A New Way Of Resumming The Finite-μ_B Taylor Series

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Outline of the Talk

1. Motivation.
2. Exponential Resummation of the Taylor Series:
   (i) Method.
   (ii) Results for the Pressure and Baryon Density.
   (iii) Breakdown of the Calculation, Phase Factor and Roots of the Partition Function.
3. Conclusions.

Finite-Density QCD: Approaches and Challenges

- Finite-Density QCD: Interesting but computationally challenging due to the Sign Problem.
- Some of the approaches that have been tried in the QCD case are:
  1. Reweighting [Z. Fodor & S. Katz (2002, 2004)] [Overlap problem].
  2. Imaginary μ [P. deForcrand & O. Philipsen (2002), M. Elia & M. Lombardo (2003), Budapest-Wuppertal (2008, 2016)] [Analytic continuation back to real μ].
  3. Taylor Expansion [Allton *et al.* (2003), R. Gavai & S. Gupta (2003)] [Higher-order coefficients difficult to calculate. Also, rate of convergence slow].
- As a result, each of these approaches requires one or another type of **resummation.**

Resummation Schemes

- Reweighting: Reweight simultaneously in more than one parameter (Multi-parameter reweighting) [Z. Fodor & S. Katz (2004), S. Ejiri (2004)].
- Imaginary μ: Analytic continuation using rational vs. polynomial functions [R. Bellwied *et al.* (2015), C. Schmidt *et al.* (2020)].
- Combine imaginary-μ results with Taylor expansion along the Line of Constant Physics [S. Borsányi *et al.* (2021)].
- Taylor Expansions: Resum Taylor series using Padé resummation [R. Gavai & S. Gupta (2008)].

In this talk, I will describe a second way of resumming the Taylor series.

Exponential Resummation of the Taylor Series

- The calculation of the Nth order Taylor coefficient requires the evaluation of terms such as $\langle \bar{D}_1^a \bar{D}_2^b \cdots \bar{D}_N^k \rangle$, where

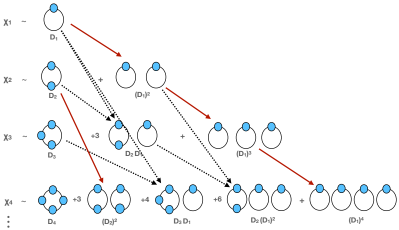
$$\bar{D}_n = \frac{D_n}{n!} = \frac{1}{n!} \frac{\partial^n}{\partial \hat{\mu}_B^n} \ln \det M\left(T, \hat{\mu}_B\right)\Big|_{\mu_B=0} \qquad (\hat{\mu}_B \equiv \mu_B/T).$$

- At Nth order, one must have $1 \cdot a + 2 \cdot b + \cdots + k \cdot N = N$. Thus for e.g. an operator such as $\bar{D}_1$ also contributes at Nth order through the term $\bar{D}_1^N$.

- It is not difficult to see that the contribution of $\bar{D}_1$ to all orders in μ_B takes the form $\exp\left(\bar{D}_1 \hat{\mu}_B\right)$. Similarly, the contribution of the operator $\bar{D}_n$ to all orders in μ_B can be written as $\exp\left(\bar{D}_n \hat{\mu}_B^n\right)$.

- One can thus resum the contribution of the first N operators to all orders in μ_B through the exponential factor $\exp\left(\bar{D}_1 \hat{\mu}_B + \bar{D}_2 \hat{\mu}_B^2 + \cdots + \bar{D}_N \hat{\mu}_B^N\right)$.

Exponential Resummation of the Taylor Series



[arXiv:2106.03165]

- In the continuum theory, D_n is the integrated n-point correlation function of the conserved charge operator: $D_n = \int d^4x_1 \dots d^4x_n \, J_0(x_1) \dots J_0(x_n)$.
- Although in both cases one only has the first N derivatives, the latter way of representing the quark determinant (red arrows) also accounts for the contribution of those derivatives to all orders in μ_B.
- Similarly, cross terms such as $D_1 D_2$ (dotted arrows) are also accounted for since $\exp(\bar{D}_1 \hat{\mu}_B + \bar{D}_2 \hat{\mu}_B^2 + \dots) = \exp(\bar{D}_1 \hat{\mu}_B) \cdot \exp(\bar{D}_2 \hat{\mu}_B^2) \cdot \dots$

Finite-μ_B Contribution to the Pressure

- Hence, the resummed estimate (to all orders in μ_B for $\bar{D}_{n\leqslant N}$) for the finite-μ_B contribution to the pressure $\Delta P(T,\mu_B) \equiv P(T,\mu_B) - P(T,0)$ is given by

$$\frac{\Delta P_N^R(T,\mu_B)}{T^4} = \frac{1}{VT^3} \ln \left\langle \exp\left(\sum_{n=1}^{N} \bar{D}_n \hat{\mu}_B^n\right)\right\rangle,$$

  where $\langle\cdot\rangle$ is the expectation value w.r.t. to an ensemble generated at $\mu_B = 0$.

- For comparison, the Taylor-expanded result for the pressure (to $\mathcal{O}(\hat{\mu}_B^N)$) is given by

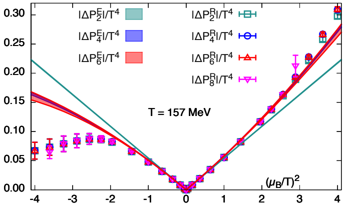
$$\frac{\Delta P_N^E(T,\mu_B)}{T^4} = \sum_{n=1}^{N} \chi_n^B \frac{\hat{\mu}_B^n}{n!}.$$

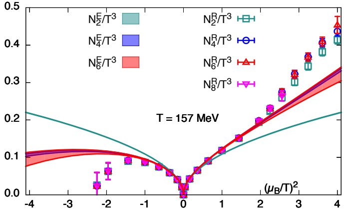
- We see that $\Delta P_N^R(T,\hat{\mu}_B) = \Delta P_N^E(T,\hat{\mu}_B) + \sum_{n>N}^{\infty} \langle \bar{D}_1^i \bar{D}_2^j \ldots \bar{D}_N^k \rangle\, \hat{\mu}_B^n$, where $1\cdot i + 2\cdot j + \cdots + N\cdot k = n$.

Setup of the Calculation

- For this calculation, we made use of the data generated the HotQCD collaboration for their calculation of the QCD Taylor coefficients.
- 2+1-flavor HISQ ensembles for several temperatures in the range 145 MeV $\lesssim T \lesssim$ 176 MeV. The light and strange quark masses were set to their respective physical values.
- For the present study, we restricted ourselves to a single lattice spacing, $N_\tau = 8$, since this had the most extensive $\left(\mathcal{O}(10^5 \text{ - } 10^6)\right)$ statistics. Spatial sites $N_\sigma = 32$ in all cases.
- The $\bar{D}_n$ were calculated stochastically using up to 2000 random vectors for $\bar{D}_1$ and 500 random vectors for $\bar{D}_2, \ldots, \bar{D}_8$.
- Note: No unbiased way of evaluating the exponential factor from the stochastic estimates for the $\bar{D}_n$. However, bias is a finite-size effect which should vanish in the limit $N_{\text{rv}} \to \infty$.

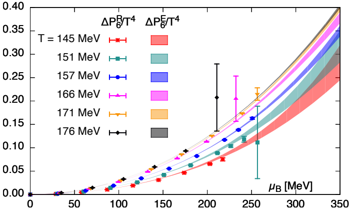
Resummed versus Taylor Expansion: Pressure and Baryon Density

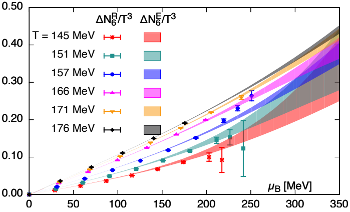




- Analysis of eighth-order Taylor coefficients ongoing. Hence we only show Taylor expansion results up to sixth order.
- 2^{nd} order resummed results agree with 4^{th} order Taylor expansion results for real μ_B (faster convergence).
- For $\hat{\mu}_B \gtrsim 1.5$, our calculation breaks down for real μ_B, while we see large deviations (compared to the Taylor series) for imaginary μ_B.

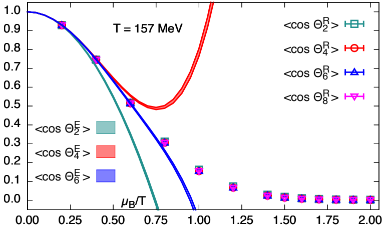
Breakdown of the Resummed Calculation for Real μ_B





- For real $\hat{\mu}_B \gtrsim 1$, the calculation can break down due to fluctuations. However, this can be overcome by increasing the statistics.
- For $\hat{\mu}_B \gtrsim 1.5$ however, breakdown persists even after statistics is increased 10-15 times.

Sign Problem and the Phase Factor



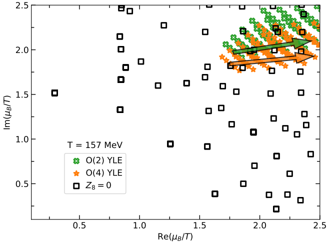
- Since even(odd) $\bar{D}_n$ are real(imaginary), we may write the determinant ratio as

$$\left\langle \exp\left(\sum_{n=1}^{N} \bar{D}_n \hat{\mu}_B^n\right) \right\rangle \equiv \left\langle R_N(\mu_B) e^{i\Theta_N(\mu_B)} \right\rangle,$$

where $e^{i\Theta_N(\mu_B)}$ is the complex phase factor and $R_N(\mu_B)$ is the phase-quenched determinant ratio.

- Expect calculation to break down in the limit $\langle e^{i\Theta_N} \rangle \to 0$. For $T \approx 157$ MeV, this happens around $\hat{\mu}_B \sim 1.5$, which is exactly where our calculation breaks down.

Roots of the Partition Function



- The breakdown of the calculation can be understood by looking at the distribution of the roots of the partition function in the complex μ_B plane [Mukherjee & Skokov (2021), Giordano & Pasztor (2019), Connelly *et al.* (2020). See also the talk by G. Basar at this conference]

- The closest (to the origin) roots of the partition function are at $|\hat{\mu}_B| \sim 1.5$. The branch cuts expected from $O(N)$ universality [Mukherjee & Skokov (2021)] lie further away.

Conclusions

- We have demonstrated a new way of resumming the QCD Taylor series in μ_B, starting from the derivatives $\bar{D}_n$ of the fermion determinant.
- The method resums the contribution of the first N derivatives $\bar{D}_1, \ldots, \bar{D}_N$ to all orders in μ_B.
- We presented our results for the pressure and the baryon number density, obtained at a single lattice spacing $N_\tau = 8$, and for temperatures in the range 145 MeV $\lesssim T \lesssim$ 176 MeV.
- We related the breakdown of our calculation, for $\hat{\mu}_B \gtrsim 1.5$, to the vanishing of the average phase factor $\langle \cos \Theta(\mu_B) \rangle$ and the zeros of the partition function in the complex μ_B plane.
- It remains to make these connections more precise and to explore their consequences for the QCD phase diagram.