Absence of inhomogeneous phases in the $2+1$-dim. Gross-Neveu model model with chiral imbalance

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CRC–TR 211
Strong-interaction matter under extreme conditions

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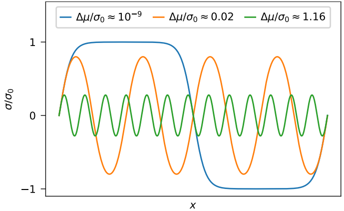
- CRC-TR 211: Phase diagram of QCD matter at finite μ and T
- Focus on chiral symmetry breaking, especially on inhomogeneous chiral order parameter

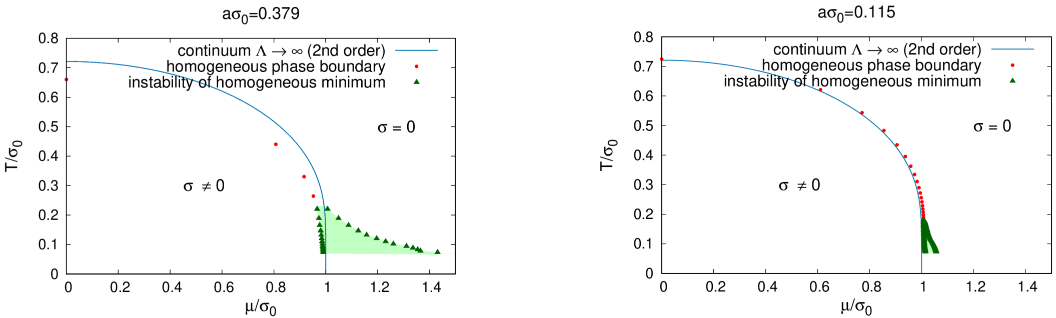
$$\langle \bar{\psi}\psi \rangle = f(\mathbf{x})$$

- Found in simple strong-interaction models at small T and large μ



- First found in 1+1-dim. GN model in mean-field approximation[1]
- Inhomogeneous order parameter also in beyond mean-field study[2]

[1] M. Thies, K. Urlichs, *Phys. Rev. D* **2003**, *67*, 125015.
[2] J. Lenz et al., *Phys. Rev. D* **2020**, *101*, 094512.



- Studied $2+1$-dim. GN model via lattice methods[3]
- Stability analysis & full lattice minimization[4]
- Inhomogeneous phase depends on regularization scheme and regulator, vanishes when removing the regulator

[3] R. Narayanan, *Phys. Rev. D* **2020**, *101*, 096001.
[4] M. Buballa, L. Kurth, M. Wagner, M. Winstel, *Phys. Rev. D* **2021**, *103*, 034503.

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- Action of the Gross-Neveu model in $2+1$ dimensions with chiral imbalance for N_f fermion flavors:

$$S[\bar{\psi}_f, \psi_f] = \int d^3x \left[\bar{\psi}_f \left(\gamma_\nu \partial_\nu + \gamma_0\,\mu + \gamma_0 \underbrace{\mathrm{i}\gamma_4\gamma_5}_{\gamma_{45}} \mu_{45}\right) \psi_f - \frac{\lambda}{2N_f} \left(\bar{\psi}_f\,\psi_f\right)^2 \right]$$

- Introduction of auxiliary bosonic field σ:

$$S[\bar{\psi}_f, \psi_f, \sigma] = \int d^3x \left[\frac{N_f}{2\lambda}\sigma^2 + \bar{\psi}_f \underbrace{\left(\gamma_\nu \partial_\nu + \gamma^0 \mu + \gamma_0\gamma_{45}\mu_{45} + \sigma\right)}_{\mathrm{Q}} \psi_f \right]$$

- $\langle \bar{\psi}(x)\psi(x)\rangle = -\frac{N_f}{\lambda}\langle\sigma(x)\rangle \Rightarrow$ refer to σ as chiral condensate.
- Mean-field approximation ($N_f \to \infty$): Neglect bosonic quantum fluctuations $\langle\sigma(x)\rangle = \sigma(x)$
- Only bosonic global minima contribute to partition function
- Restrict to spatially dependent condensates $\sigma(x) = \sigma(\mathbf{x})$

Symmetries and chiral imbalance

- Discrete chiral symmetries (4×4 Representation of Euclidean Dirac algebra)

$$\psi_f \to \gamma_4 \psi_f\,, \quad \bar{\psi}_f \to -\bar{\psi}_f \gamma_4\,, \quad \sigma \to -\sigma,$$
$$\psi_f \to \gamma_5 \psi_f\,, \quad \bar{\psi}_f \to -\bar{\psi}_f \gamma_5\,, \quad \sigma \to -\sigma$$

- $\gamma_{45} = \mathrm{i}\gamma_4\gamma_5$ generates continuous (chiral) symmetry
- Dirac-Operator is block-diagonal

$$Q[\mu, \mu_{45}, \sigma] = \begin{pmatrix} Q^{(2,+)}[\mu + \mu_{45}, \sigma] & 0 \\ 0 & Q^{(2,-)}[\mu - \mu_{45}, \sigma] \end{pmatrix}$$

- Dirac operators build from irreducible fermion representation

$$Q^{(2,\pm)}[\mu, \sigma] = \pm\tau_2(\partial_0 + \mu) \pm \tau_3\partial_1 \pm \tau_1\partial_2 + \sigma$$

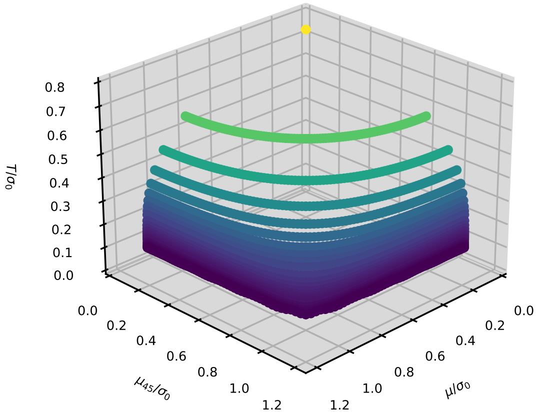
- $\mu \neq 0, 0 \leqslant \mu_{45} \leqslant \mu$ increases chiral imbalance, i.e. difference between $\mu_L = \mu + \mu_{45}$ for upper 2 comp. and $\mu_R = \mu - \mu_{45}$ for lower 2 comp.
- What are the effects on the respective (in-)homogeneous phases?

$\Rightarrow$ Study with two different lattice regularizations using naive fermions and different coupling to σ

Homogeneous phase diagram, $\sigma(\mathbf{x}) = \text{const.}$

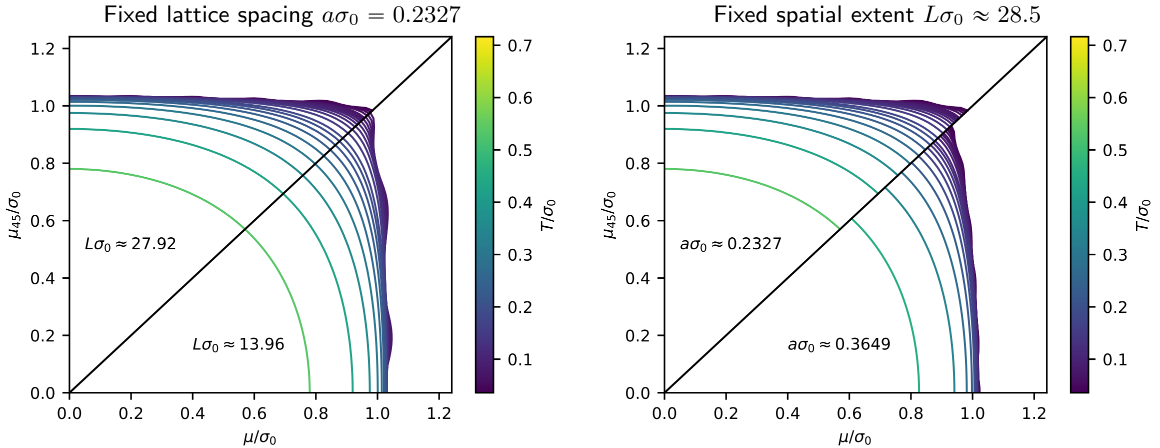
- $\sigma(\mathbf{x}) = \bar{\sigma} = \text{const.}$, Minimization of lattice action, identical for both discretization
- Theoretically observed symmetry $\mu_{45} \leftrightarrow \mu$ & $\mu \rightarrow -\mu$ & $\mu_{45} \rightarrow -\mu_{45}$

$$a\sigma_0 = 0.2327, L\sigma_0 = 27.92$$



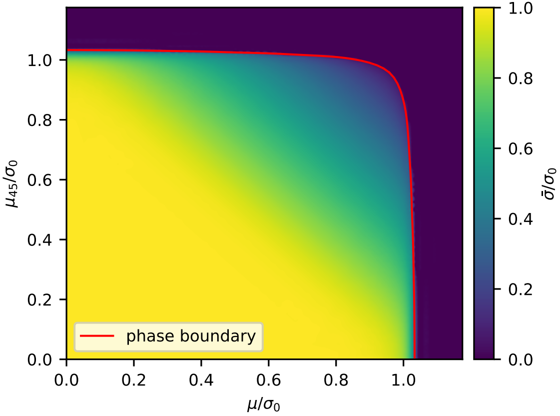
Homogeneous phase diagram, $\sigma(\mathbf{x}) = \text{const.}$

- Results for $\mu_{45} = 0$ are already quite close to continuum results[5]



[5] K. Klimenko, *Z. Phys. C* **1988**, *37*, 457.

The homogeneous order parameter $\bar{\sigma}$

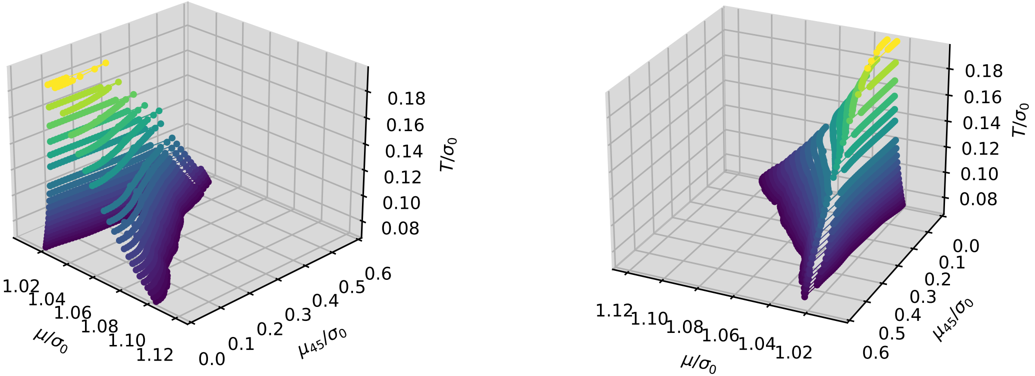


- Order parameter at $T/\sigma_0 = 0.0716$ with $a\sigma = 0.2327, L\sigma_0 = 27.92$
- Plateau for $\mu_L/\sigma_0 = \mu/\sigma_0 + \mu_{45}/\sigma_0 \leqslant 1.0$, where $\bar{\sigma} \approx \sigma_0$, then continuous decrease of $\bar{\sigma}$
- Competition of $|\mu_L/\sigma_0| > 1.0$ and $|\mu_R/\sigma| < 1.0$ leads to continuous decrease

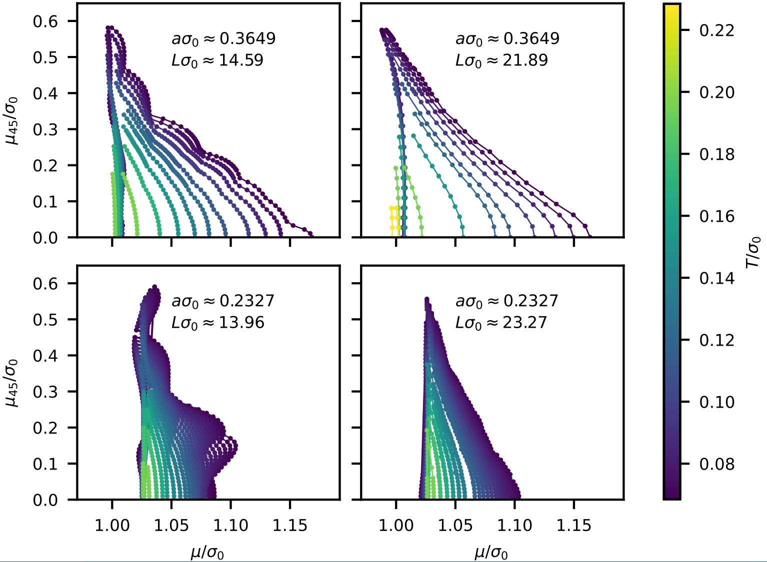
Stability analysis of $\sigma = \bar{\sigma}$

- Stability only depends on spatial momentum of inhom. perturbation
- Obtain instability region of $\sigma = \bar{\sigma}$ for one of two discretizations around $\mu/\sigma_0 \gtrsim 1.0$ & $\mu_{45}/\sigma_0 \lesssim 0.56$

$$a\sigma_0 = 0.2327, \; L\sigma_0 = 23.27$$



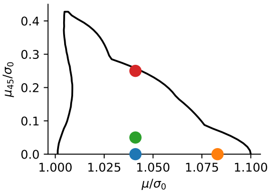
Stability analysis on multiple lattices

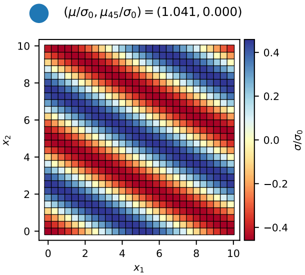


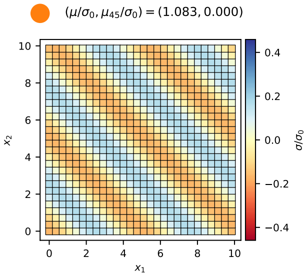
Minimization with respect to $\sigma = \sigma(\mathbf{x})$

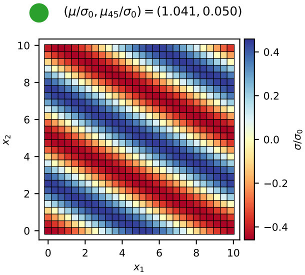
Within instability region

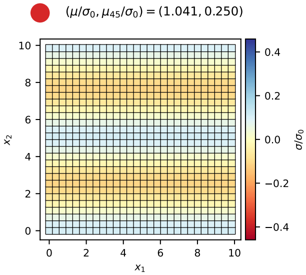
$$a\sigma_0 \approx 0.3649,$$
$$L\sigma_0 = 10.22,$$
$$T/\sigma_0 = 0.114$$











Conclusion

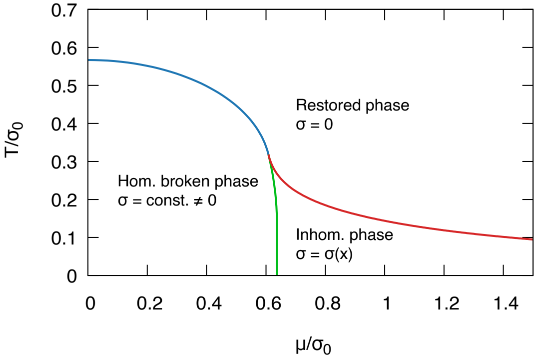
- No inhomogeneous phase in the renormalized $2+1$-dim. GN for all T, μ, μ_{45}
- Chiral imbalance disfavors inhomogeneous condensates in this model
- Strong regulator and regularization scheme dependence of inhomogeneous order parameter
    - Explore regulator dependence e.g. in $3+1$-dim. Four-fermi theories (non-renormalizable)

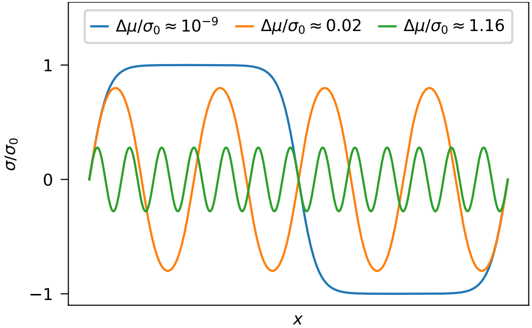
Outlook

- Yukawa and FF models in $1+1$, $2+1$, $3+1$d via mean-field (lattice) approach
- Study bosonic fluctuation effects based on these results

Appendix

Phase diagram of $1+1$-dim. GN in the large-N_f limit





Chiral condensate with $\mu = \mu_c + \Delta\mu$ at $T = 0$

Chirality operator

- Definition of 2+1-dimensional sGN with irreducible representation of fermions
  Dirac matrices as Pauli matrices

$$\gamma^0 = \sigma_1,\ \gamma^1 = \sigma_2,\ \gamma^2 = \sigma_3$$

$$\gamma^0 = -\sigma_1\ \gamma^1 = -\sigma_2,\ \gamma^2 = -\sigma_3$$

$\Rightarrow$ Non-trivial γ_5 not available

- Which symmetry is spontaneously broken by the condensate ?

Parity in 2+1 dimensions

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- Parity as inversion of all spatial coordinates equivalent to rotation

$$
\begin{aligned}
(x_0, x_1, x_2)^T &\xrightarrow{P} (x_0, x_1, -x_2)^T \\
\psi &\xrightarrow{P} -\mathrm{i}\gamma_2\psi \\
\bar{\psi} &\xrightarrow{P} -\mathrm{i}\bar{\psi}\gamma_2
\end{aligned}
$$

- Obtain $\sigma \xrightarrow{P} -\sigma$
- Non-vanishing σ indicates spontaneous breaking of parity

Chiral symmetry transformation

- Use four component spinors via combination of two inequivalent irreducible spinors ($\tau_i \equiv$ Pauli matrices in isospin space)

$$\gamma_\nu = \tau_3 \otimes \sigma_{\nu+1}, \quad \gamma_4 = \tau_1 \otimes \mathbb{1},$$
$$\gamma_5 = -\tau_2 \otimes \mathbb{1}, \quad \gamma_{45} = i\gamma_4\gamma_5 = \text{diag}(\mathbb{1}, -\mathbb{1})$$

- Parity to be defined via tensor product with τ_1
- Mass term $\propto \bar{\psi}\psi$ now invariant under parity

Chiral transformations

- Symmetries of free massless fermions in 2+1 dimensions $(U(2N_f))$

$$\psi_f \to e^{i\theta\Gamma}\psi_f \qquad \Gamma \in \{\mathbb{1}, \gamma_{45}, \gamma_4, \gamma_5\}$$

- For the Gross-Neveu model only a subgroup is realized

$$\psi_f \to \gamma_5\psi_f,\ \bar{\psi}_f \to -\bar{\psi}_f\gamma_5 \tag{3}$$

$$\psi_f \to \gamma_4\psi_f,\ \bar{\psi} \to -\bar{\psi}_f\gamma_4 \tag{4}$$

- Together with this discrete transformation we have continuous symmetries

$$\psi_f \to e^{i\phi\gamma_{45}}\psi_f,\ \bar{\psi}_f \to \bar{\psi}_f e^{-i\phi\gamma_{45}} \tag{5}$$

$$\psi_f \to e^{i\alpha}\psi_f,\ \bar{\psi}_f \to \bar{\psi}_f e^{-i\alpha} \tag{6}$$

- Combination of (5) with (3) reproduces (4)

Discretization of the effective action

$$S_{\text{eff}}[\sigma] = N_f \left[\frac{1}{2\lambda} \int \mathrm{d}^3 x \sigma^2 - \ln \det \underbrace{\left(\gamma_\nu \partial_\nu + \gamma^0 \mu + \gamma_0 \gamma_{45} \mu_{45} + \sigma \right)}_{\mathrm{Q}} \right]$$

- Discretization via naive fermions in momentum space

$$Q(p,q) = \mathrm{i}\delta_{p,q} \sum_{\nu=0}^{2} \gamma_\nu \sin\left(p_\nu - \delta_{\nu,0}\, \mathrm{i}(\mu + \gamma_{45}\mu_{45}) \right) + \delta_{p_0,q_0} \tilde{W}_2(\mathbf{p}-\mathbf{q})\, \tilde{\sigma}(\mathbf{p}-\mathbf{q}),$$

$$\frac{S_{\text{eff}}[\sigma]}{N_{\text{f}}} = \frac{\beta}{2\lambda} \sum_{\mathbf{p}} \sigma^2(\mathbf{p}) - \frac{1}{8} \sum_{n_0=-N_t+1}^{N_t} \ln\left(\det\left(\mathrm{Q}(n_0, \mathbf{p}; n_0, \mathbf{q}) \right) \right)$$

- Why is $W_2(\mathbf{p}-\mathbf{q}) \neq 1$? Factor of $1/8$?

Naive fermions in the GN model

- Naive discretization of fermion field ψ
  - Introduces 2^d fermion doubler in d dimensions (not a problem as $N_\mathrm{f} \to \infty$)
  - $\Rightarrow$ Adapt fermion content of the theory by $1/8$
  - Wrong momentum relation in kinetic term (can be corrected by a transformation in subflavor space to new field coordinates χ for free fermions)
- $\tilde{W}_2(\mathbf{p}-\mathbf{q}) = 1$: Naively discretized interaction term $\sum_\mathbf{x} \bar{\psi}(\mathbf{x})\sigma(\mathbf{x})\psi(\mathbf{x})$ in terms of χ field coordinates
  - Mixes couplings between different flavors, introduces additional interactions, e.g. $\chi_1\gamma_1\sigma\chi_2$ etc. $\rightarrow$ represents different continuum interactions[6]
- Cutout wrong interaction terms

$$\sum_{n_0=-N_t+1}^{N_t} \sum_{\mathbf{p},\mathbf{q}} \bar{\chi}_f(n_0,\mathbf{p}) W_2(\mathbf{p}-\mathbf{q})\sigma(\mathbf{p}-\mathbf{q})\chi_f(n_0,\mathbf{q})$$

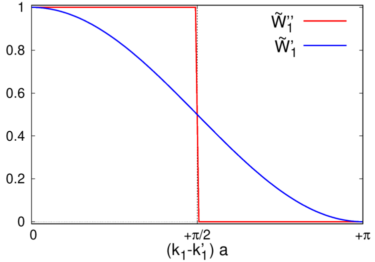
- Weight function fulfills
  - $\tilde{W}_2(\mathbf{k}) \rightarrow 1$ for $(|k_1|,|k_2|) \approx (0,0)$
  - $\tilde{W}_2(\mathbf{k}) \rightarrow 0$ for $(|k_1|,|k_2|) \approx (\pi,0), (|k_1|,|k_2|) \approx (0,\pi), (|k_1|,|k_2|) \approx (\pi,\pi)$

[6] Y. Cohen, S. Elitzur, E. Rabinovici, *Nucl. Phys. B* **1983**, *220*, 102–118.

Two choices for W_2

- Two choices, which represent the correct theory
- Cosine function in momentum space "soft cutout":

$$W_2'(\mathbf{x}-\mathbf{y}) = \prod_{\nu=1,2} \underbrace{\left(\frac{1}{4}\delta_{x_\nu,y_\nu-1} + \frac{1}{2}\delta_{x_\nu,y_\nu} + \frac{1}{4}\delta_{x_\nu,y_\nu+1}\right)}_{W_1'(x_\nu-y_\nu)}$$



- Step function in momentum space "hard cutout":

$$W_2''(\mathbf{x}-\mathbf{y}) = \prod_{\nu=1,2} \underbrace{\frac{1}{N_s}\left(1 + \sum_{n=1}^{N_s/4-1} 2\cos\left(\frac{2\pi n(x_\nu-y_\nu)}{L}\right) + \cos\left(\frac{\pi(x_\nu-y_\nu)}{2}\right)\right)}_{W_1''(x_\nu-y_\nu)}$$