The 't Hooft-Veneziano limit of the Polyakov loop models

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I. Polyakov loop models

1. Definition of traced Polyakov loop (PL):

$$W(x) \;=\; \mathrm{Tr} \prod_{t=1}^{N_t} U_0(x,t) \; .$$

2. Physical meaning (F_q - free energy of heavy quark):

$$\langle W(x) \rangle \;=\; e^{-\beta F_q} \; .$$

$\langle W(x) \rangle$ serves as an exact (approximate) order parameter of a deconfinement phase transition in pure (with dynamical quarks) gauge theory.

3. Two-point correlation: potential between quark–anti-quark pair

4. N-point correlation: N-quark (baryon) potential

One conventional way to reveal a phase structure of LGT is to construct an effective model for the PL and then to study phase transitions in this model.

Finite-temperature effective model

The following spin-like model describes the Polyakov loop interaction in the finite-temperature LGT (simplest version)

$$S = \beta \sum_{x,n} \mathrm{Re} W(x) W^*(x+e_n) + \sum_x \sum_{f=1}^{N_f} \left(h_r^f W(x) + h_i^f W^*(x) \right) \quad (1)$$

$\beta = \beta(g^2)$; $h_r^f = h_r(m_f, \mu)$ and $h_i^f = h_i(m_f, \mu)$ are functions of quark mass m_f and baryon chemical potential μ. For one flavor

$$h_r \; = \; h(m)\, e^{\mu} \; , \; h_i \; = \; h(m)\, e^{-\mu}$$

More ”advanced” approximation to finite-temperature QCD has been derived by J. Greensite, R. Hollwieser (2016): non-local Polyakov loop model for pure gauge theory

$$S = \beta \sum_{x,y} \mathrm{Re}\, W(x)\, K(x-y)\, W^*(y) \quad (2)$$

Our goal is to study these and similar PL models in the combined large N, N_f limit called the ’t Hooft-Veneziano limit at finite temperature and non-zero baryon chemical potential.

II. Construction of the large N, N_f solution

The 't Hooft-Veneziano limit: $g \to 0, N \to \infty, N_f \to \infty$ such that the product $g^2 N$ and the ratio $N_f/N = \kappa$ are kept fixed. For N_f degenerate flavors: $N_f h(m) \to N\kappa h(m) = N\alpha$. The method used to obtain exact solution in this limit does not rely on the large N factorization and applies equally well to all PL models if the effective action depends on the fundamental and/or adjoint PLs.

- Insert the unity in every lattice site

$$\int dtds\, \delta\left(Nt - \text{Re}\, W(x)\right)\, \delta\left(Ns - \text{Im}\, W(x)\right) \,. \tag{3}$$

- Calculate the resulting group integral in the large N limit

$$\mathcal{I}(a,b) \;=\; \int dW \exp\left[aW + bW^*\right] \tag{4}$$

with a, b - complex parameters

Group integral and partition function

$$\mathcal{I}(a,b) \;=\; \sum_{r=0}^{\infty}\sum_{q=-\infty}^{\infty} (ab)^r \sum_{\lambda\vdash r} \frac{d(\lambda)d(\lambda+|q|^N)}{r!(r+N|q|)!}\, c^{|q|N}, \qquad (5)$$

$c=a$ if $q>0$ and $c=b$ if $q<0$. $\lambda=(\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_N\geq 0)$ is a partition of r. $d(\lambda)$ - dimension of symmetric group S_r. Sum over partitions can be evaluated in the large N limit. Partition function reads (u(x)=q(x)/N)

$$Z=\prod_x \int_0^1 \rho(x)d\rho(x)\int_0^{2\pi}\frac{d\omega(x)}{2\pi}\int_{-\infty}^{\infty} du(x)\; e^{N^2 S_{eff}}\;. \qquad (6)$$

The effective action is given by ($W(x)=\rho(x)e^{i\omega(x)}$, $\alpha=\kappa h$)

$$\begin{aligned} S_{eff} \;&=\; S_g(\{\rho(x)e^{i\omega(x)},\rho(x)e^{-i\omega(x)}\})+\alpha\sum_x \rho(x)\cos\omega(x) \\ &+\; \mu\sum_x u(x)+\sum_x V(\rho(x),\omega(x),u(x))\;, \qquad (7)\end{aligned}$$

$$V \;\sim\; \lim_{N\to\infty} N^{-2}\;\ln\mathcal{I}\;.$$

III. Phase transitions

Partition function and all observables are calculated by saddle-point method: one has to look for a translation invariant solutions.

Pure gauge case, $\alpha = 0$: first order confinement-deconfinement phase transition. The expectation value of the PL jumps from zero to $1/2$ at the critical point $d\beta = 1$.
$\alpha = \kappa h(m) \neq 0$: the system undergoes the third order phase transition of the GWW type along the critical line

$$d\beta + \alpha = 1 \; . \tag{8}$$

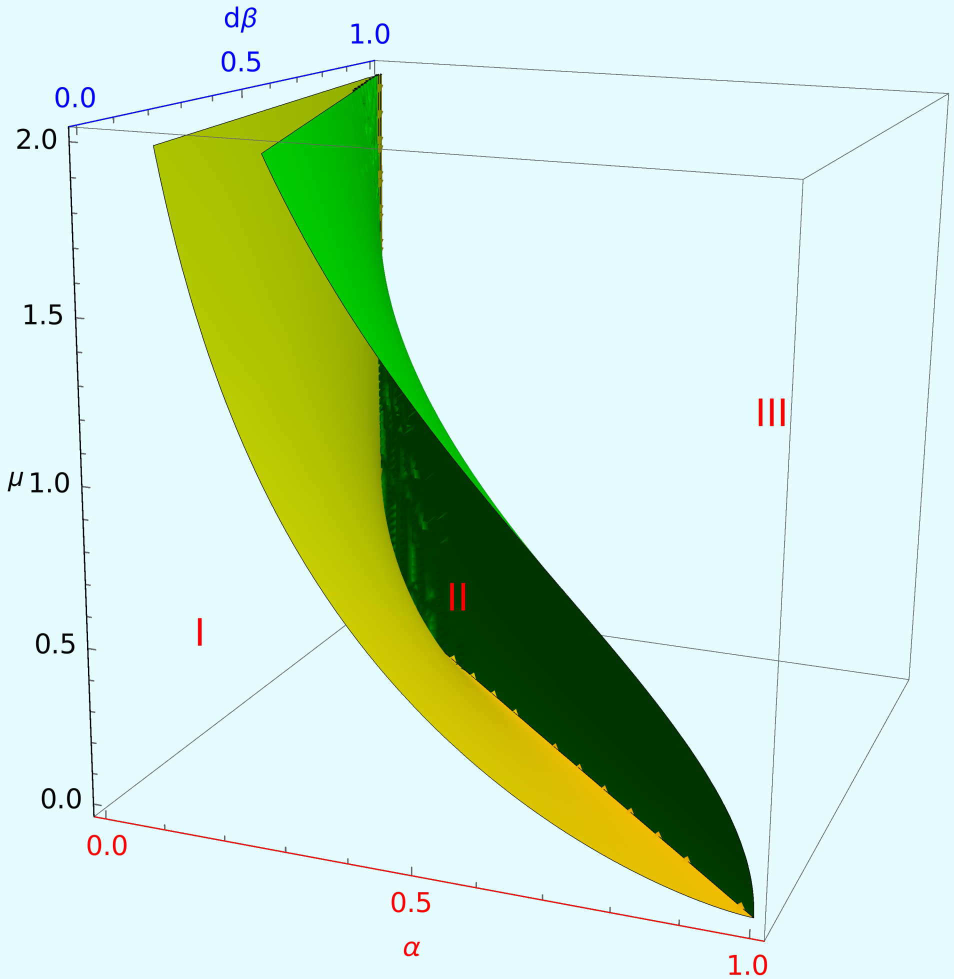
This agrees with the mean-field solution of $U(N)$ PL model at large N
- at $\mu = 0$: P. H. Damgaard and A. Patkós, Phys.Lett. B172 (1986) 369;
- at $\mu \neq 0$: C. Christensen, Phys.Lett. B714 (2012) 306.

$SU(N)$ integrals $\mathcal{I}(a, b)$ differ from $U(N)$ ones at large N if $\mu \neq 0$:
O. Borisenko, V. Chelnokov, S. Voloshyn, Nucl.Phys. B960 (2020) 115177.

Phase diagram at $\mu \neq 0$

Critical surface (yellow) of the 3rd order phase transition:

$$\mu = \ln\left(1 + \sqrt{1 - z^2}\right) - \ln z - \sqrt{1 - z^2}\,, \; z = \frac{\alpha}{1 - d\beta}\,. \qquad (9)$$



Region I: no dependence on μ, $SU(N)$ free energy coincides with $U(N)$ one. Mass spectrum is real.

Region II: non-trivial dependence on μ. Non-zero particle density and complex masses.

Region III: Masses are real. $z \approx \cosh^{-1} \mu$.

IV. Correlation functions and screening masses

The correlation function of an arbitrary form

$$\Gamma(\eta, \bar{\eta}) = \langle \prod_x W(x)^{\eta(x)} W^*(x)^{\bar{\eta}(x)} \rangle \quad (10)$$

is evaluated by integrating over Gaussian fluctuations around saddle-point solution. At large but finite N the properties of Γ depend on the dimension and presence/absence of external field h.

Below I discuss two examples in dimension $d = 3$.

N-point function and baryon potential

When $h = 0$ and in the confinement phase

$$\Gamma_N(\sigma) \sim \sum_x \prod_{i=1}^{N} G_{x,x(i)}(\sigma) \, , \; \sigma = \sqrt{\frac{2}{\beta}(1 - d\beta)} \, . \qquad (11)$$

$x(i)$ - position of N static quarks. Green function

$$G_{x,x'} = \frac{\text{const}}{R^{\frac{d}{2}-1}} K_{\frac{d}{2}-1}(\sigma R) \, , \; R^2 = \sum_{n=1}^{d} (x_n - x'_n)^2 \, .$$

Calculating $\Gamma_N(\sigma)$ in the continuum reduces to the geometric median problem: find a point y which minimizes $\sum_{i=1}^{N} \sqrt{\sum_{n=1}^{d} (y_n - x_n(i))^2}$.
If $N = 3$ this gives famous Y law for the baryon potential

$$\Gamma_3(\sigma) \sim \exp[-\sigma Y] \, . \qquad (12)$$

Complex masses and oscillating decay

Connected part of the Polyakov loop correlation in Regions II and III

$$\langle\, W(0)W^*(R)\,\rangle_c = MM^*(G_R(m_1) + G_R(m_2))\,. \qquad (13)$$

M is magnetization, $G_R(m_i)$ are diagonal correlators in the correlation matrix at $\mu \neq 0$. If $\mu = 0$, $m_{1,2}$ correspond to chromo-electric and chromo-magnetic masses. If $\mu \neq 0$, in the Region II the masses are complex: $m_1 = m_2^* = m_r + im_i$. This leads to an exponential oscillating decay of the correlations

$$\langle\, W(0)W^*(R)\,\rangle_c \sim \mathrm{e}^{-m_r R} \cos m_i R\,. \qquad (14)$$

In the Region III the masses are real, $m_2 > m_1$. No phase transition separating Regions II and III has been found.

Phase with oscillating decay was shown to exist in $(1+1)d$ LGT with heavy quarks H. Nishimura, M. Ogilvie, K. Pangeni, Phys.Rev. D93 (2016) 094501 and in $Z(3)$ spin model in a complex external field O. Akerlund, P. de Forcrand, T. Rindlisbacher, JHEP 10 (2016) 055

V. Conclusion

- Certain class of $SU(N)$ group integrals differ from $U(N)$ ones even in the large N limit
- The problem of complex mass generation at finite baryon density deserves further thorough investigation