A Fresh Look at the Chemical Potential on the Lattice*

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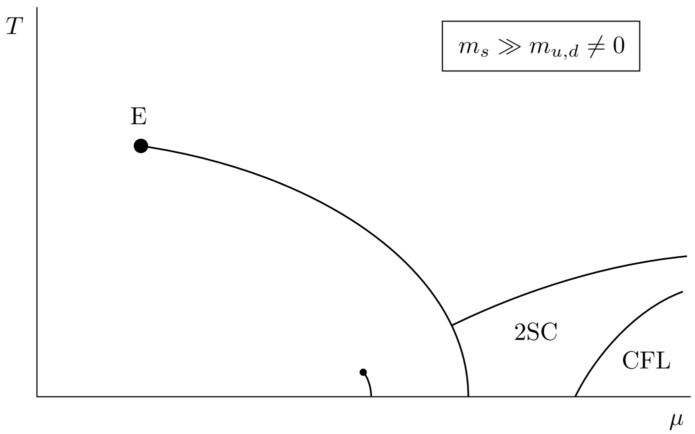
*Based on arXiv:2012.07798

Introduction

♠ Based on symmetries and models, expected QCD Phase Diagram: A fundamental aspect – Critical Point in T-μ_B plane.



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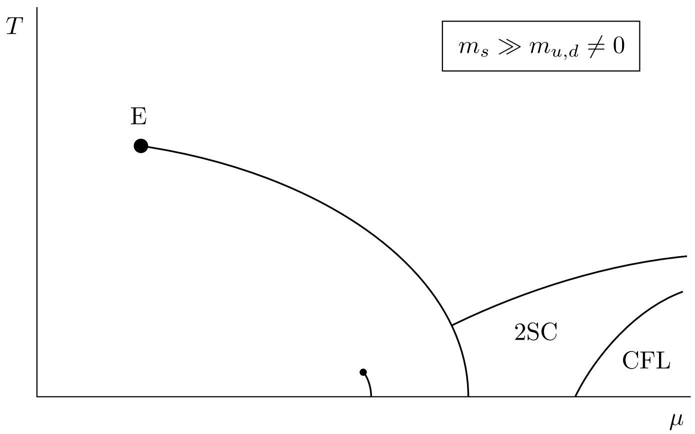

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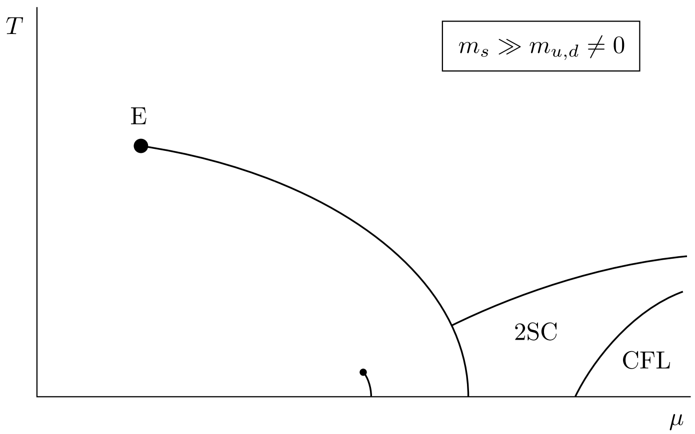

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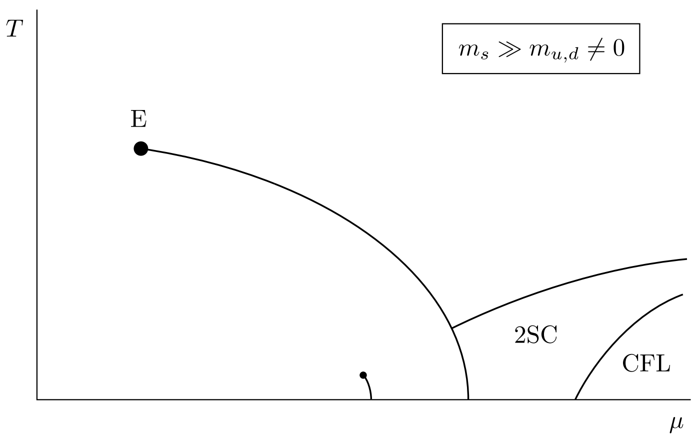

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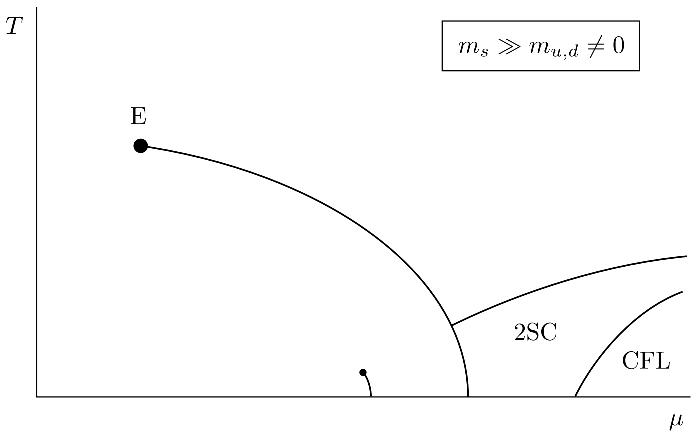

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- Sixth order fluctuations for $O(4)$-criticality (Friman et al. EPJ 2011; Star Collaboration arXiv:2105.14698)

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♡ Recall that the naıvely discretized fermionic action is

$$S^F = \sum_{x,x'} \bar{\psi}(x) \left[\sum_{\mu=1}^{4} D^\mu(x,x') + ma\delta_{x,x'} \right] \psi(x'),$$

where

$$D^\mu(x,x') = \frac{1}{2}\gamma^\mu \left[U_x^\mu \delta_{x,x'-\hat{\mu}} - U_{x'}^{\mu\dagger} \delta_{x,x'+\hat{\mu}} \right].$$

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♢ Conserved charge is the natural point-split form
$N = \sum_x \bar{\psi}(x)\gamma^4[U_x^{4\dagger}\psi(x+\hat{4}) + U_x^4\psi(x-\hat{4})]/2.$

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♣ This leads to μ-dependent a^{-2} divergences in $a \to 0$ limit in energy density and quark number density even in the free theory!

$$\begin{aligned} \epsilon &= c_0 a^{-4} + c_1 \mu^2 a^{-2} + c_3 \mu^4 + c_4 \mu^2 T^2 + c_5 T^4 \\ n &= d_0 a^{-3} + d_1 \mu a^{-2} + d_3 \mu^3 + d_4 \mu T^2 + d_5 T^3. \end{aligned} \tag{1}$$

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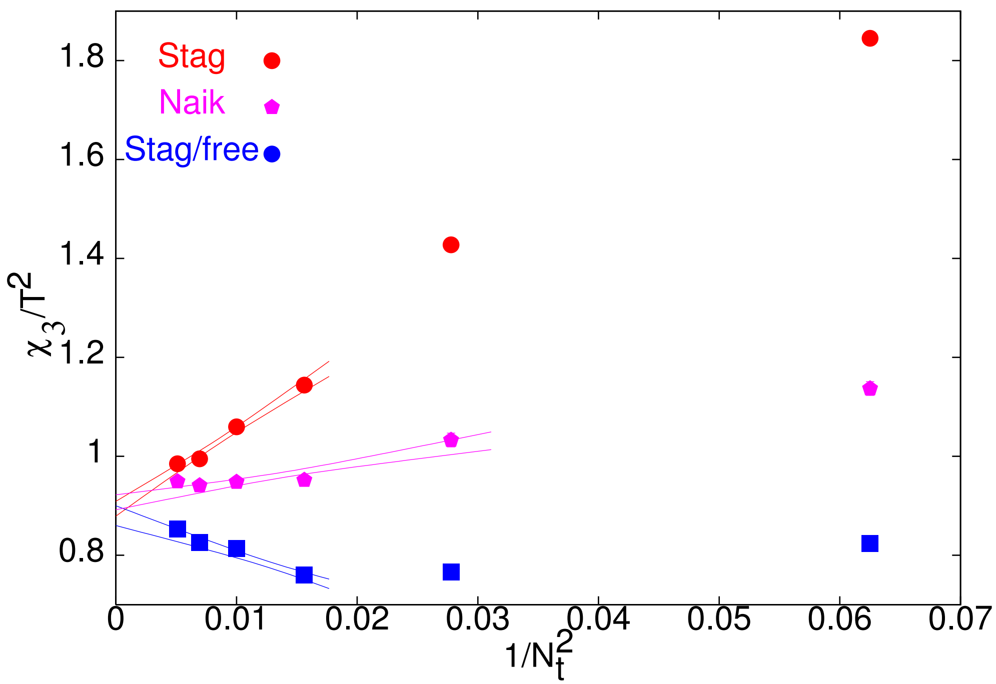
♠ Subtracting off vacuum contribution at $T = 0 = \mu$, eliminates the leading divergence in each case but the μ-dependent divergence persists.

♡ Hasenfratz-Karsch (PLB 1983) & Kogut et al. (PRD 1983) proposed to modify the weights to $\exp(\pm a\mu)$ to obtain finite results while simultaneously Bilić-Gavai (EPJC 1984) showed $(1 \pm a\mu)/\sqrt{(1 - a^2\mu^2)}$ also lead to finite results.

$\diamondsuit$ Indeed, in general *any* set of functions f, g, satisfying $f(a\mu) \cdot g(a\mu) = 1$ with $f(0) = f'(0) = 1$ suffices to eliminate the μ-dependent divergence(Gavai, PRD 1985).

♢ Indeed, in general *any* set of functions f, g, satisfying $f(a\mu) \cdot g(a\mu) = 1$ with $f(0) = f'(0) = 1$ suffices to eliminate the μ-dependent divergence(Gavai, PRD 1985).

♡ Note that the analytical proof was *only* for free quarks & thus pert. theory. Numerical computations had to be performed to show that it worked for the non-perturbative interacting case as well (Gavai-Gupta PRD 67, 034501 (2003)) :



♡ As $\chi \sim \partial n/\partial\mu$, its divergence is quadratic but μ-independent.

♠ At fixed temperature $T^{-1} = N_t \cdot a$, $a \to 0$ is equivalent to $N_t \to \infty$.

♢ Question : Why, and how, does lattice introduce this divergence? or Does it really ?

♣ It was argued [Hasenfratz-Karsch (PLB 1983)] that the divergence arises on the lattice due to the lack of a "formal" gauge symmetry: In continuum theory, μ term appears as a 4^{th} component of a constant gauge field. All the forms above restore this formal symmetry on lattice.

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♡ It turns out to be **Wrong !** *Latticization does not introduce divergence. It merely assists in spotting what exists already*(RVG-Sharma PLB 2015).

Universality or Uniqueness ?

(Gavai, arXiv: 2012.07798)

◇ Why are there three lattice actions when QCD has one ?

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f_L(\mu a) &= 1 + \mu a \ , & g_L(\mu a) = 1 - \mu a \\
f_E(\mu a) &= \exp(\mu a) \ , & g_E(\mu a) = \exp(-\mu a) \\
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♠ Technically, all these actions differ by **only irrelevant** terms, which are $\mathcal{O}(\mu^2 a^2)$ or higher.

♡ As $a \to 0$, all three reduce to the continuum QCD action. Universality then should assure us that physics should be the same for all three in that limit.

♡ *Paradox :* Irrelevant terms vanish from action as $a \to 0$ but do eliminate divergences. Apparent violation of universality ? !!

♣ Recalling that the divergences **do exist** in the continuum, as demonstrated in RVG-Sharma, PLB 2015, one wonders how/whether they will reappear for the modified actions with f_E or f_S as $a \to 0$.

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- Let us examine the quark number susceptibility at $\mu = 0$ in some details (Gavai, arXiv: 2012.07798):

$$\chi = \frac{1}{N_t N_s^3 a^2} \left[\left\langle \left(\mathrm{Tr} M^{-1} M' \right)^2 \right\rangle + \left\langle \mathrm{Tr} \left(M^{-1} M'' - M^{-1} M' M^{-1} M' \right) \right\rangle \right] , \quad (3)$$

Here M is the quark matrix (inverse propagator) and $M'(M'')$ is its first(second) derivative with respect to μ, all evaluated at $\mu = 0$. None of them depends on a explicitly $[S_F(a\mu) = \bar\psi M(a\mu) \psi]$.

- All the terms inside the bracket are dimensionless. As $a \to 0$, *all, & each,* of them, must vanish as a^2 for a nontrivial continuum limit of χ.

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- It is thus ruled out that the M'' term vanishes faster than a^2 for the other two actions and the results remain different for these actions even in the continuum limit, in spite of M'' arising from an **irrelevant** term in the action.

- It is clear that this problem of *non-universal* results worsens as one computes higher order fluctuations. While $f''_E(0) = f''_S(0)$, it can be shown that $f'''_E(0) = f''''_E(0) = 1$ but $f'''_S(0) = 3$ & $f''''_S(0) = 9$. Indeed, $f^n_E(0) \neq f^n_S(0)$ for $n > 2$. (Gavai, arXiv: 2012.07798)

- The 3^{rd} and 4^{th} orders are crucially used in locating the QCD critical point, both theoretically and experimentally. (See, e.g., Gavai, Contemp. Phys. 2016)

- The sixth order fluctuations have been advocated as signatures for the $O(4)$ criticality as well. (See, e.g., STAR coll., arXiv: 2105.14698)

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♡ Remember that universality ensured the pressure P to be the same **after** the continuum limit is taken. The computations of χ and higher order fluctuations described above is at finite a, *i.e.*, **before** the $a \to 0$ limit.

♠ The n^{th} derivatives of P is thus contaminated by the corresponding $\mu^n a^n$ terms in P for $n \geq 2$ which eventually are irrelevant terms in pressure, but not in the susceptibilities.

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♢ Further Problem : Neither with f_E nor with f_S, one has a conserved charge anymore ! Easy to prove this by deriving current conservation equation for $\mu \neq 0$ (Gavai, arXiv: 2012.07798).

♠ As a result, $Z \neq \sum_n z^n Z_n^C$ on the lattice for them. Possible only in the continuum limit of $a \to 0$.

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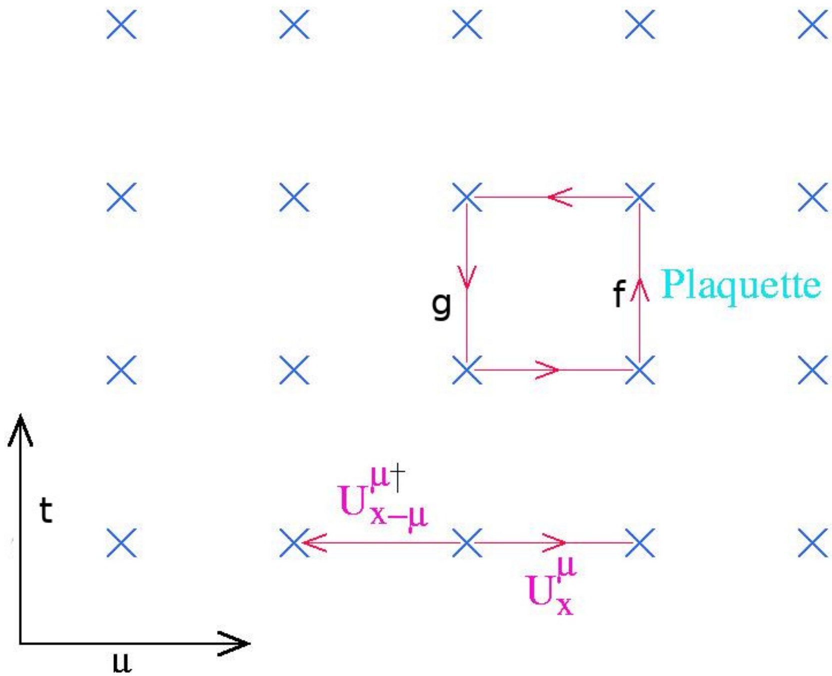
♡ Only for the linear μ-case one has an exactly conserved charge on the lattice, and thus an *exact* canonical partition function $Z = \sum_n z^n Z_n^C$ on the lattice.



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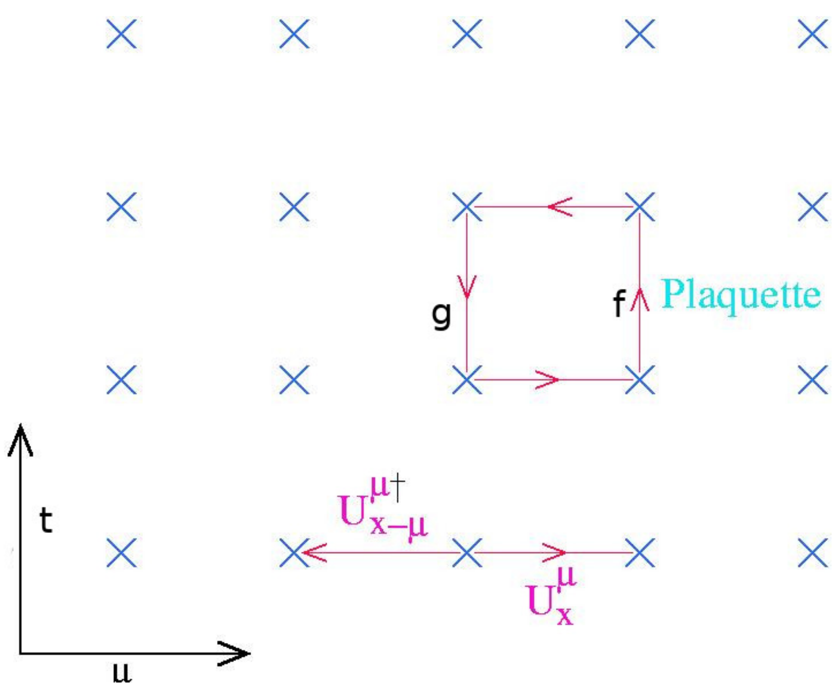

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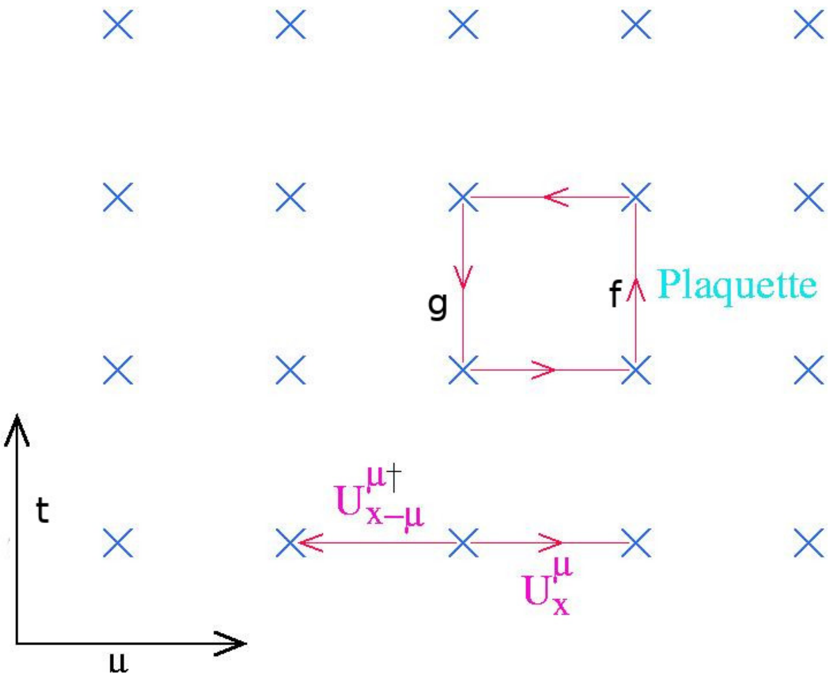

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♣ Moreover, as $a \to 0$ small quark loops must start contributing. How is this possible? Do the small loops sum up to zero/constant ?

♣ The same universality violation issue again ! At least, crucial to verify that it is obeyed for all three actions.

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- Insisting on Chiral invariance for overlap quarks at finite density leads to a linear μ-dependent action. $f \cdot g = 1$ prescription breaks the chiral invariance for overlap/Domain wall quarks and will also violate universality.