

Lattice QCD equation of state at finite chemical potential from an alternative expansion scheme*

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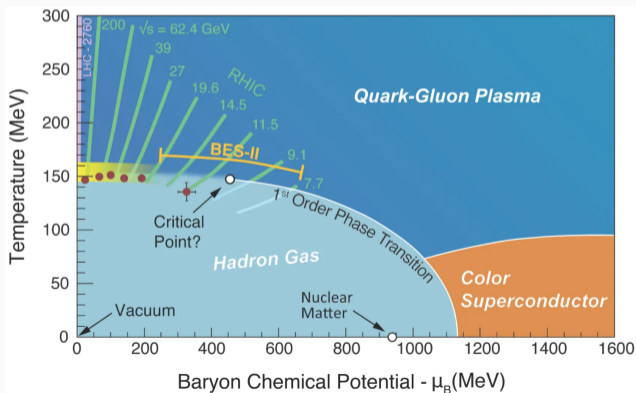
with:

S. Borsányi, Z. Fodor, J. N. Guenther, R. Kara, S. D. Katz, A. Pásztor, C. Ratti, K. K. Szabó

The phase diagram of QCD

Different phases of QCD matter (in equilibrium) are depicted in (temperature vs baryo-chemical potential) phase diagram

- Early Universe-like conditions at $\mu_B = 0$ (matter-anti-matter symmetry)
- Transition from hadron gas to QGP at $\mu_B = 0$ is a **smooth crossover** at $T \simeq 155 - 160$ MeV
- At larger μ_B , the transition is believed to become of first order \rightarrow **critical point**
- Investigations from first principles:
 - **Lattice QCD**
 - Perturbative methods (HTL, etc.)
 - Functional methods (FRG, DSE, etc.)

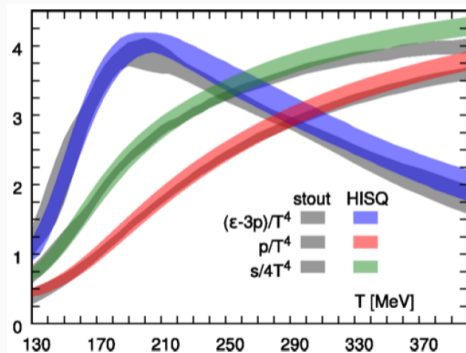


Lattice QCD: equation of state (EoS)

- ★ A crucial input to hydrodynamic simulations of e.g., heavy-ion collisions
- ★ Known at $\mu_B = 0$ to high precision for a few years now (continuum limit, physical quark masses) \rightarrow Agreement between different calculations

From grandcanonical partition function \mathcal{Z}

- * **Pressure:** $p = -k_B T \frac{\partial \ln \mathcal{Z}}{\partial V}$
- * **Entropy density:** $s = \left(\frac{\partial p}{\partial T} \right)_{\mu_i}$
- * **Charge densities:** $n_i = \left(\frac{\partial p}{\partial \mu_i} \right)_{T, \mu_{j \neq i}}$
- * **Energy density:** $\epsilon = Ts - p + \sum_i \mu_i n_i$
- * More (**Fluctuations**, etc...)



WB: Borsányi *et al.*, PLB 370 (2014) 99-104, HotQCD: Bazavov *et al.* PRD 90 (2014) 094503

Lattice QCD at finite μ_B

Lattice QCD suffers from the **sign problem** at finite chemical potential

- Taylor expansion around $\mu_B = 0$

$$\frac{p(T, \mu_B)}{T^4} = \sum_{n=0}^{\infty} c_{2n}(T) \left(\frac{\mu_B}{T}\right)^{2n}, \quad c_n(T) = \frac{1}{n!} \chi_n^B(T, \mu_B = 0)$$

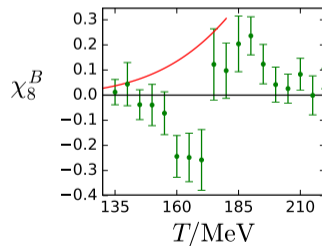
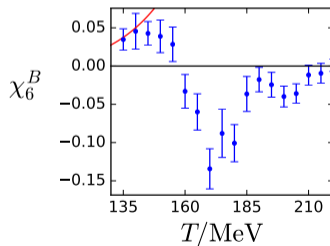
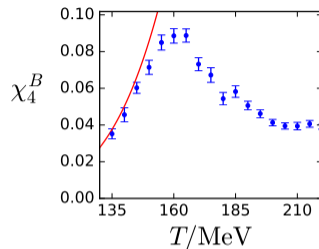
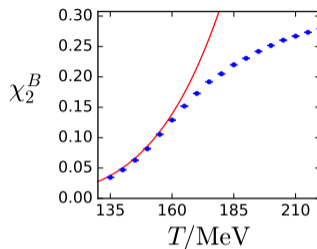
- Analytical continuation from imaginary μ_B
- Other methods to work around the sign problem still in exploratory stages
 - Reweighting techniques
 - Complex Langevin
 - Lefschetz thimbles
 - ...
- **The equation of state:** lattice results for the Taylor coefficients are currently available up to $\mathcal{O}(\hat{\mu}_B^8)$, but the reach is still limited to $\hat{\mu}_B \lesssim 2 - 2.5$ despite great computational effort (**WB: Borsányi et al. JHEP 10 (2018) 205, HotQCD: Bazavov et al. PRD101 (2020), 074502**)

Lattice QCD at finite μ_B - Taylor coefficients

- Fluctuations of baryon number are the Taylor expansion coefficients of the pressure

$$\chi_{ijk}^{BQS}(T) = \left. \frac{\partial^{i+j+k} p/T^4}{\partial \hat{\mu}_B^i \partial \hat{\mu}_Q^j \partial \hat{\mu}_S^k} \right|_{\vec{\mu}=0}$$

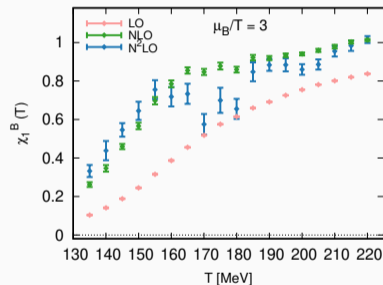
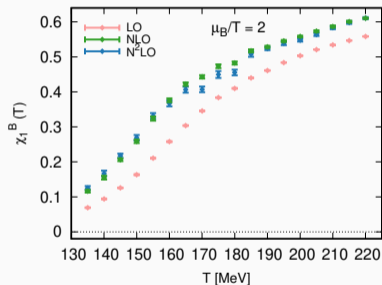
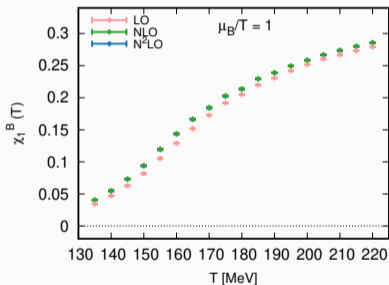
- Signal extraction is increasingly difficult with higher orders, especially in the transition region
- Higher order coefficients present a more complicated structure



WB: Borsányi *et al.* JHEP 10 (2018) 205;
(also e.g., HotQCD: Bazavov *et al.* PRD101 (2020), 074502)

Lattice QCD at finite μ_B - Taylor expansion

- Thermodynamic quantities at large chemical potential become problematic
- Higher orders do not help with the convergence of the series

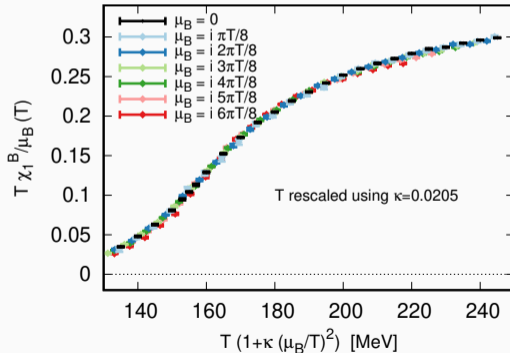
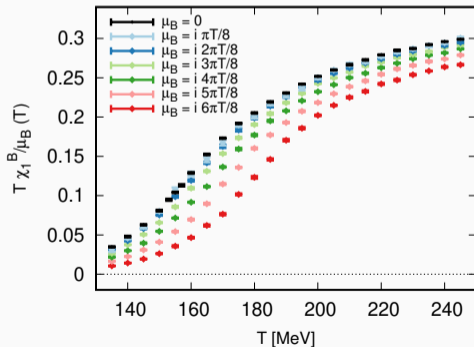


- Inherent problem with Taylor expansion: carried out at $T = \text{const.}$ This doesn't cope well with $\hat{\mu}_B$ -dependent transition temperature
- Can we find an alternative expansion to improve finite- $\hat{\mu}_B$ behavior?

An alternative approach

From simulations at imaginary μ_B we observe that $\chi_1^B(T, \hat{\mu}_B)$ at (imaginary) $\hat{\mu}_B$ appears to be differing from $\chi_2^B(T, 0)$ mostly by a rescaling of T :

$$\frac{\chi_1^B(T, \hat{\mu}_B)}{\hat{\mu}_B} = \chi_2^B(T', 0), \quad T' = T (1 + \kappa \hat{\mu}_B^2)$$

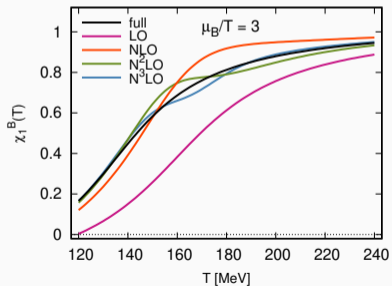
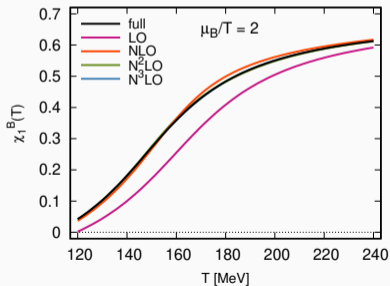
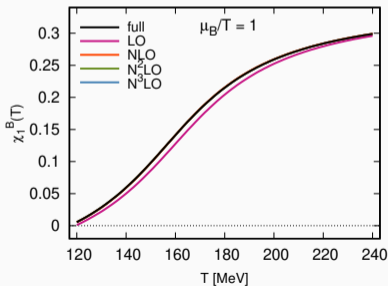


Taylor expanding a (shifting) sigmoid

Assume we have a sigmoid function $f(T)$ which shifts with $\hat{\mu}$, with a simple T -independent shifting parameter κ . How does Taylor cope with it?

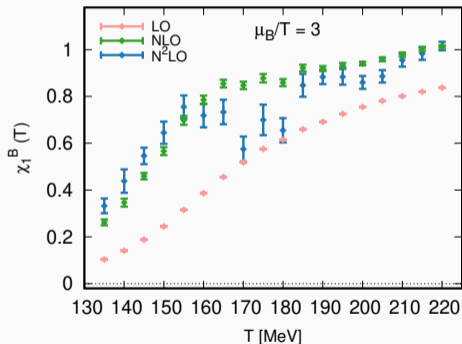
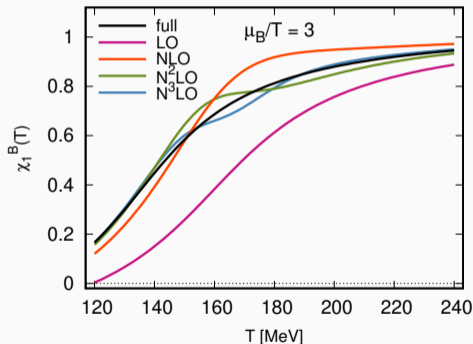
$$f(T, \hat{\mu}) = f(T', 0), \quad T' = T(1 + \kappa \hat{\mu}^2),$$

We fitted $f(T, 0) = a + b \arctan(c(T - d))$ to $\chi_2^B(T, 0)$ data for a 48×12 lattice



Taylor expanding a (shifting) sigmoid

- The Taylor expansion seems to have problems reproducing the original function (left)
- Quite suggestive comparison with actual Taylor-expanded lattice data (right)



- Problems at T slightly larger than $T_{pc} \Rightarrow$ influence from structure in χ_6^B and χ_8^B

Rigorous formulation

- We have observed the $\hat{\mu}_B$ -dependence seems to amount to a simple T - rescaling
- A simplistic scenario with a single T - independent parameter κ does not provide a systematic treatment which can serve as an alternative expansion scheme
- We allow for more than $\mathcal{O}(\hat{\mu}^2)$ expansion of T' and let the coefficients be T -dependent:

$$\frac{\chi_1^B(T, \hat{\mu}_B)}{\hat{\mu}_B} = \chi_2^B(T', 0) , \quad T' = T (1 + \kappa_2(T) \hat{\mu}_B^2 + \kappa_4(T) \hat{\mu}_B^4 + \mathcal{O}(\hat{\mu}_B^6))$$

- **Important:** we are simply re-organizing the Taylor expansion via an expansion in the shift

$$\Delta T = T - T' = (\kappa_2(T) \hat{\mu}_B^2 + \kappa_4(T) \hat{\mu}_B^4 + \mathcal{O}(\hat{\mu}_B^6))$$

- Comparing the (Taylor) expansion in $\hat{\mu}_B$ and our expansion in ΔT order by order, we can relate $\chi_n^B(T)$ and $\kappa_n(T)$

Rigorous formulation

Equating same-order terms one finds:

$$\kappa_2(T) = \frac{1}{6T} \frac{\chi_4^B(T)}{\chi_2^{B'}(T)}$$

$$\kappa_4(T) = \frac{1}{360\chi_2^{B'}(T)^3} \left(3\chi_2^{B'}(T)^2 \chi_6^B(T) - 5\chi_2^{B''}(T)\chi_4^B(T)^2 \right)$$

- In principle, the procedure can be carried over systematically, however higher order terms still suffer from cancellations and can prove challenging
- Instead, we only use the first relation and combine it with simulations at imaginary- $\hat{\mu}_B$ to extract $\kappa_2(T)$, $\kappa_4(T)$

Determine κ_n

I. Directly determine $\kappa_2(T)$ at $\hat{\mu}_B = 0$ from the previous relation

II. From our imaginary- $\hat{\mu}_B$ simulations ($\hat{\mu}_Q = \hat{\mu}_S = 0$) we calculate:

$$\frac{T' - T}{T \hat{\mu}_B^2} = \kappa_2(T) + \kappa_4(T) \hat{\mu}_B^2 + \mathcal{O}(\hat{\mu}_B^4) = \Pi(T)$$

III. Calculate $\Pi(T, N_\tau, \hat{\mu}_B^2)$ for $\hat{\mu}_B = in\pi/8$ and $N_\tau = 10, 12, 16$

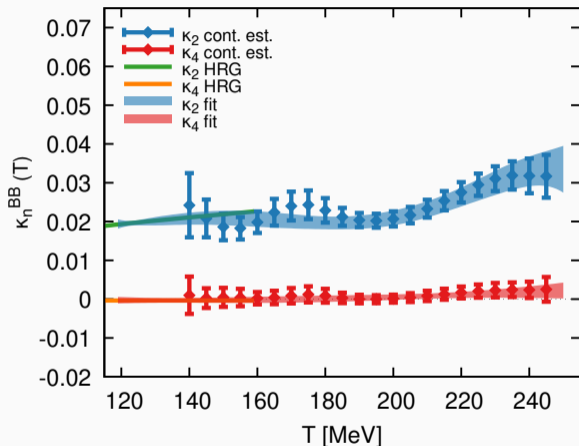
IV. Perform a combined fit of the $\hat{\mu}_B^2$ and $1/N_\tau^2$ dependence of $\Pi(T)$ at each temperature, yielding a continuum estimate for the coefficients

\Rightarrow The $\mathcal{O}(1)$ and $\mathcal{O}(\hat{\mu}_B^2)$ coefficients of the fit are $\kappa_2(T)$ and $\kappa_4(T)$

The results for $\kappa_2(T)$, $\kappa_4(T)$

Our initial guess was not far-off:

- Fairly constant $\kappa_2(T)$ over a large T -range
- Clear separation in magnitude between $\kappa_2(T)$ and $\kappa_4(T)$ hints at better convergence
- Agreement with the HRG model results at low temperatures
- Polynomial fits of $\kappa_2(T)$ and $\kappa_4(T)$ before use in thermodynamics (good fit qualities)



NOTE: polynomial fits take into account both statistical and systematic correlations.

Thermodynamics at finite (real) μ_B

Thermodynamic quantities at finite (real) μ_B can be reconstructed from the same ansatz:

$$\frac{n_B(T, \hat{\mu}_B)}{T^3} = \hat{\mu}_B \chi_2^B(T', 0)$$

with $T' = T(1 + \kappa_2^{BB}(T) \hat{\mu}_B^2 + \kappa_4^{BB}(T) \hat{\mu}_B^4)$.

From the baryon density n_B one finds the pressure:

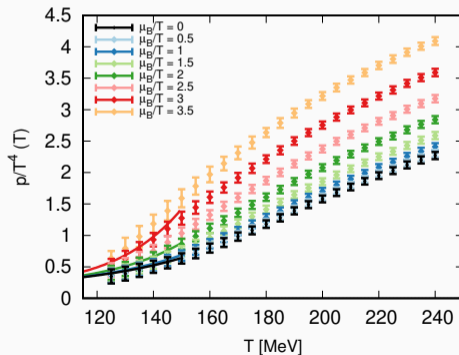
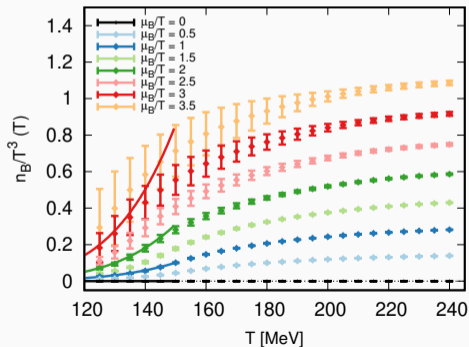
$$\frac{p(T, \hat{\mu}_B)}{T^4} = \frac{p(T, 0)}{T^4} + \int_0^{\hat{\mu}_B} d\hat{\mu}'_B \frac{n_B(T, \hat{\mu}'_B)}{T^3}$$

then the entropy, energy density:

$$\frac{s(T, \hat{\mu}_B)}{T^4} = 4 \frac{p(T, \hat{\mu}_B)}{T^4} + T \left. \frac{\partial p(T, \hat{\mu}_B)}{\partial T} \right|_{\hat{\mu}_B} - \hat{\mu}_B \frac{n_B(T, \hat{\mu}_B)}{T^3}$$
$$\frac{\epsilon(T, \hat{\mu}_B)}{T^4} = \frac{s(T, \hat{\mu}_B)}{T^3} - \frac{p(T, \hat{\mu}_B)}{T^4} + \hat{\mu}_B \frac{n_B(T, \hat{\mu}_B)}{T^3}$$

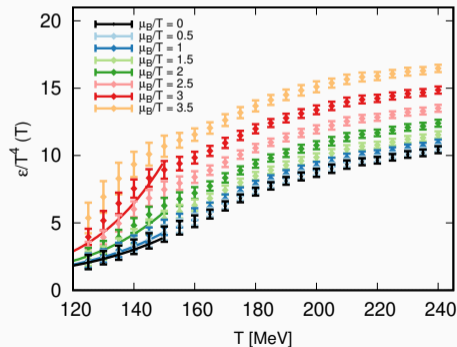
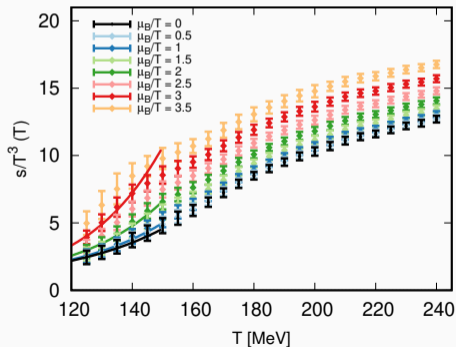
Thermodynamics at finite (real) μ_B

- We reconstruct thermodynamic quantities up to $\hat{\mu}_B \simeq 3.5$ with uncertainties well under control
- Agreement with HRG model calculations at small temperatures
- No pathological (non-monotonic) behavior is present



Thermodynamics at finite (real) μ_B

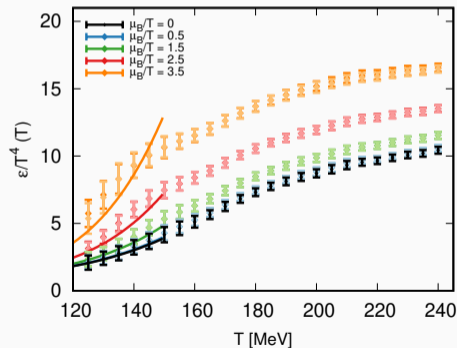
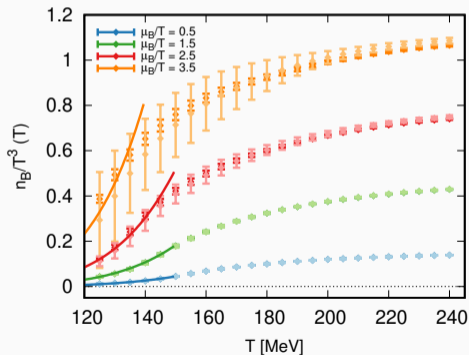
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Thermodynamics at finite (real) μ_B

- We also check the results without the inclusion of $\kappa_4(T)$ (darker shades)
- Including $\kappa_4(T)$ only results in added error, but does not “move” the results

→ Good convergence



Summary

- The EoS for QCD at large chemical potential is highly demanded in heavy-ion collisions community, especially for hydrodynamic simulations
- Historical approach of Taylor expansion for EoS has shortcomings
 - Because of technical/numerical challenges
 - Because of phase structure of the theory
- An alternative expansion scheme tailored to the specific behavior of relevant observables seems a better approach (better convergence)
- Thermodynamic quantities up to $\hat{\mu}_B \simeq 3.5$ have very reasonable uncertainties
- Just like Taylor, systematically improvable

Summary

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THANK YOU!

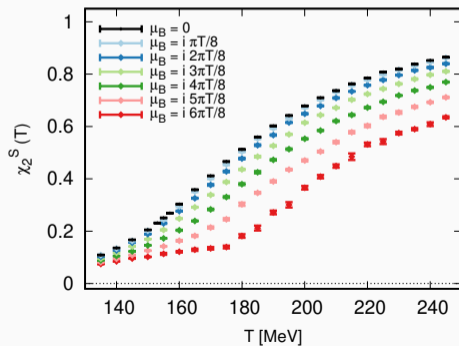
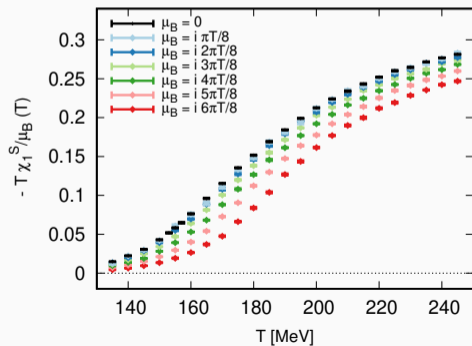
BACKUP

An alternative approach

The other (BS) second order susceptibilities display a very similar scenario:

$$\frac{\chi_1^S}{\hat{\mu}_B}(T, \hat{\mu}_B) = \chi_{11}^{BS}(T', 0) ,$$

$$\chi_2^S(T, \hat{\mu}_B) = \chi_2^S(T', 0)$$



Rigorous formulation

Similar relations can be derived analogously from:

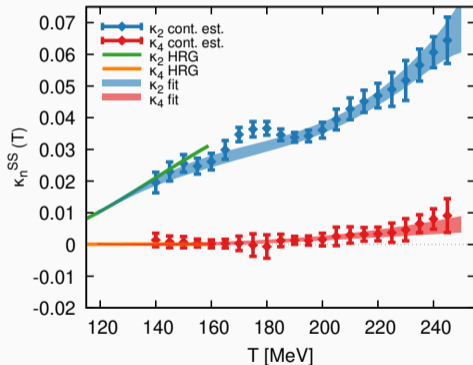
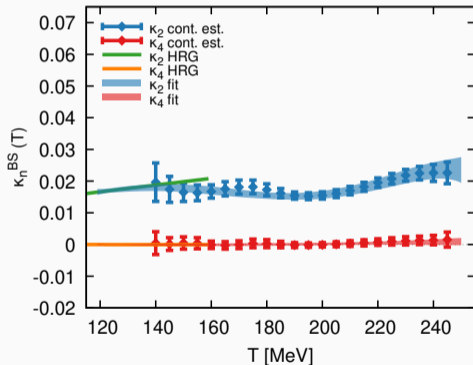
$$\frac{\chi_1^S}{\hat{\mu}_B}(T, \hat{\mu}_B) = \chi_{11}^{BS}(T', 0), \quad \chi_2^S(T, \hat{\mu}_B) = \chi_2^S(T', 0)$$

yielding:

$$\begin{aligned} \kappa_2^{BS}(T) &= \frac{1}{6T} \frac{\chi_{31}^{BS}(T)}{\chi_{11}^{BS'}(T)} & \kappa_2^S(T) &= \frac{1}{2T} \frac{\chi_{22}^{BS}(T)}{\chi_2^{S'}(T)} \\ \kappa_4^{BS}(T) &= \frac{1}{360\chi_{11}^{BS'}(T)^3} \left(3\chi_{11}^{BS'}(T)^2 \chi_{51}^{BS}(T) \right. & \kappa_4^S(T) &= \frac{1}{24\chi_2^{S'}(T)^3} \left(\chi_2^{S'}(T)^2 \chi_{42}^{BS}(T) \right. \\ & \quad \left. - 5\chi_{11}^{BS''}(T)\chi_{31}^{BS}(T)^2 \right) & & \quad \left. - 3\chi_2^{S''}(T)\chi_{22}^{BS}(T)^2 \right) \end{aligned}$$

The results for $\kappa_2(T)$, $\kappa_4(T)$

A similar picture appears for κ_n^{BS} and κ_n^{SS}



NOTE: polynomial fits take into account both statistical and systematic correlations.

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