Recent progress in the tensor renormalization group

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Cases with the sign problem

finite density QCD

early universe, neutron star,..

chiral gauge theory

The SM, GUT,..

supersymmetry

AdS/CFT, string theory,…

θ -vacuum

strong CP problem

real time simulation

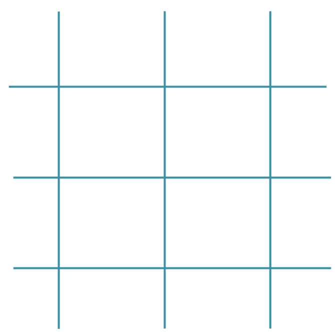
Schwinger-Keldysh formalism

It is important to solve the sign problem.

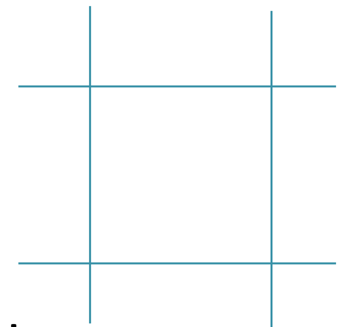
Tensor renormalization group (TRG)

[Levin-Nave, 2007]

tensor network

$Z =$ 

coarse graining
of tensor network



no statistical process & no sign problem

Previous tensor network (TN) studies in field theory:

Kuramashi, Takeda, Meurice, Jansen, Banuls, D. C.-J. Lin, Catterall, Unmuth-Yockey, Bazavov, Sakai, Yoshimura, Jha, Oba, Akiyama, Nakayama, Fukuma, Matsumoto,…

c.f. improved MC method:
reweighting method, complex Langevin, Lefschetz thimble, …

Features of TRG

(1) no sign problem, no statistical errors

(2) systematic error from finite Dcut

(3) partition function is directly calculable

(4) computational cost is $\log(V) \times D_{cut}^{p}$

easy to take the thermodynamic limit

large internal DOF or higher dimensions
→ high cost

It may be possible to calculate parameters that the MC method is not good at.

TRG approach to field theory

1. fermions

2. scalars

3. gauge fields

4. algorithms for d=3,4

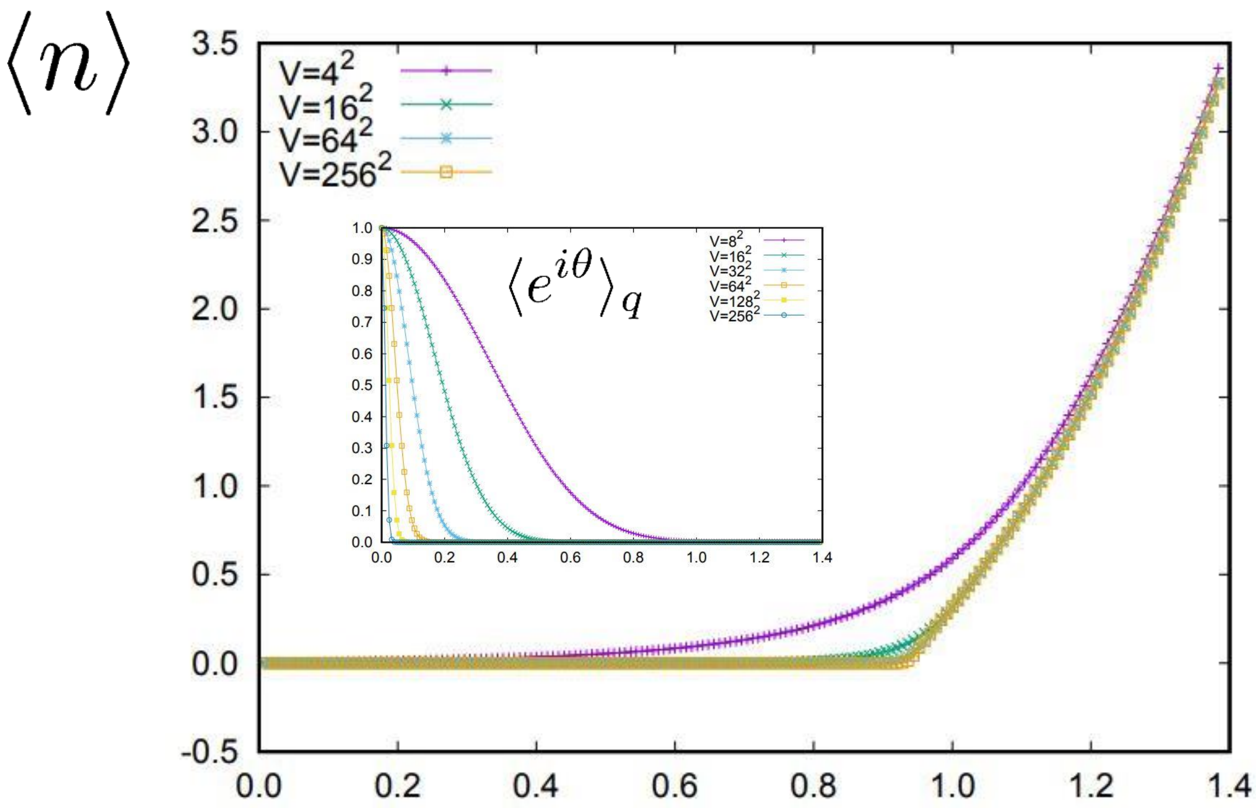
5. large internal DOF (Nc, Nf>>1)

EASY HARD

fermions U(1),SU(2) $\gtrsim$ 2 $>>$ 4 $>>$ SU(N) (N>2) $>>>$ 5

2D scalar field theory at finite density

[D.K.-Kuramashi-Nakamura-Sakai-Takeda-Yoshimura, 2019]



$\lambda = 1$

$m^2 = 0.01$

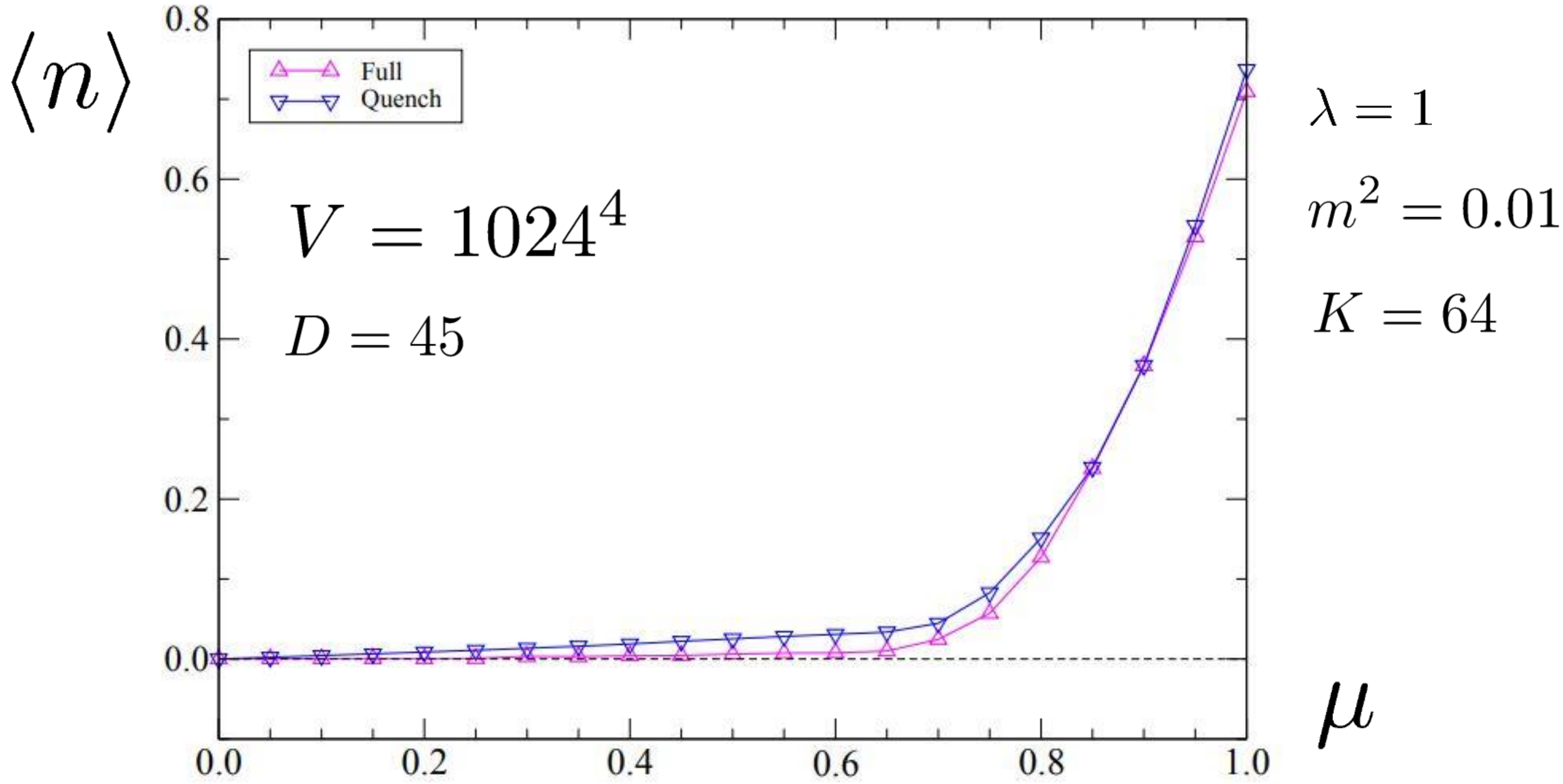
$K = 64$

$D_{cut} = 64$

Silver Blaze phenomena is clearly observed for large volume lattices.

4D scalar field theory at finite density

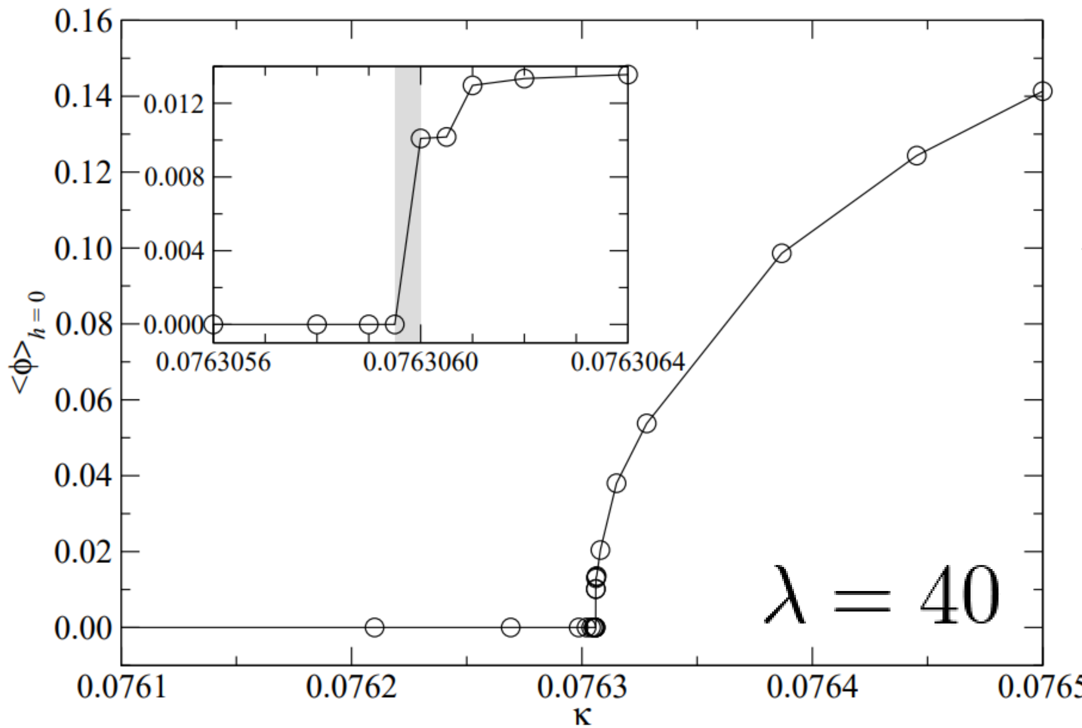
[Akiyama-DK-Kuramashi-Yamashita-Yoshimura, 2020]



The number density obtained on V=1024^4(!)
shows behavior like a constant (Silver blaze).

4D real scalar field theory

[Akiyama-Kuramashi-Yoshimura, 2021]



$V = 1024^4$

$D = 55$

$$\kappa_c^{(ATRG)} = 0.0892625(375) \qquad \lambda = 5$$

$$\kappa_c^{(MC)} = 0.08893(20)$$

[Akerlund et al. 2013]

The critical coupling constant is precisely obtained.

TRG for fermions

- The Grassmann TRG [Gu et al. 2010]

c.f. Schwinger model (Shimizu-Kuramashi, 2014)
Gross-Neveu model (Takeda-Yoshimura, 2014)

- A reformulation of GTRG [Akiyama-DK (2020), 2005.07570]

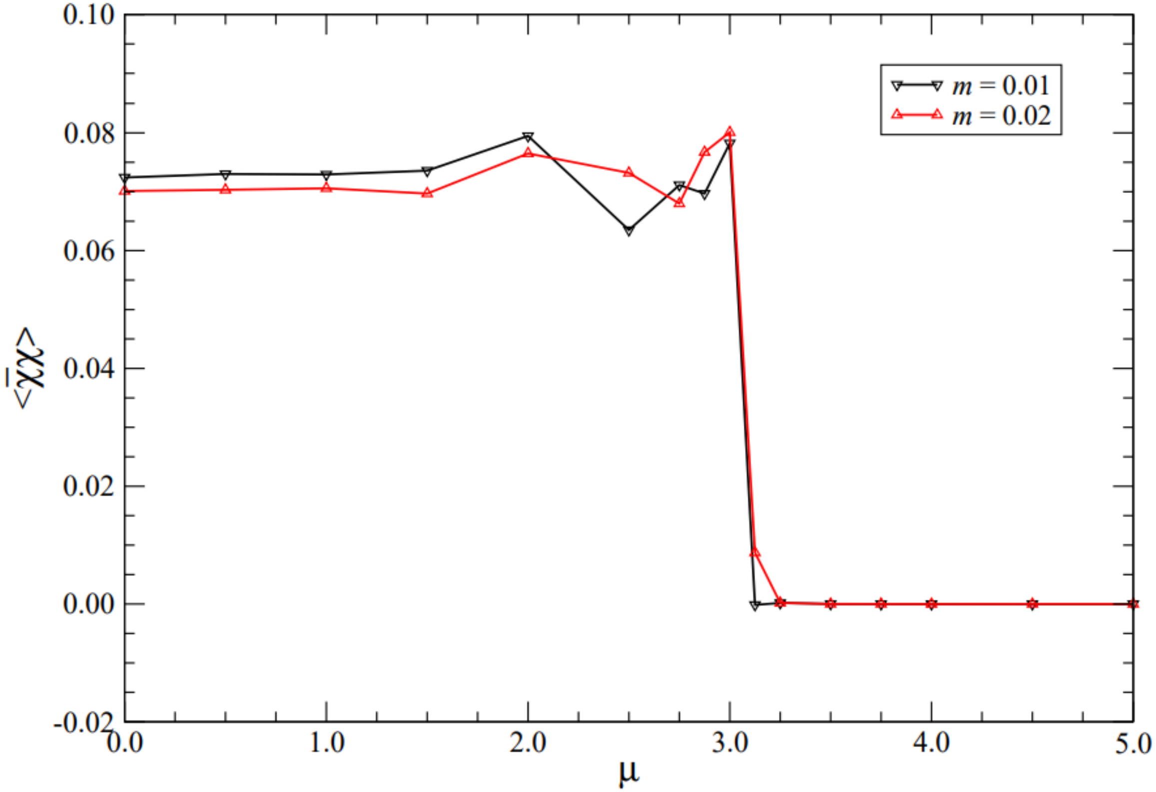
$$\mathcal{T}_{\psi_1\psi_2\psi_3\psi_4} = \sum_{i,j,k,l=0}^{1} T_{ijkl}\psi_1^i\psi_2^j\psi_3^k\psi_4^l$$

Tensor network representations for general lattice fermions including Wilson fermions, staggered fermions, domain wall fermions, are obtained.

4D NJL model

[Akiyama-Kuramashi-Yamashita-Yoshimura, 2021]

$$V = 1024^4, D = 55$$



Computations with TRG have also become possible for 4D NJL.

TRG for gauge theories

- character expansion

$$e^{\alpha \mathrm{Retr}(U)} = \sum_R d_R \lambda_R(\alpha) \chi_R(U)$$

- U(1) lattice theory

$$e^{\frac{1}{g^2}\cos(F_{12}(x))} = \sum_{n=-\infty}^{\infty} I_n(1/g^2) e^{inF_{12}(x)}$$

$$= \sum_{n=-\infty}^{\infty} I_n(1/g^2) e^{inA_2(x+\hat{1})} e^{-inA_2(x)} e^{-inA_1(x+\hat{2})} e^{inA_1(x)}$$

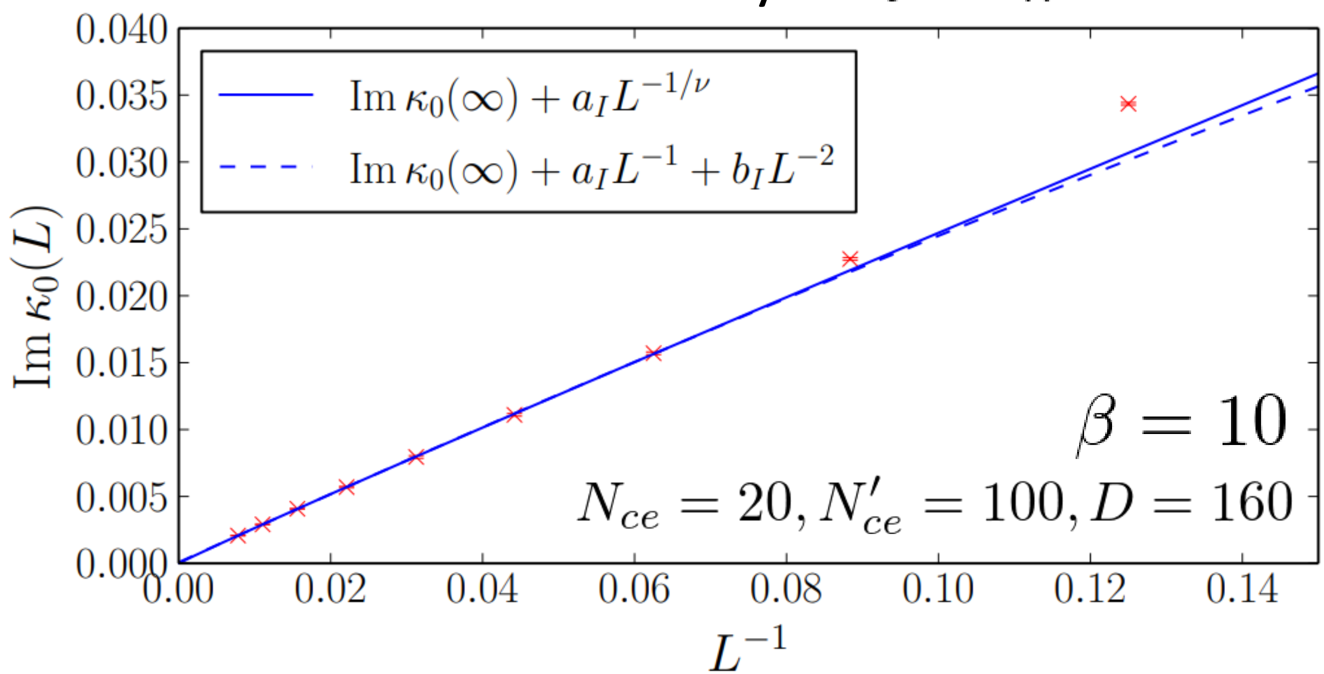
$$\int dA_1 dA_2 \quad \Longrightarrow \quad Z = \mathrm{Tr} \prod_n T_{i_n j_n i'_n j'_n}$$

c.f. Schwinger model (Shimizu-Kuramashi, 2014)
CP(1) (Takeda-Kawauchi, 2016)
U(1) with θ-term (Kuramashi-Yoshimura, 2019)

Schwinger model with θ-term

[Shimizu and Kuramashi, 2014]

Fischer zero analysis $\theta = \pi$

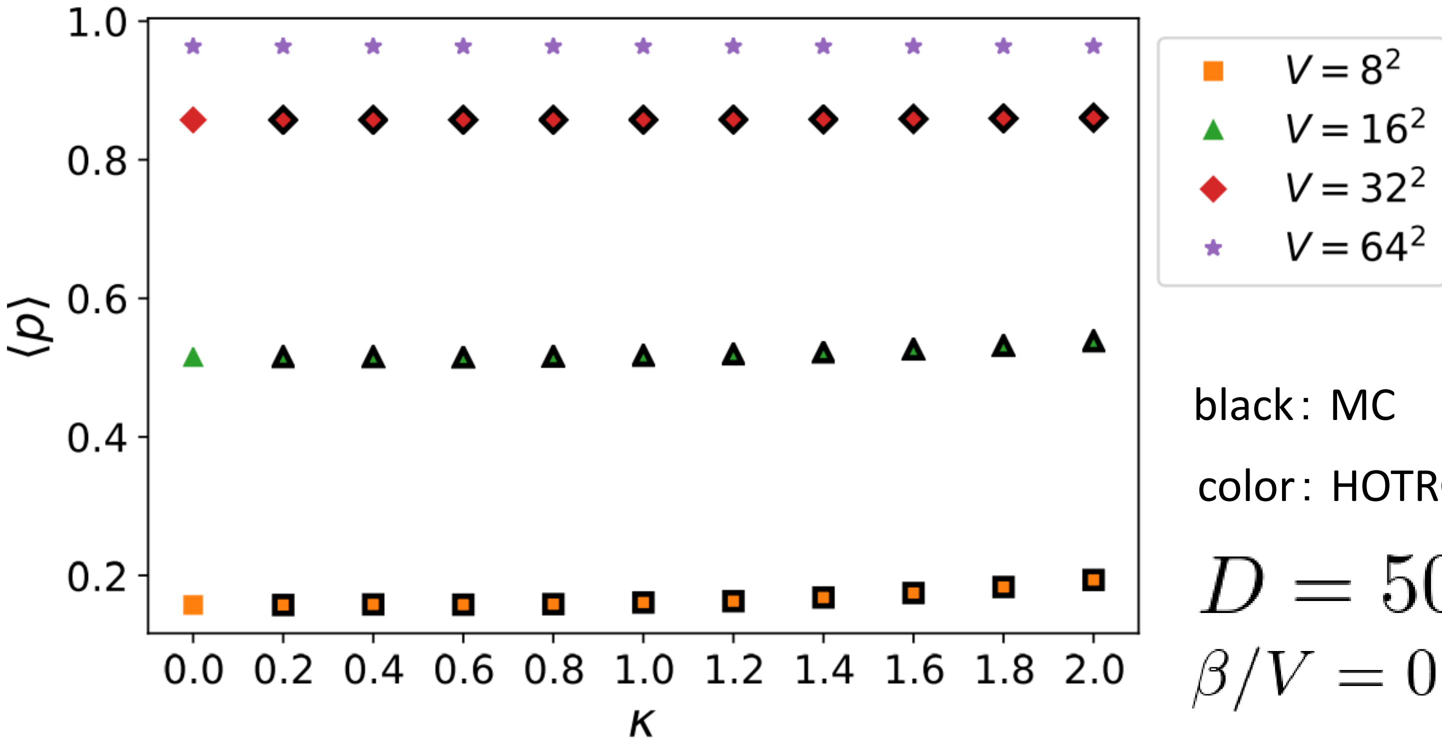


The same critical exponent ($\nu = 1$)
as 2d Ising model is obtained.

2d SU(2) gauge-Higgs model $(\lambda = \infty)$

[Bazavov, Catterall, Jha, Unmuth-Yockey, 2019]

average plaquette



black : MC

color : HOTRG

$D = 50$

$\beta/V = 0.01$

A tensor for SU(2) theory is defined by the CE

The TRG method well reproduces the MC results.

TN of 2d YM theory without the CE

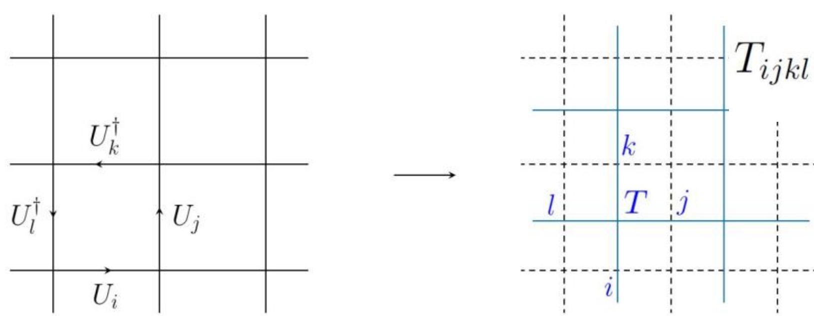
[Fukuma, DK and Matsumoto, ariXiv: 2107.14149]

- group integrals are discretized by K random gauge fields $\mathring{G} = \{U^{(1)}, U^{(2)}, \ldots, U^{(K)}\}$ as

$$\int dU f(U) \approx \frac{1}{K} \sum_{i=1}^{K} f(U^{(i)})$$

- TN with finite bond dimension

$$Z = \int DU e^{-(\beta/N) \sum_n \mathrm{Re}\,\mathrm{tr}(1-U_P(n))} \approx \mathrm{Tr} \prod_{n \in \Gamma} T_{i_n j_n i'_n j'_n}$$



$$T_{ijkl} = \frac{1}{K^2} e^{-(\beta/N)\,\mathrm{Re}\,\mathrm{tr}\,(1-U_i U_j U_k^\dagger U_l^\dagger)}$$

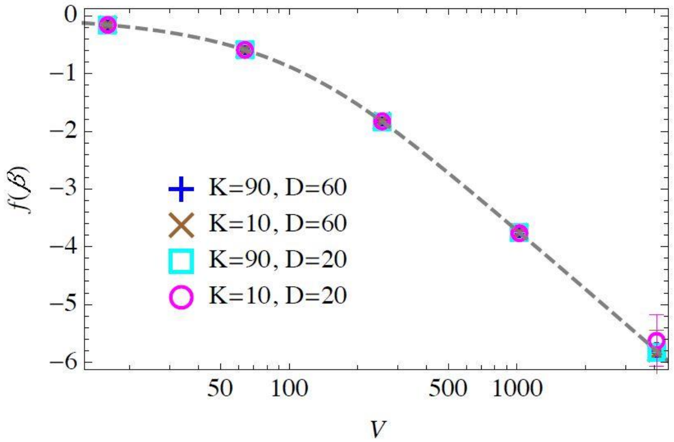
applicable for any gauge group w/o the CE

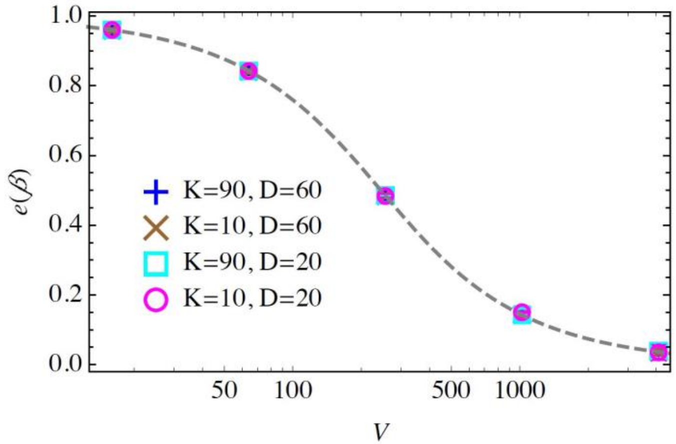
SU(2)

[Fukuma, DK and Matsumoto, ariXiv: 2107.14149]

$$f(\beta) = \frac{1}{V}\ln Z(\beta)$$

$$e(\beta) = -\frac{\partial}{\partial\beta} f(\beta)$$





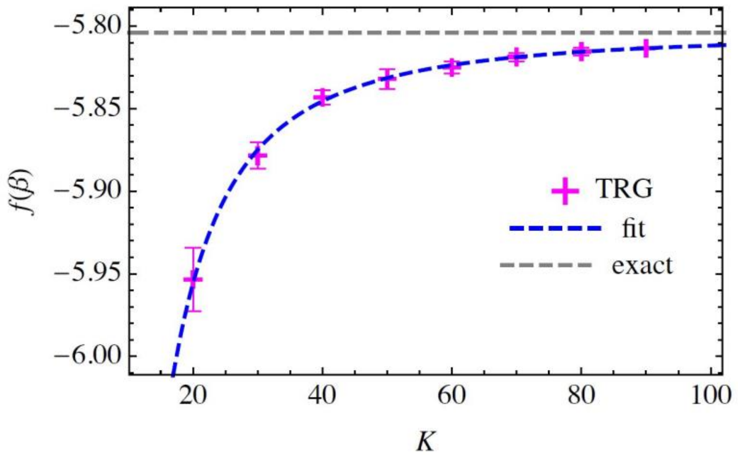
Numerical results of free energy density and energy density agree well with the exact values of 2d YM.

$$Z = \sum_R \lambda_R^V(\beta)$$

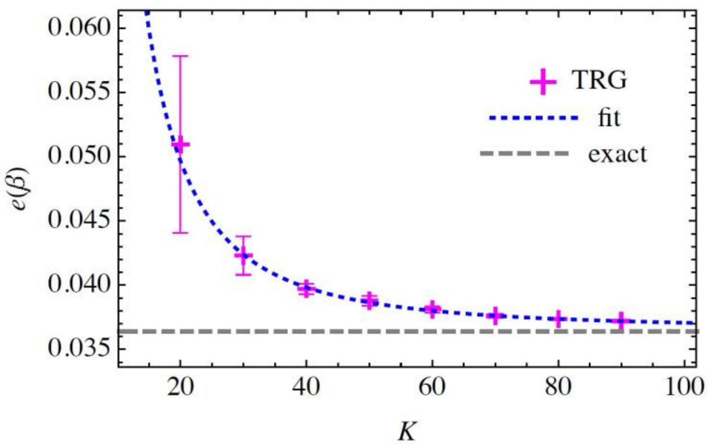
SU(2): the large K limit

[Fukuma, DK and Matsumoto, ariXiv: 2107.14149]

Free energy density (D=60)



energy density (D=60)



Fit results at D=60 by a fitting formula $g(K) = \mu + \alpha K^{-p}$

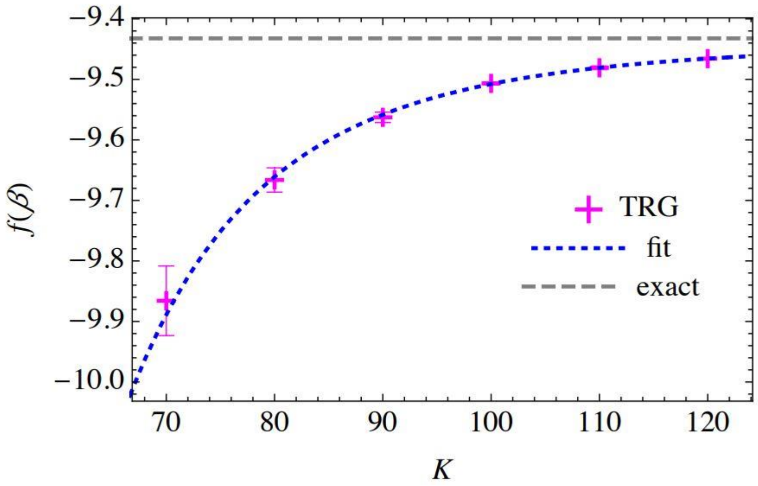
	(exact)	μ	α	p	χ^2/DOF
$f(\beta)$	-5.8040	$-5.8046^{+0.004}_{-0.003}$	-43^{+26}_{-60}	$1.88^{+0.27}_{-0.28}$	0.24
$e(\beta)$	0.03639	$0.03655^{+0.0005}_{-0.0003}$	5^{+24}_{-4}	$2.0^{+0.5}_{-0.5}$	0.11

Fit results at D=60 are close to the exact value.
The extrapolation significantly improves the accuracy.

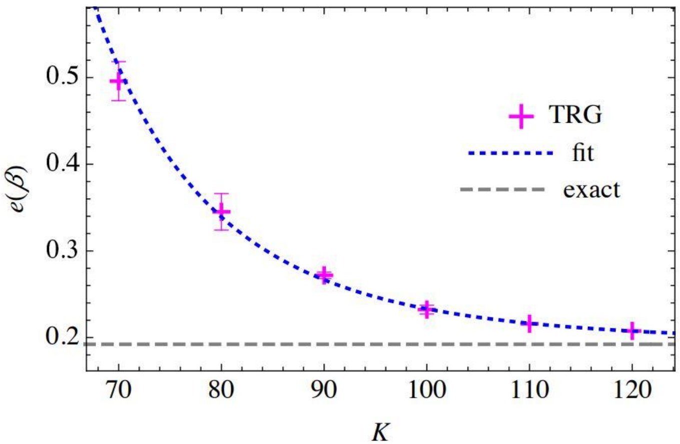
SU(3): the large K limit

[Fukuma, DK and Matsumoto, ariXiv: 2107.14149]

Free energy density (D=90)



energy density (D=90)



Fit results at D=90 by a fitting formula $g(K) = \mu + \alpha K^{-p}$

	(exact)	μ	α	p	χ^2/DOF
$f(\beta)$	-9.4323	$-9.4400^{+0.0019}_{-0.0043}$	$-0.3^{+0.2}_{-1.7} \times 10^{10}$	$5.31^{+0.44}_{-0.01}$	0.21
$e(\beta)$	0.1923	$0.1941^{+0.0017}_{-0.0008}$	$2.2^{+5.6}_{-1.6} \times 10^{10}$	$5.88^{+0.29}_{-0.01}$	1.18

Fit results at D=90 are close to the exact value.

Our TN representation for SU(3) properly works!

TRG in higher dimensions

[DK-Nakayama (2019), 1912.02414] [DK-Oba-Takeda(2021), 2107.08769]

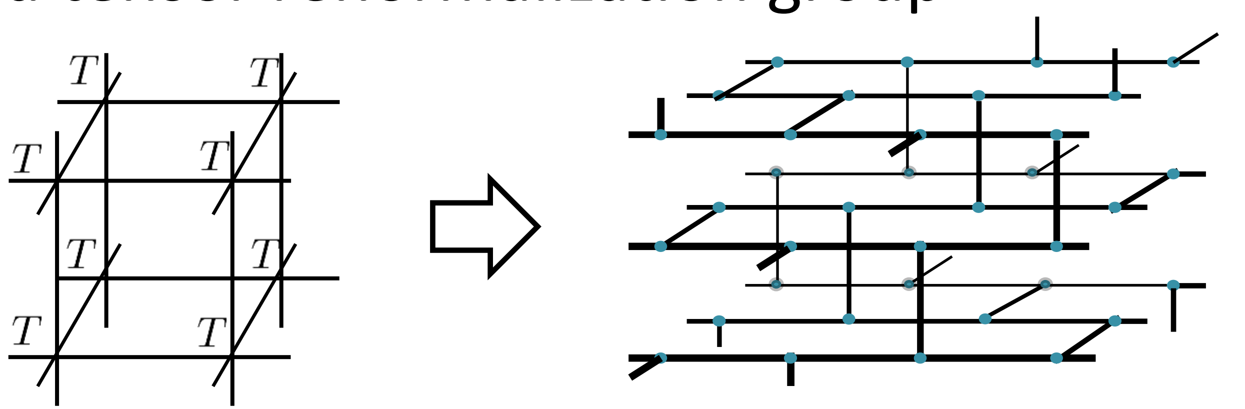
- hidden structure of a tensor

$$T_{ijklmn} = \sum_a W^{(1)}_{ai} W^{(2)}_{aj} W^{(3)}_{ak} W^{(4)}_{al} W^{(5)}_{am} W^{(6)}_{an}$$

(for instance, d=3)

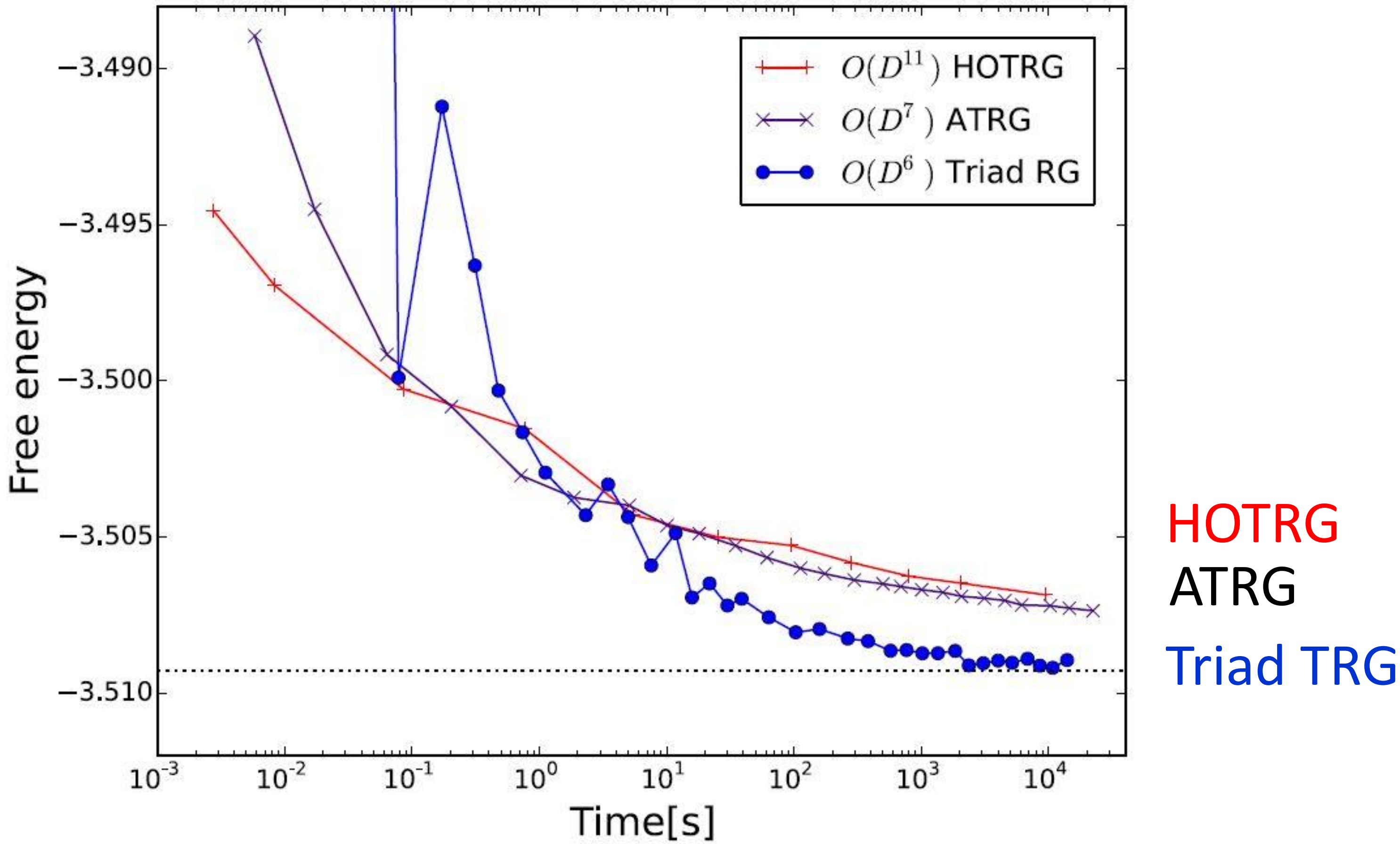
$$= \sum_{a,b,c} A_{ija} B_{akb} C_{blc} D_{cmn}$$

- triad tensor renormalization group



The computational cost of TRG is naturally reduced on transformed networks made only of rank-3 tensor.

Numerical tests on 3D Ising model



The triad TRG method show a better performance in 3D Ising model.

Current status (2021/7/30) and future issues

fermions

OK

gauge fields

OK: U(1), SU(2),
SU(3) (d=2)

For SU(3) in d=3 and 4,
some ideas are needed.

scalar fields

basically OK

… but improvements of
numerical integrations
are needed.

good algorithms in d=3,4

ATRG, triad TRG(d=3), …

many issues such as efficient
algorithms, transformed TN, ….

large internal DOF
(including QCD)

many tasks, and the difficulty
level is high. There may be
great progress within 5 years(?)

it seems to be related to the
development of quantum
computers.

Thank you!