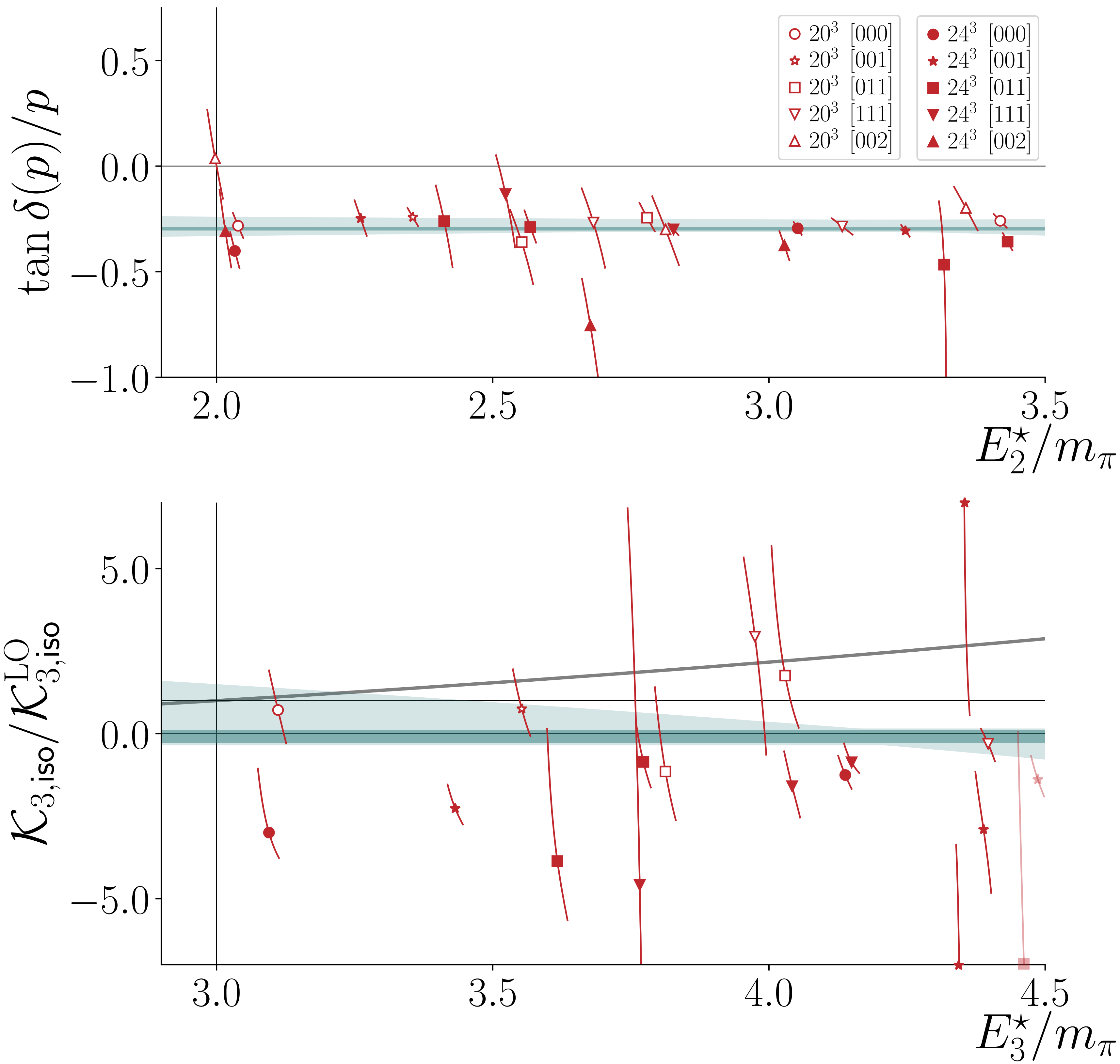


❑ Workflow outline



MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

K matrix fits



Finite-volume formalism relates energies to K matrices

One-to-one for $K_{\text{df},3}$ depending only on $E_{\text{cm}} = E^\star$

Fit both two and three-body K to various polynomials

Cut on the CM energy in the fits

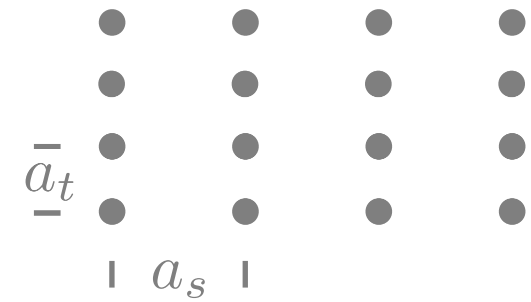
$K_{\text{df},3}$ is scheme dependent (removed upon converting to $\mathcal{M}_3$)

$$\pi^+\pi^+\pi^+ \to \pi^+\pi^+\pi^+$$

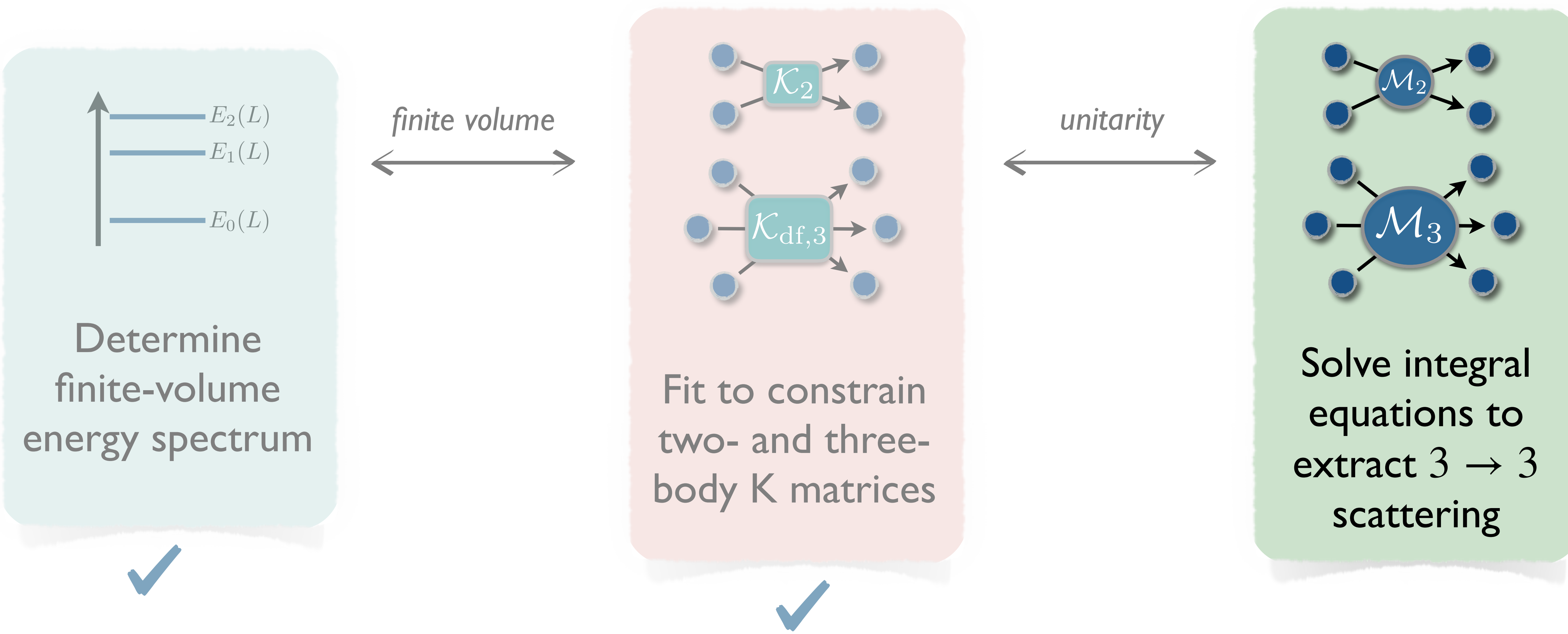
lattice details

$N_f = 2+1 \qquad a_s/a_t = 3.444(6) \qquad L_s/a_s = 20, 24$

$m_\pi \approx 400\text{MeV} \qquad a_s \approx 0.12\text{fm}$



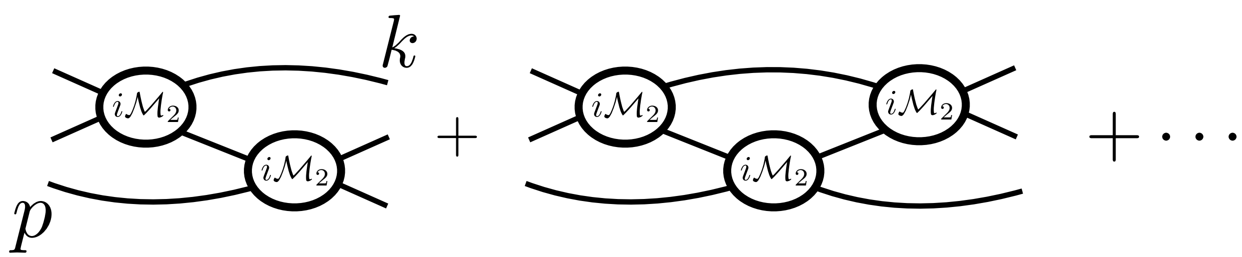
- [ ] Workflow outline



MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

Integral equation

$$\mathcal{M}_3^{\mathsf{un}}(E_3^\star, \boldsymbol{p}, \boldsymbol{k}) = \mathcal{D}^{\mathsf{un}}(E_3^\star, \boldsymbol{p}, \boldsymbol{k}) + \mathcal{E}^{\mathsf{un}}(E_3^\star, \boldsymbol{p})\mathcal{T}(E_3^\star)\mathcal{E}^{\mathsf{un}}(E_3^\star, \boldsymbol{k})$$

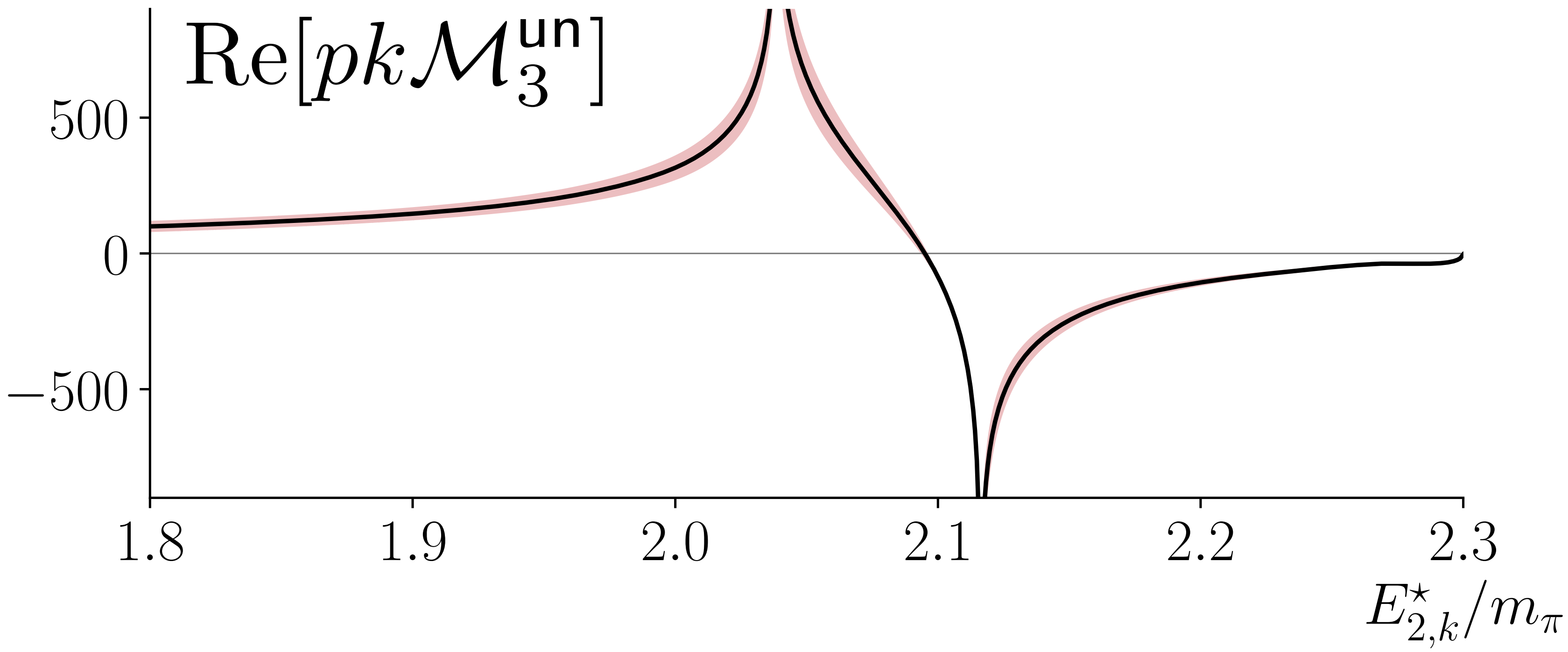


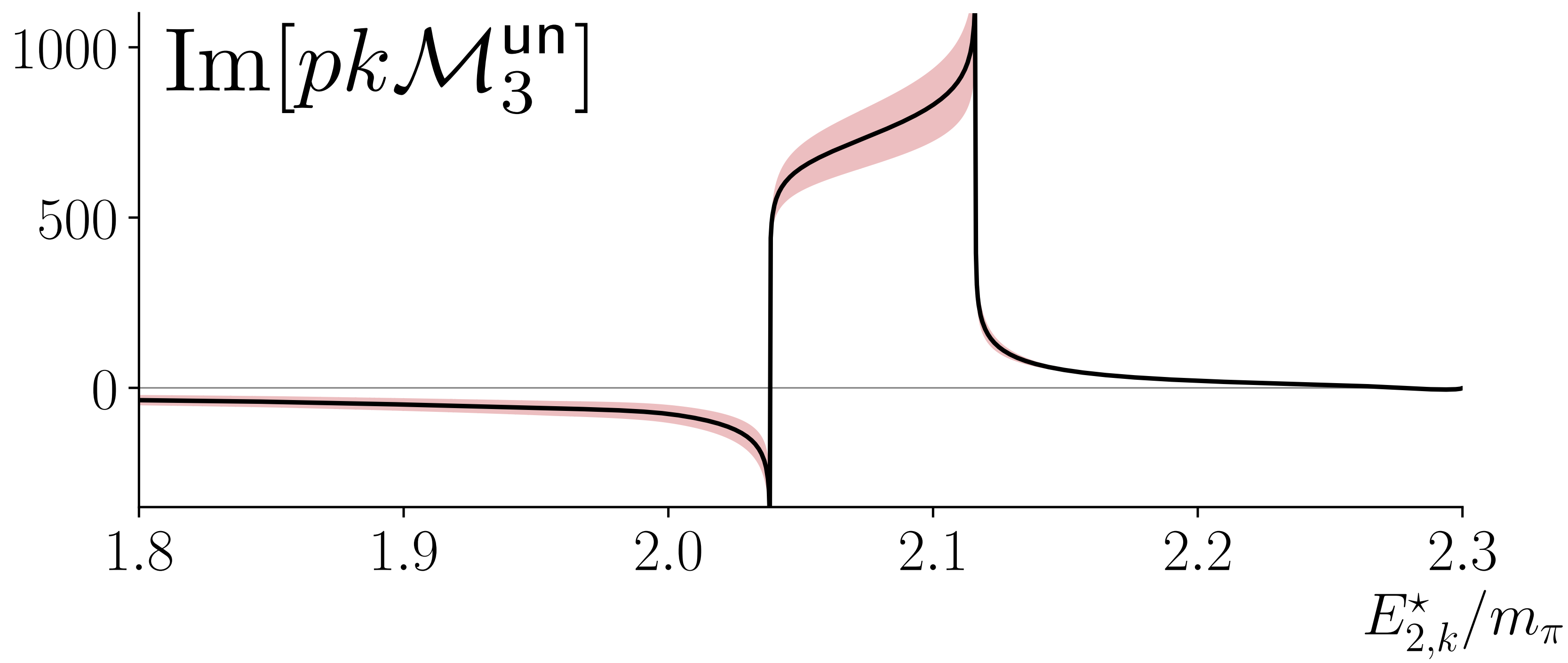
Vanishes for $K_{\text{df},3} = 0$

$$\boldsymbol{D}(N, \epsilon) = -\boldsymbol{\mathcal{M}} \cdot \boldsymbol{G}(\epsilon) \cdot \boldsymbol{\mathcal{M}} - \boldsymbol{\mathcal{M}} \cdot \boldsymbol{G}(\epsilon) \cdot \boldsymbol{P} \cdot \boldsymbol{D}(N, \epsilon)$$

$$\mathcal{D}^{\mathsf{un}}(E_3^\star, \boldsymbol{p}, \boldsymbol{k}) = \lim_{\epsilon \to 0} \lim_{N \to \infty} \boldsymbol{D}_{pk}(N, \epsilon)$$

Integral equation





Total angular momentum = 0

Two-particle sub-system angular momentum = 0

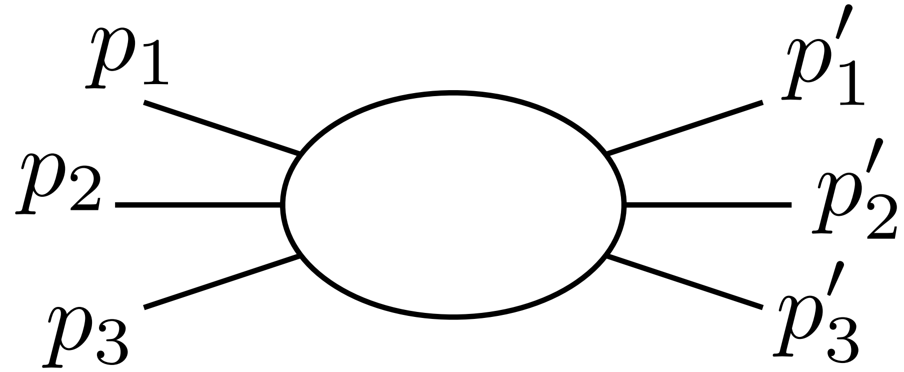
Plot at fixed $E_3^\star$ and p

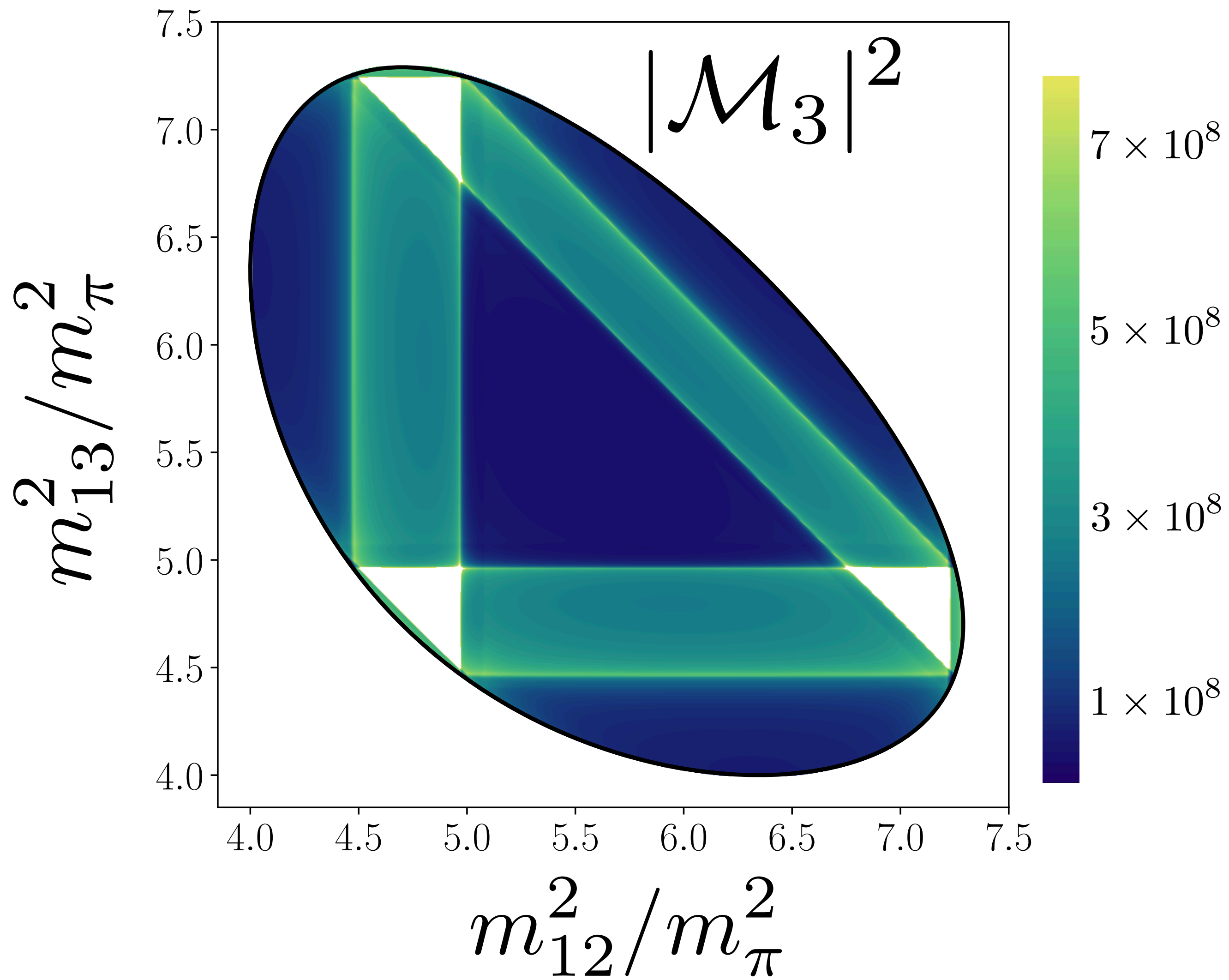
Both two- and three-body uncertainties estimated

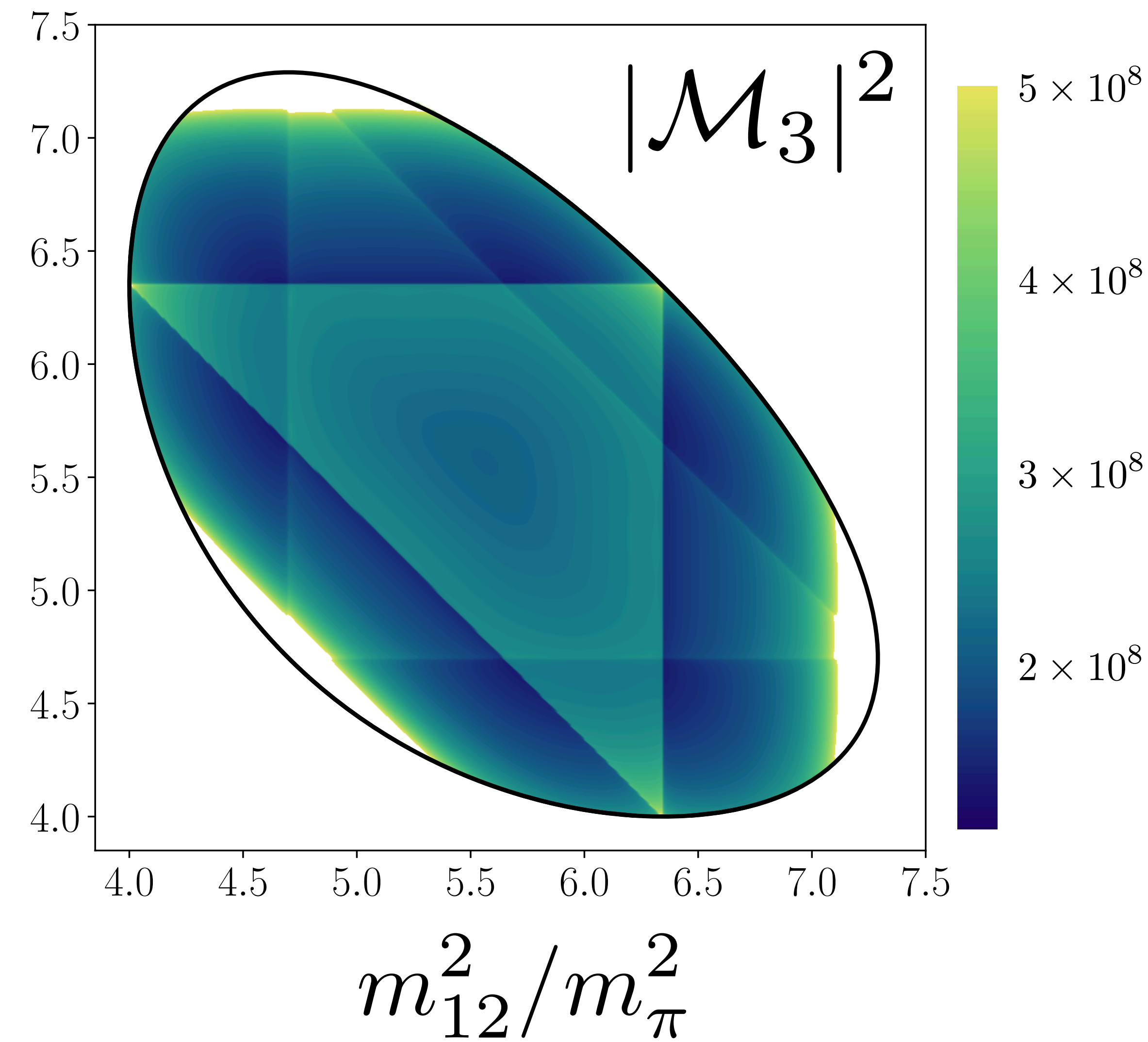
Still need to symmetrize

MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

$$\mathcal{M}_3 = \sum_{i,j\in\{1,2,3\}} \mathcal{M}_3^{\mathsf{un}}(p'_i, p_j)$$







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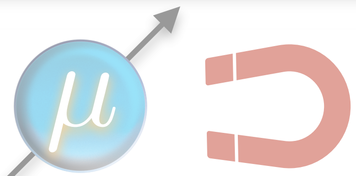
Finite-volume landscape

(Important) $e^{-\mu L}$ effects

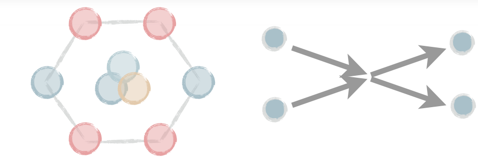


Dictated by analyt. cont. of observables

Relevant for precision & when $1/\mu$ is enhanced

Convergence of $e^{-\alpha_n \mu L}$ series can be delicate

2 → 2 scattering

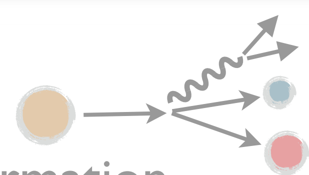


Sets the standard of production-ready formalism

$E_n(L)$ has power-like dependence on all open channels

Shows the utility of the skeleton expansion framework

(1+)$\mathscr{J}$ → 2 transitions

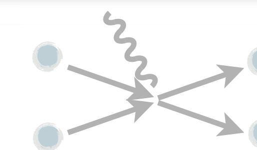


Eigenvectors of q.c. matrix = useful information

Looks like a relation on states:

$$|n, L\rangle = a(L)\,|\pi\pi, \text{out}\rangle + b(L)\,|K\bar{K}, \text{out}\rangle + \cdots$$

2+$\mathscr{J}$ → 2 scattering



Sub-processes complicate volume effects

≥3 d.o.f.s in/out → rich analytic structure

relation on states is naive!

2 → 3, 3 → 3 scattering

intermediate, scheme-dep. quantities = unavoidable

satisfying bridge to amplitudes community

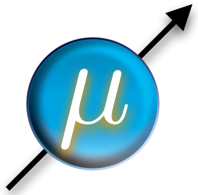
(1+)$\mathscr{J}$ → 3 transitions

Müller - Fri 6:30
Romero-Lopez - Fri 6:45

Finite-volume landscape

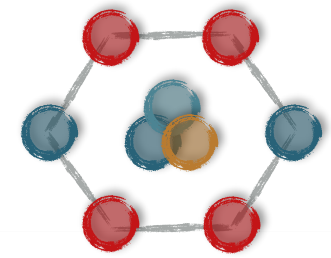
(Important) $e^{-\mu L}$ effects

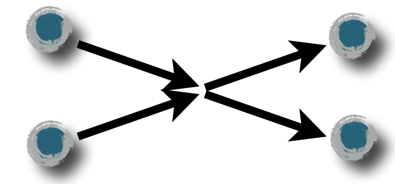
$(g-2)_\mu$, PDFs, deuteron, $\cdots$



$2 \to 2$ scattering

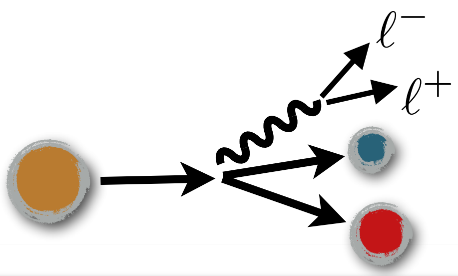
$\rho(770)$, $\sigma/f_0(500)$, $K^\star$, κ, a_0, Δ, $\cdots$





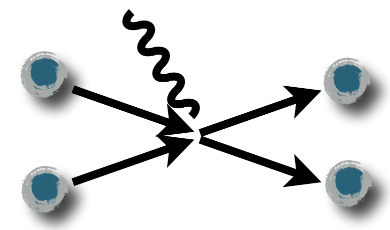
(1+)$\mathscr{J} \to 2$ transitions

$K \to \pi\pi$, $\gamma^\star \to \pi\pi$, $\cdots$



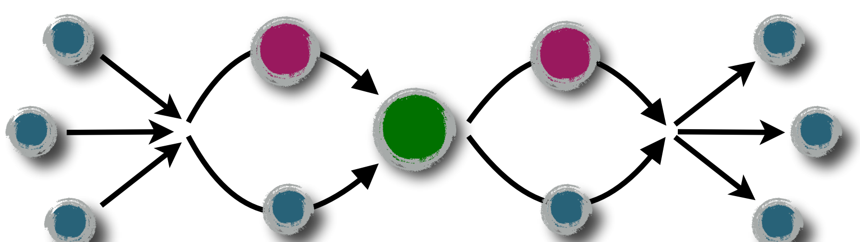
2+$\mathscr{J} \to 2$ scattering

$\rho\gamma \to \rho$, $\pi\pi \to \gamma\pi\pi$, $\cdots$



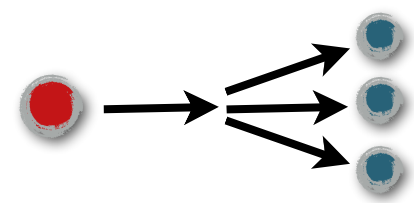
$2 \to 3$, $3 \to 3$ scattering

$N\pi\pi \to N(1420) \to N\pi\pi$, $\cdots$



(1+)$\mathscr{J} \to 3$ transitions

$K \to \pi\pi\pi$, $\gamma^\star \to \pi\pi\pi$, $\cdots$



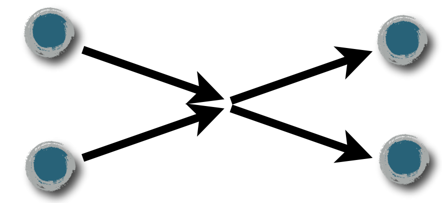
Thanks!


LATTICE 2015
The 33rd International Symposium on Lattice Field Theory
July 18, 2015, Kobe International Conference Center

Back-up slides

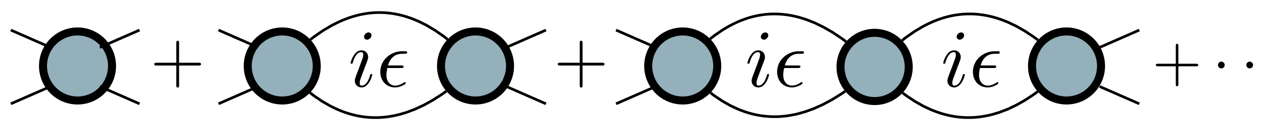
Analyticity

- Instead of $|\mathcal{M}(s)|^2$ → analytically continue the ***amplitude*** itself



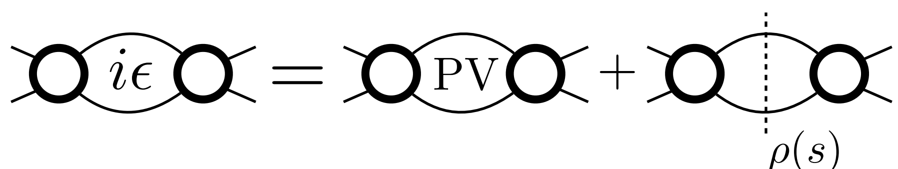
For two-particle energies $(2m)^2 < s < (4m)^2$, what is the analytic structure?

$\mathcal{M}(s) \equiv$



propagating pion

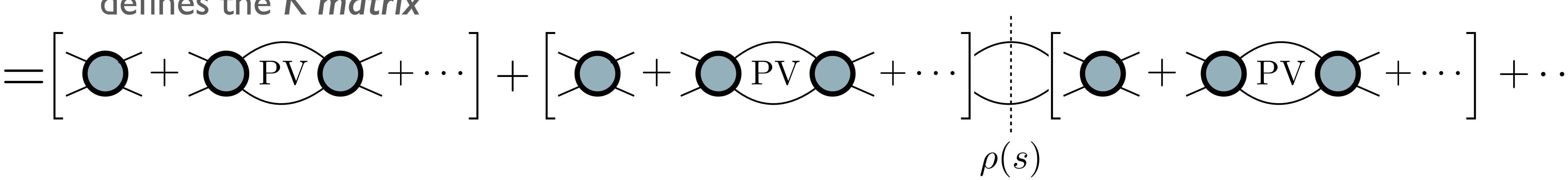
non-analytic:
on-shell particles = singularities



$\rho(s) \propto i\sqrt{s-(2m)^2}$

cutting rule

defines the *K matrix*



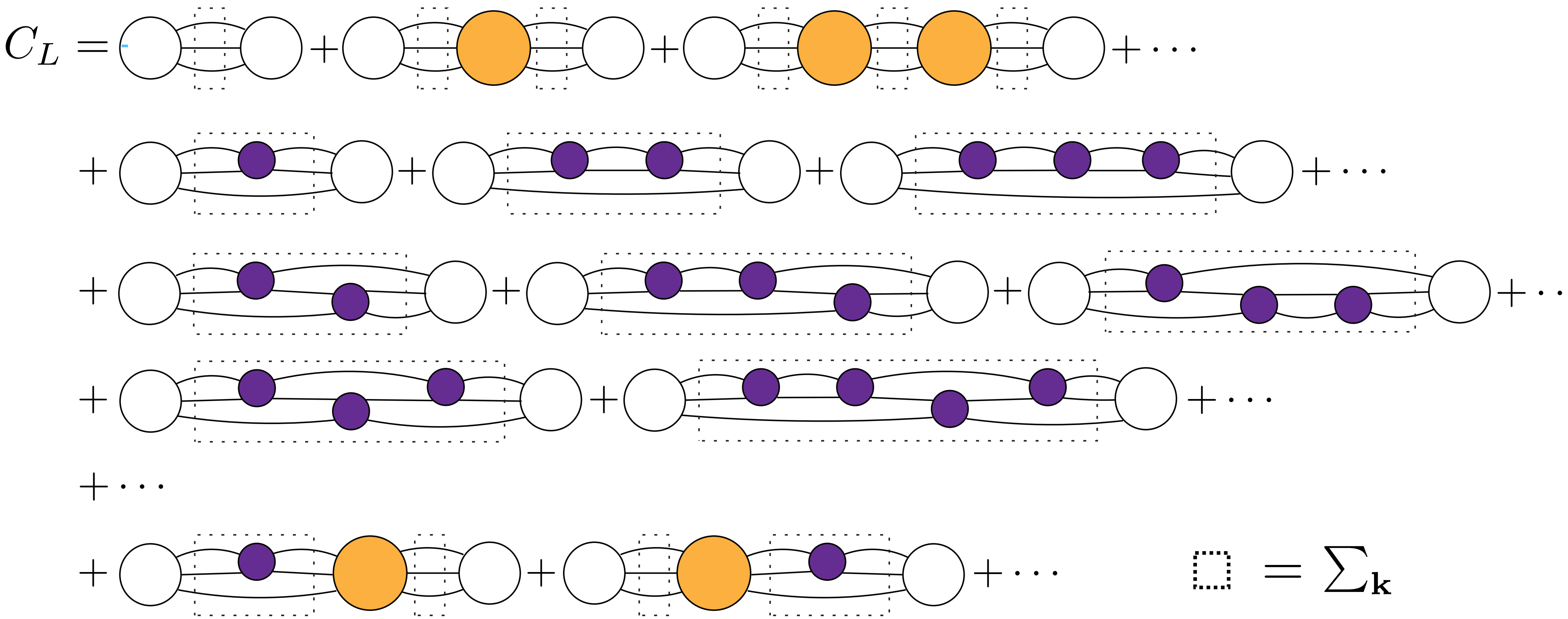
$$= \mathcal{K}(s) + \mathcal{K}(s)\rho(s)\mathcal{K}(s) + \cdots = \frac{1}{\mathcal{K}(s)^{-1} - \rho(s)}$$

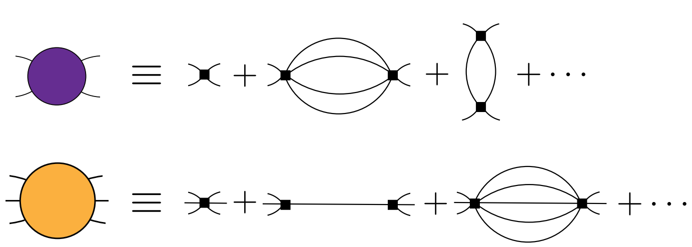
branch-cut singularity

$\sqrt{s-(2m)^2}$

3-particle derivation

- Study 3-body correlator in an ***all-orders skeleton expansion***



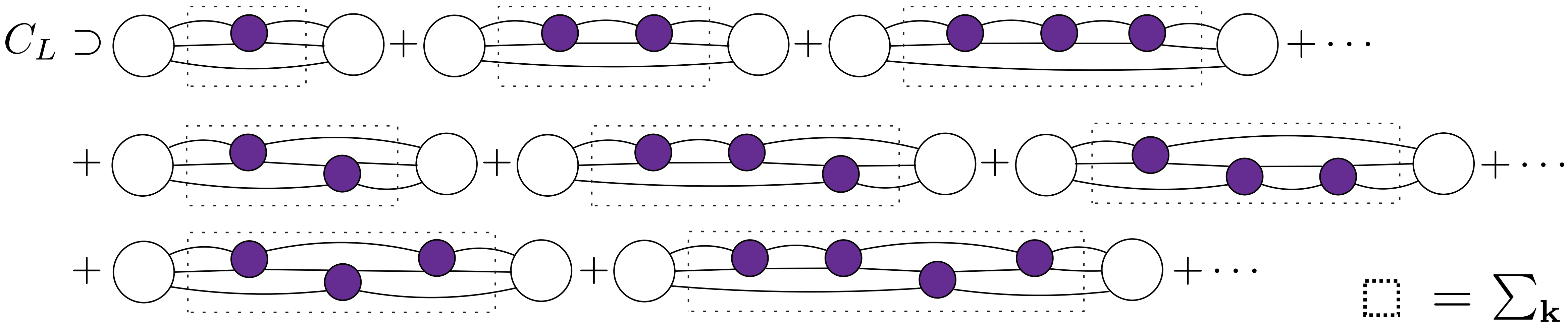


kernels have suppressed *L* dependence

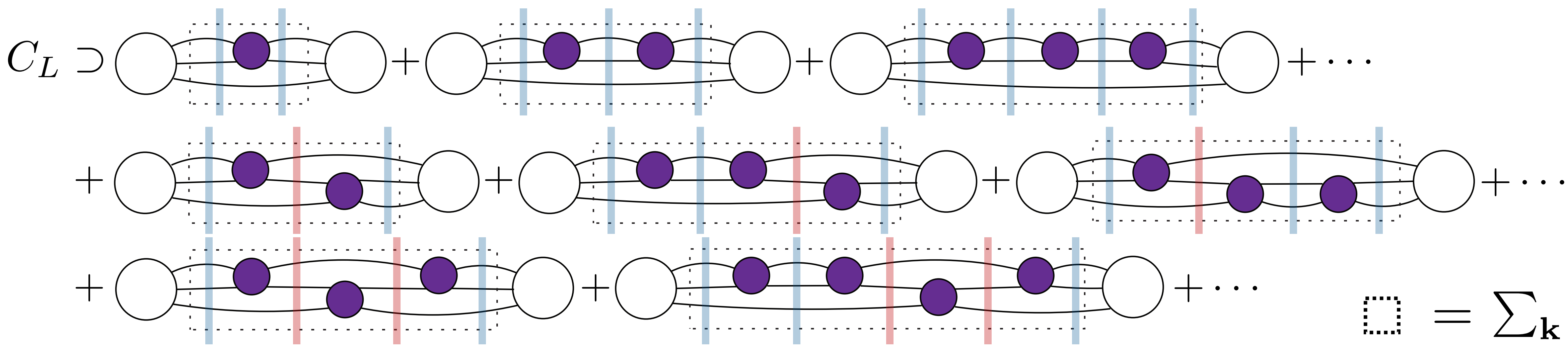
lines = fully dressed hadrons

• MTH, Sharpe (2014) •

Two types of cuts



Two types of cuts



$$\mathbf{A}_3' \, \mathbf{F} \, \mathbf{K}_2 \, \mathbf{F} \, \mathbf{A}_3 + \mathbf{A}_3' \, \mathbf{F} \, [\mathbf{K}_2 \, \mathbf{F}]^2 \, \mathbf{A}_3 + \mathbf{A}_3' \, \mathbf{F} \, [\mathbf{K}_2 \, \mathbf{F}]^3 \, \mathbf{A}_3 + \cdots = \mathbf{A}_3' \, \mathbf{F} \frac{1}{1 - \mathbf{K}_2 \mathbf{F}} \mathbf{K}_2 \mathbf{F} \, \mathbf{A}_3$$

$$\mathbf{A}_3' \, \mathbf{F} \, \mathbf{K}_2 \, \mathbf{G} \, \mathbf{K}_2 \, \mathbf{F} \, \mathbf{A}_3 + \cdots$$

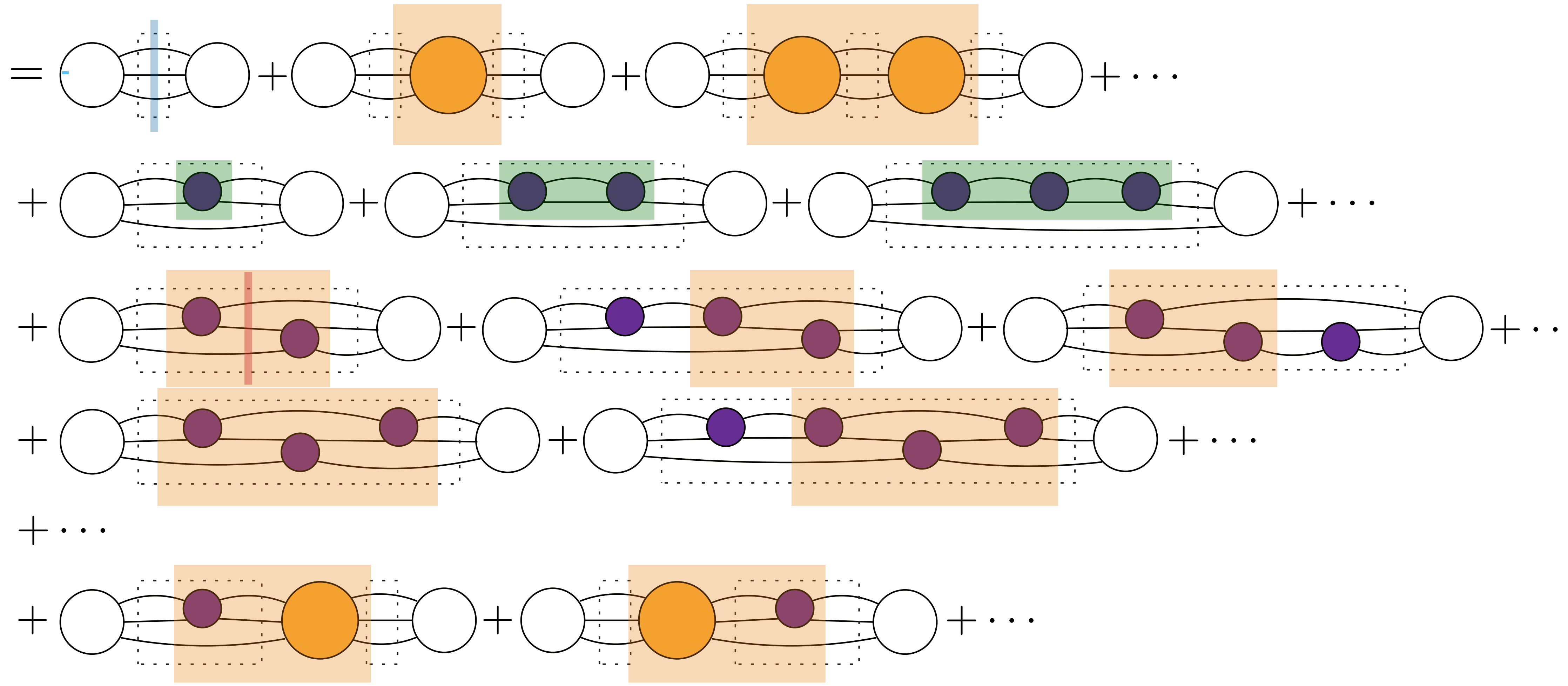
have not yet considered entire diagram contributions

missing contributions from ***off-shellness***

missing smooth terms (short-distance parts)

Short-distance parts & summation

$C_L =$



$$C_L - C_\infty = \mathbf{A}'_3\,\mathbf{F}_{33}\,\mathbf{A}_3 + \mathbf{A}'_3\,\mathbf{F}_{33}\,\mathbf{K}_{\mathrm{df},3}\,\mathbf{F}_{33}\,\mathbf{A}_3 + \cdots$$

$$= \mathbf{A}'_3\,\frac{1}{\mathbf{F}_{33}^{-1} + \mathbf{K}_{\mathrm{df},3}}\,\mathbf{A}_3$$

$$\mathbf{F}_{33} \equiv \frac{1}{3}\mathbf{F} + \mathbf{F}\,\mathbf{K}_2\,\frac{1}{1-(\mathbf{F}+\mathbf{G})\,\mathbf{K}_2}\,\mathbf{F}$$

no term left behind