$\mathcal{PT}$ symmetry & patterns in finite-density QCD

Stella Schindler

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Collaborators:
Leandro Medina
Michael Ogilvie (prior talk)
Moses Schindler

Support:

Based on:

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The big picture

Finite-density QCD has a $\mathcal{PT}$-type symmetry: invariance under $\mathcal{CK}$

(simultaneous charge and complex conjugation)

Field of $\mathcal{PT}$-symmetric physics

- Non-Hermitian Hamiltonians with antilinear symmetries
- Well understood in many contexts
- 1000s of experimental realizations and physical applications

QCD phase structure

- Non-Hermitian, which causes a sign problem
- Not fully understood at nonzero μ
- Important theoretical and experimental implications

Can we apply tools from the field of $\mathcal{PT}$ symmetry to major open questions of lattice QCD?

$\mathcal{PT}$ symmetry

Invariance under <u>any</u> combination of linear & antilinear operators "$\mathcal{P}$" & "$\mathcal{T}$"

(In this context, $\mathcal{P}$ & $\mathcal{T}$ need not represent the parity/time operators of particle physics)

Non-Hermitian but well behaved

- Eigenvalues are either real or part of a complex-conjugate pair
- All real spectrum: "unbroken $\mathcal{PT}$ symmetry"
- Analogues of traditional Hermitian properties: unitarity, orthogonality, etc.

Applied physics:

Balanced loss & gain

Quantum mechanics:

$V(x) = V^*(-x)$

Finite-density QCD:

$\det M(\mu) = \det M^*(-\mu)$

CM Bender & S Boettcher, PRL (1998).
CM Bender et al., *PT Symmetry* (WorldScientific 2019).
PN Meisinger & MC Ogilvie, Ph Trans R Soc A (2013).
MA Schindler, **ST Schindler**, MC Ogilvie, to appear in J Phys Conf Series (2021).

Circumventing $\mathcal{PT}$ sign problems

Example $\mathcal{PT}$ scalar QFT:

$$S[\chi(x)] = \sum_x \frac{1}{2}\left(d_\mu \chi(x)\right)^2 + V[\chi(x)] - ih(x)\chi(x)$$

with

$$V[\chi(x)] = V^*[-\chi(x)]$$

Step 1. Rewrite kinetic term as:

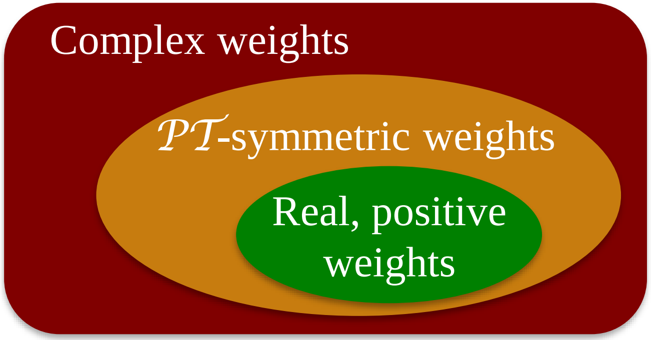
$$\exp\left[\frac{1}{2}(\partial_\mu \chi)^2\right] = \int d\pi_\mu(x) \exp\left[\frac{1}{2}\pi_\mu(x)^2 + i\pi_\mu(x)\partial_\mu\chi(x)\right]$$

Step 2. Take Fourier transform of potential: $\tilde{V}$

Step 3. Do lattice integration by parts to remove χ:

$$Z = \int \prod_x d\pi_\mu(x) \exp\left\{-\sum_x \frac{1}{2}\pi_\mu^2(x) + \tilde{V}(\partial \cdot \pi(x) - h(x))\right\}$$

The space of lattice actions



Intuitively, take a Fourier transform in the path integral with respect to the field χ. (Not the position variable x.)

The Fourier transform of a $\mathcal{PT}$-symmetric function is <u>always</u> real:

$$f(x) = f^*(-x) \leftrightarrow \tilde{f}(\omega) \in \mathbb{R}$$

Caveat: Fourier transform of e^{-V} is <u>not</u> always positive definite.

(But we're working on it! Stay tuned.)

L Medina & MC Ogilvie, PoS: LATTICE (2018).

Simulating a $\mathcal{PT}$ ϕ^4 model

$\mathcal{PT}$-symmetric extension of ϕ^4 model:

$$S = \sum_x \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}(\partial_\mu \chi)^2 + \frac{1}{2} m_\chi^2 \chi^2 - ig\phi\chi + \lambda(\phi^2 - v^2)^2 + h\phi$$

Dual, simulatable form:

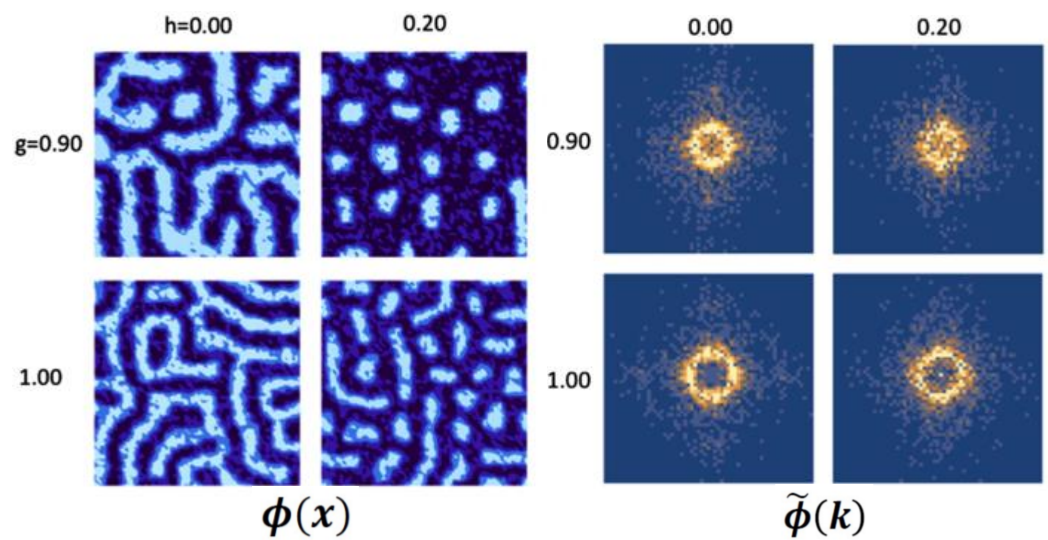
$$\tilde{S} = \sum_x \frac{1}{2}\left[\partial_\mu \phi(x)\right]^2 + \frac{1}{2}\pi_\mu^2(x) + \frac{\left[\partial \cdot \pi(x) - g\phi\right]^2}{2m_\chi^2} + \lambda(\phi^2 - v^2)^2 + h\phi$$

$\mathcal{PT}$ "breaking" → conjugate eigenvalues

- First impulse: look for disorder lines
  [e.g., Patel, PRD (2012); Akerlund, de Forcrand, & Rindlisbacher, JHEP (2016).]
- Unexpectedly found patterned fields
- Below: patterns for a range of couplings (g, h) at fixed λ and m_χ

Patterns are common at all scales in physics. Lattice simulations show their origin in this model:

- Persistent inhomogeneous behavior: violation of spectral positivity
- Ring-shaped Fourier transform: Lifshitz instability



Medina & Ogilvie (2018).
MA Schindler, **ST Schindler**, L Medina, MC Ogilvie, PRD (2020).

Calculating $\mathcal{PT}$-QFT phase structure

- Look at the mass matrix evaluated at a homogeneous solution
- Check for stability

Multi-component case:

$$\mathcal{M} = \left.\begin{pmatrix} \frac{\partial^2 V}{(\partial \phi^a)^2} & \frac{\partial^2 V}{\partial \phi^a \partial \chi^b} \\ \frac{\partial^2 V}{\partial \phi^a \partial \chi^b} & \frac{\partial^2 V}{(\partial \chi^b)^2} \end{pmatrix}\right|_{(\phi_0^a, \chi_0^b)}$$

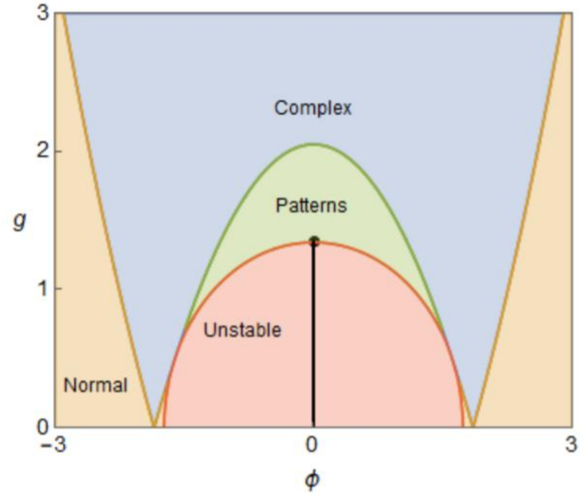
Region	$\det(\mathcal{M})$	Zeros of $\det(q^2 + \mathcal{M})$	Behavior of the propagator
Normal	Positive	All zeros are negative	Exponential decay
$\mathcal{PT}$ broken	Positive	One or more pairs of zeros are complex conjugates	Modulated exponential decay
Patterned	Positive	Even # of positive zeros	Stable to $q^2 = 0$ fluctuations, but unstable to $q^2 > 0$ fluctuations
Unstable	Positive	Odd # of positive zeros	Instability to $q^2 = 0$ fluctuations
Unstable	Negative	—	Instability to $q^2 = 0$ fluctuations

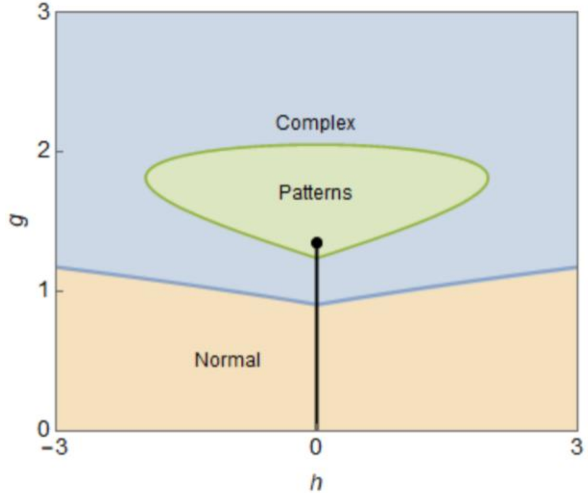
Schindler^2, Medina, & Ogilvie (2020).
Schindler^2 & Ogilvie (2021).
For $\mathcal{PT}$ Goldstones, see e.g., A Fring & T Taira, N Phys B (2020).

$\mathcal{PT}$ ϕ^4 phase diagram

$$S = \sum_x \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}(\partial_\mu \chi)^2 + \frac{1}{2} m_\chi^2 \chi^2 - ig\phi\chi + \lambda(\phi^2 - v^2)^2 + h\phi$$

- Tree-level analytics using ϕ as a parameter agree well with lattice simulations using h as a parameter
- Critical point location
  - **ϕ-g plane (top):** on boundary of patterned region
  - **g-h plane (bottom):** inside patterned region
  - Why? 1st order critical line is branch cut.
- In the g-h plane, the critical endpoint is inside the patterned region, which in turn is inside the sinusoidally-modulated region, which is bounded by a disorder line





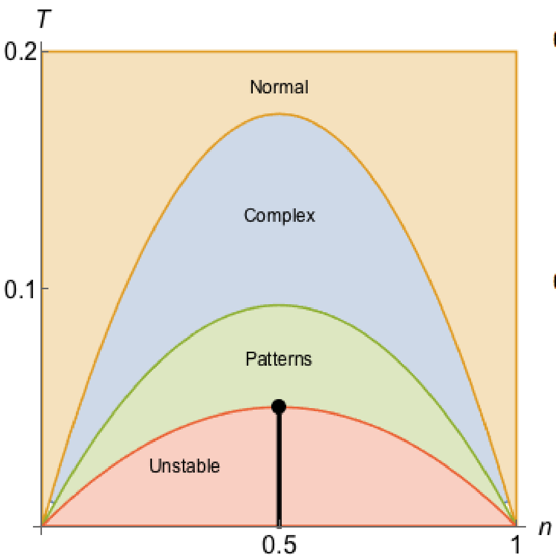
Schindler^2, Medina, & Ogilvie (2020). **Schindler**^2 & Ogilvie (2021).

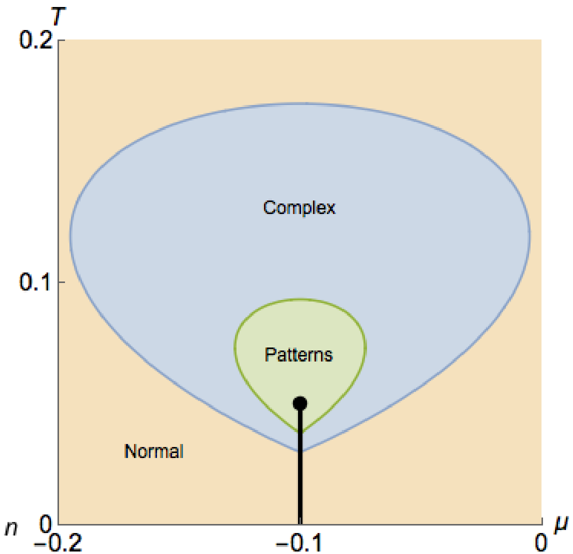
Applying $\mathcal{PT}$ methods to finite-density QCD

$$S = -2N_C N_F \sum_{\vec{x}} \log\left[1 + z e^{\chi(\vec{x}) + i\phi(\vec{x})}\right] + \sum_{\vec{x}} \left[\frac{1}{2\beta g_\chi^2}\left[(\nabla\chi)^2 + m_\chi^2 \chi^2\right] + \frac{1}{2\beta g_\phi^2}\left[(\nabla\phi)^2 + m_\phi^2 \phi^2\right] \right]$$

Simple heavy quark model:

- First term is heavy quark determinant, second term from gluonic interactions
- Parametrize Polyakov loop as $P = e^{\chi + i\phi}$ and $z = e^{\beta(\mu - M)}$
- Plots for fermion density n and μ vs. $T = 1/\beta$





Around the critical endpoint, we find:

- Patterns of confined & deconfined phase (green)
- Sinusoidally-modulated phase (blue)
- Disorder line (between blue/orange)
- Universal behavior in Z(2) pattern formation

Schindler^2 & Ogilvie (2021).
Schindler^2 & Ogilvie (in progress).
See also Y Park and ME Fisher, PRE (1999).

More general heavy quark model

$$S = -2N_C N_F \sum_{\vec{x}} \log\left[1 + z e^{\chi(\vec{x}) + i\phi(\vec{x})}\right] + \sum_{\vec{x},\vec{y}} \left[\frac{1}{2\beta}\chi(x) V_A^{-1}(x,y)\chi(y) + \frac{1}{2\beta}\phi(x) V_R^{-1}(x,y)\phi(y)\right]$$

(Same fermion term as before; general gluon potential $V = V_R + V_A$ with attractive/repulsive pieces)

Simple approximate criterion for stability against pattern formation:

$$1 + \beta \mathcal{X}_Q \tilde{V}_{qq}(k) > 0$$

(χ_Q is quark number susceptibility)

- Straightforward calculation
- Easy to extend to the continuum
- **All terms are computable on the lattice**

Towards QCD?

- Started simple; many improvements possible
- Apply to more complicated/complete models
- Extensions of algorithm (other integral transforms, methods for $i\phi^3$ universality class in progress, etc.)

Schindler^2 & Ogilvie (in progress).

Key takeaways

1. Finite-density QCD has a $\mathcal{PT}$-type symmetry

- Manifest in the lattice fermion determinant: $\det M(\mu) = \det M^*(-\mu)$

2. Motivates lattice & analytic methods for finite-density models

- Demonstrated in a complex extension of ϕ^4 theory and a heavy quark model

3. Potential physical implications

- Patterns of confined/deconfined matter, sinusoidal modulation, disorder lines

$\mathcal{PT}$ symmetry provides a promising new path forward towards understanding the phase structure of finite-density QCD