Perturbative predictions for color superconductivity on the lattice

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In collaboration with

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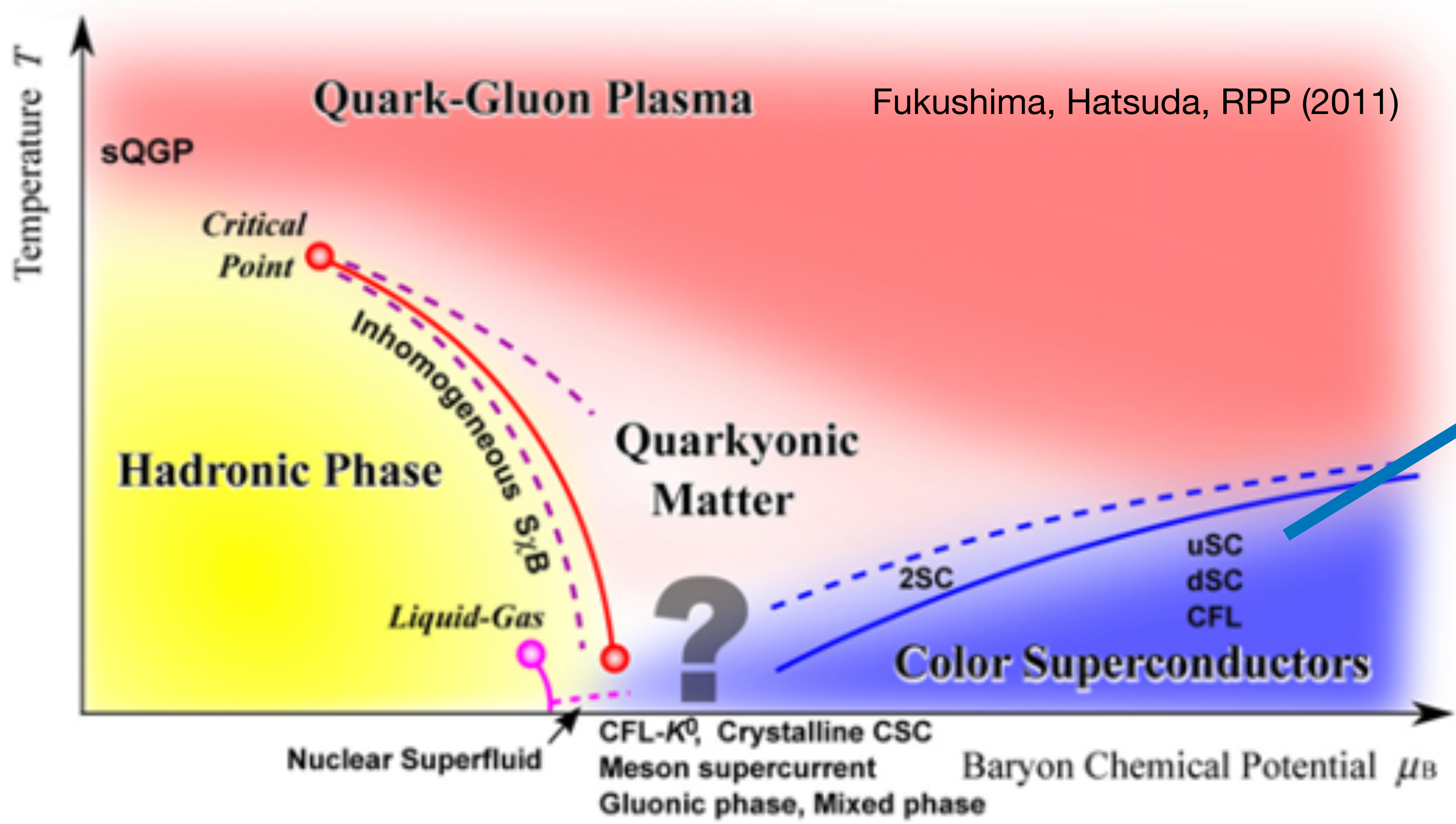
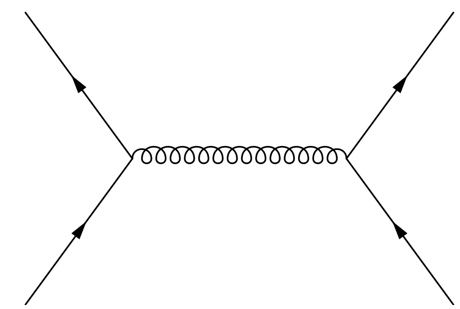
Color superconductivity (CSC)

Barrois, NPB (1977); PhD thesis (1979), Frautschi (1978)
Bailin, Love, PR (1984), Alford, Rajagopal, Wilczek, PLB (1998)



Fukushima, Hatsuda, RPP (2011)



Cooper instability in color-antitriplet channel

(attractive channel)

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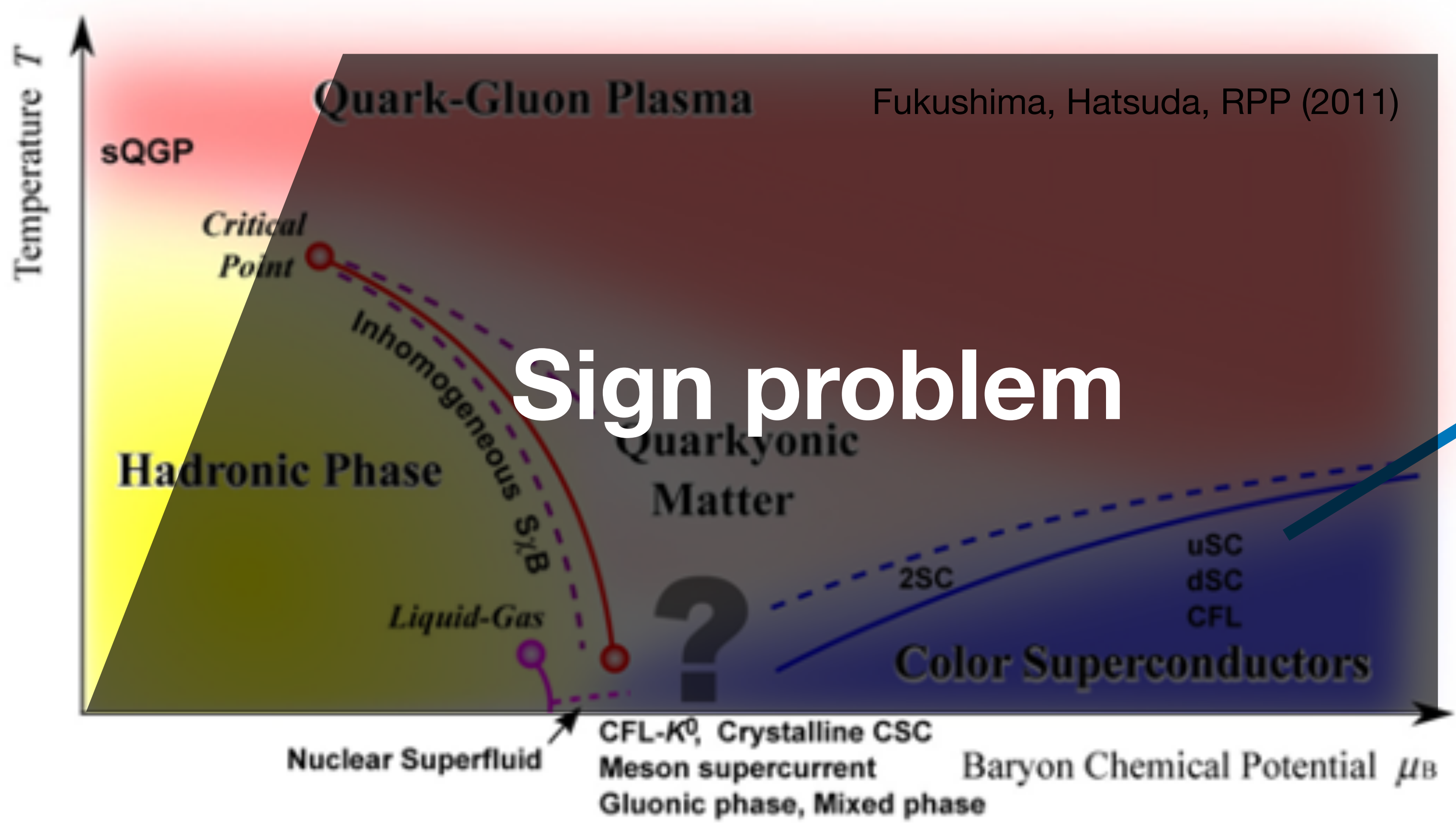

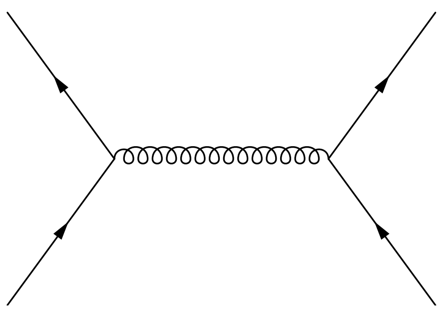

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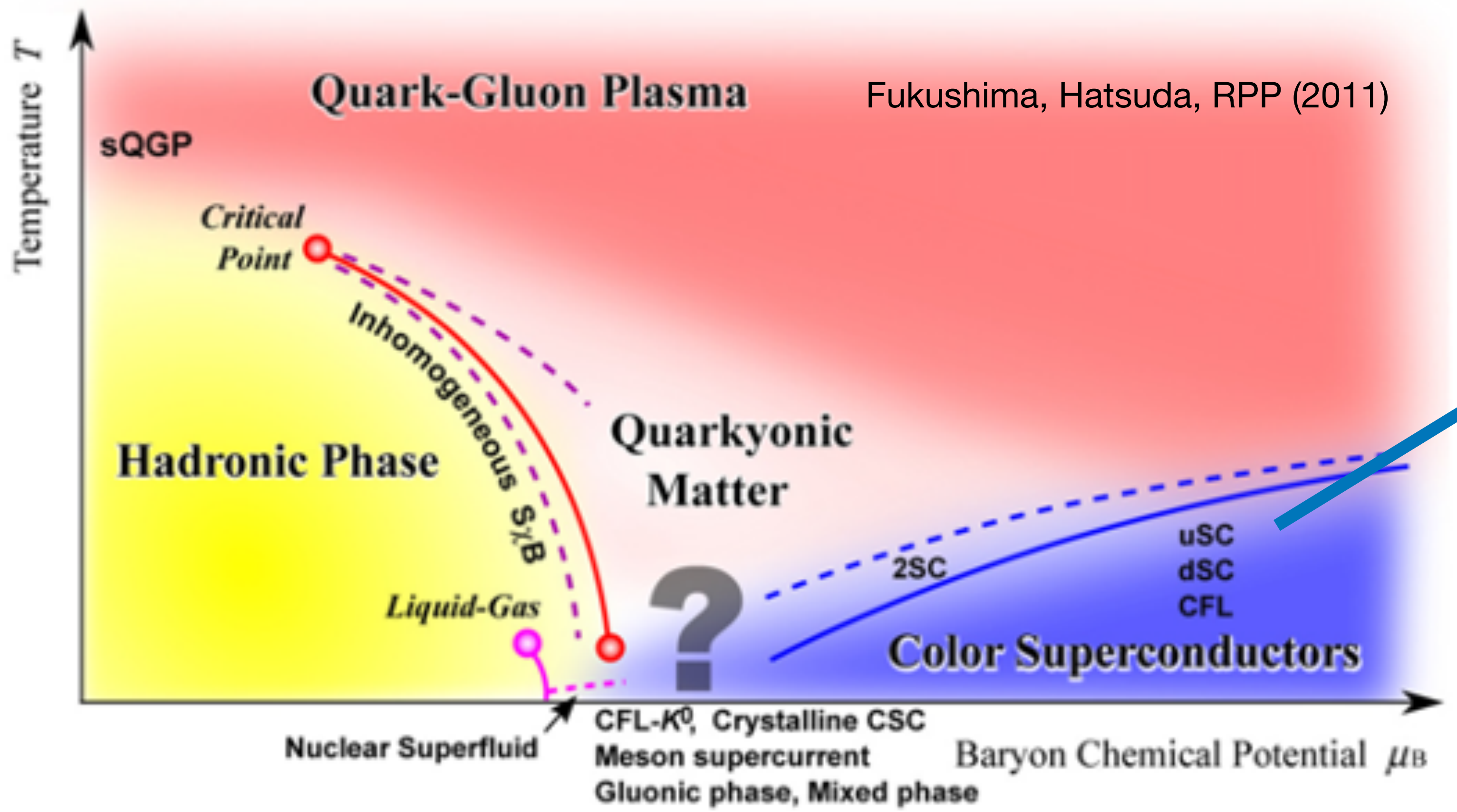

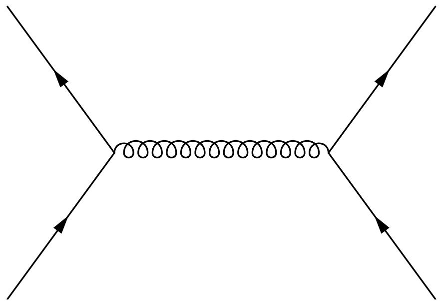

Cooper instability in color-antitriplet channel
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Exploration of CSC by first-principles lattice-QCD methods?

Candidates of methods to overcome sign problem:

- **Complex Langevin method** → **Tsutsui's talk** (application to CSC)
- Lefschetz thimble
- Tensor renormalization group
- Path optimization

…

Perturbative study of CSC on the lattice

Perturbative study

- **Valid for small coupling g (high density, small volume, …)**
- Useful for **cross check** of results by first-principles lattice-QCD methods

Study in infinite continuum case based on perturbative QCD

(Review) Alford, Schmitt, Rajagopal, Schäefer RMP (2008)

Gap eq.

$$\Delta_{k,r} = \frac{g^2}{4}\int \frac{d^3q}{(2\pi)^3} \sum_s Z(\epsilon_{q,s}) \frac{\Delta_{q,s}}{\epsilon_{q,s}} \tanh\left(\frac{\epsilon_{q,s}}{2T}\right) \times \left[D_\ell(p)\, \mathcal{T}^s_{00}(\mathbf{k},\mathbf{q}) + D_t(p,\epsilon_{q,s},\epsilon_{k,r})\, \mathcal{T}^s_t(\mathbf{k},\mathbf{q}) \right]$$

- Functional equation for gap $\Delta_{q,s}$

  Some ansatz on $\Delta_{q,s}$ is necessary.
- Gauge-symmetry-breaking cutoff

Analysis on the finite lattice

- **Gap eq. is reduced to coupled equations for finite number of elements**
  - Suitable for numerical study
- **Lattice regularization does not break gauge symmetry**

Our main results

A new method to analyze CSC quantitatively based on the lattice QCD perturbation

- Application of Nambu formalism to the lattice system
- Analysis around critical point
    - **First application of (generalized) Thouless criterion to QCD**
    - **Numerical method without any ansatz on the form of Cooper pairs**

Quantitative prediction of parameter region with which CSC occurs

- $N_{\mathrm{f}} = 4$ staggered fermions (with corrections of results reported in APLAT 2020)
- Wilson fermions ($N_{\mathrm{f}} \geq 2$)
- Critical coupling constant

Prediction for the structure of Cooper pairs

- Symmetry in flavor space
- Momentum modes composing Cooper pairs

Method

Dyson eq.
in Nambu basis

$$\mathbf{\Sigma} = \mathbf{S}^{-1} - \mathbf{S}_{\mathrm{free}}^{-1}$$

Emerged by $\Sigma_{12(21)}$

$$\mathbf{S} = \begin{pmatrix} \langle \psi \bar{\psi} \rangle & \langle \psi \psi \rangle \\ \langle \bar{\psi} \bar{\psi} \rangle & \langle \bar{\psi} \psi \rangle \end{pmatrix}$$

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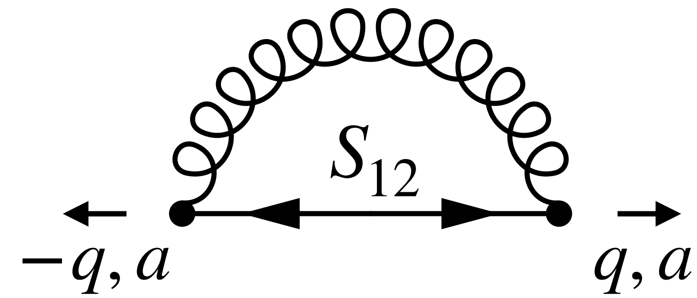
Gap eq.
in lowest order

$$\Sigma_{12(qaa')} =$$



q: momentum
a, a': internal d.o.f.

- Complicated non-liner eq. for Σ_{12}...

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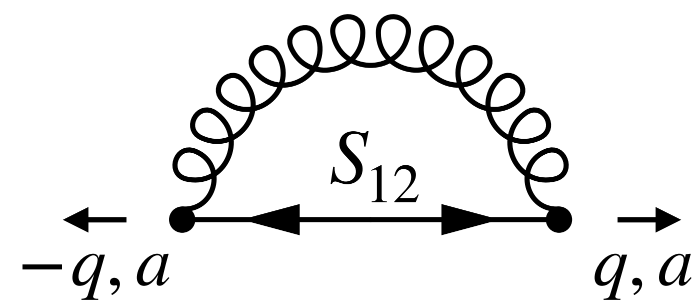

q: momentum
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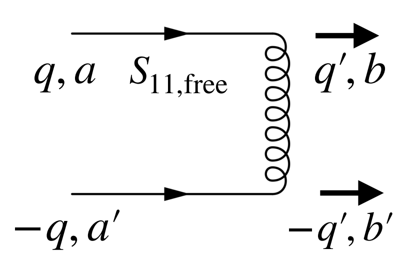
- Complicated non-liner eq. for Σ_{12}…
- **However, it is reduced to linear eq. near critical point, where Σ_{12} becomes small but nonzero value.**

Critical point
(Thouless criterion)

$$\frac{1}{V} \sum_{q'bb'} M_{(qaa')(q'bb')} \Sigma_{12(q'bb')} = \beta \Sigma_{12(qaa')}$$

$\beta = 2N_c / g^2$

$$\frac{1}{\beta} M_{(qaa')(q'bb')} =$$



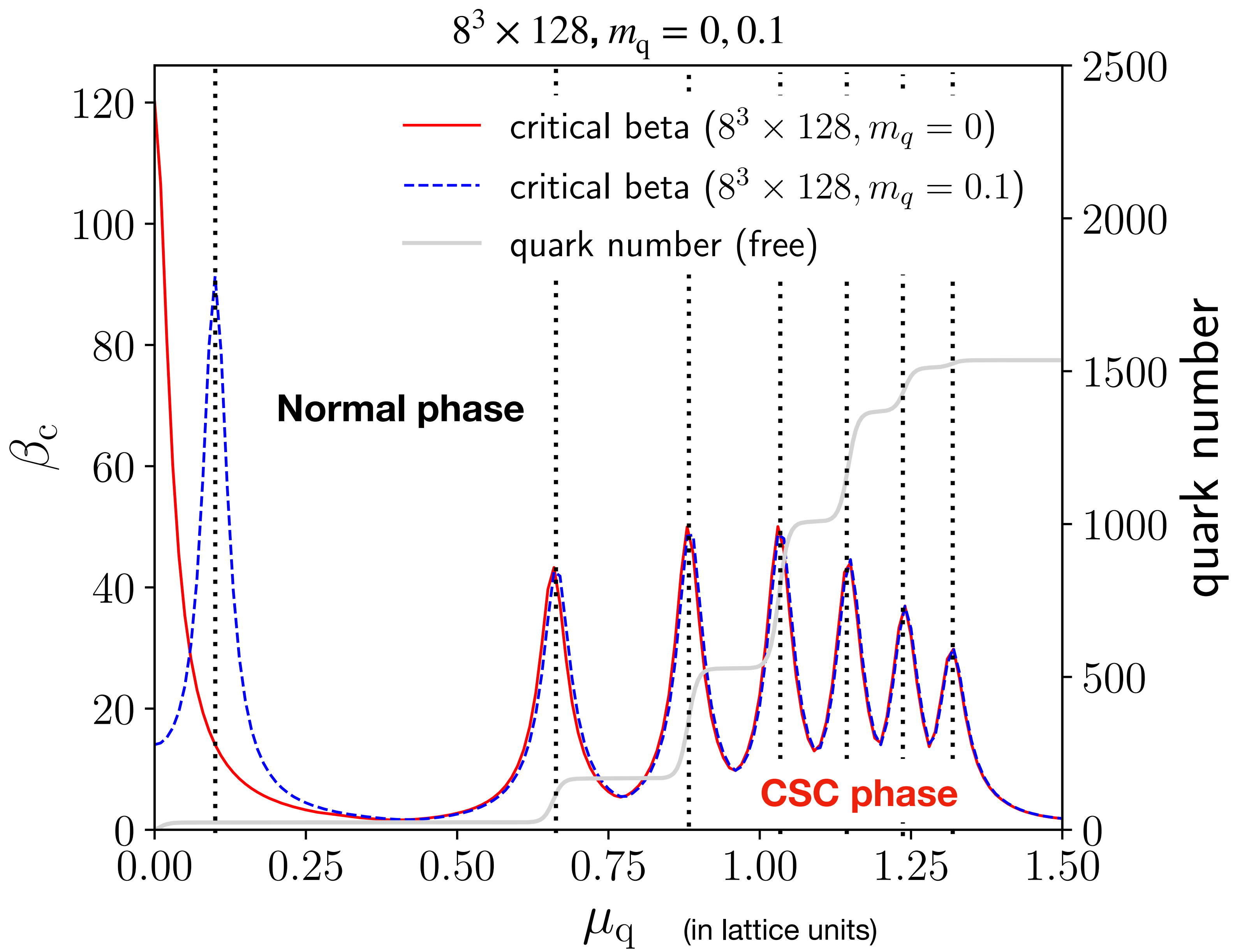
Max. eigenvalue of M = Critical β for CSC

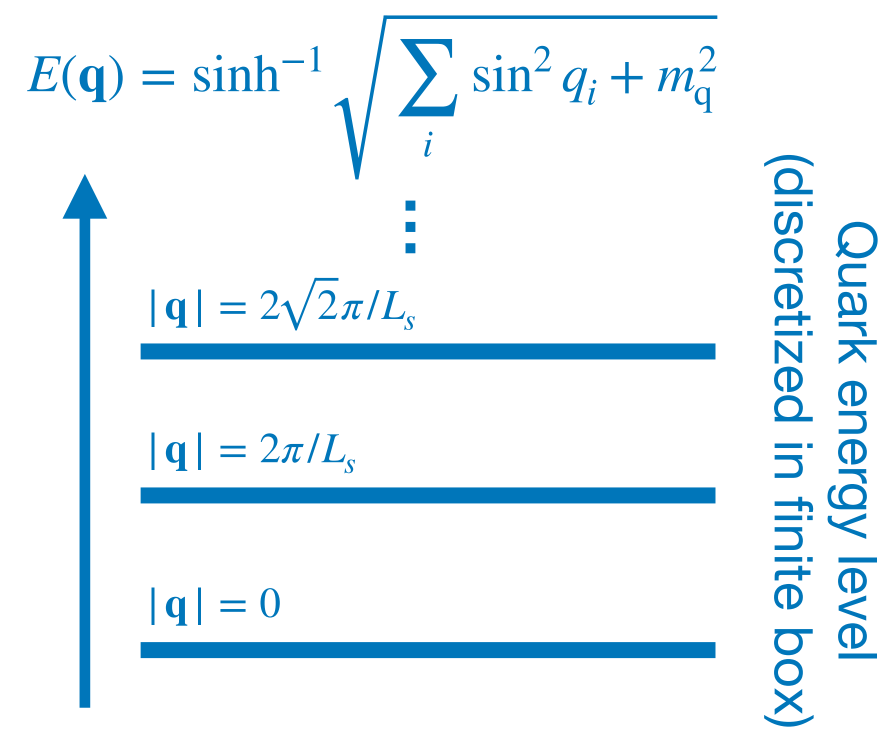
Eigenvector = Shape of gap Σ_{12}

Analysis near critical point without any ansatz!

Numerical method: power iteration

Critical β with $N_\mathrm{f} = 4$ staggered fermion





- **CSC phase extends to larger β when $\mu_q = E(\mathbf{q})$.**

  Cooper pairs are easy to form when modes exist around μ_q
- **Finite mass shifts the energy levels and thus the peak positions.**
- Peak near origin may be the finite-size effect.

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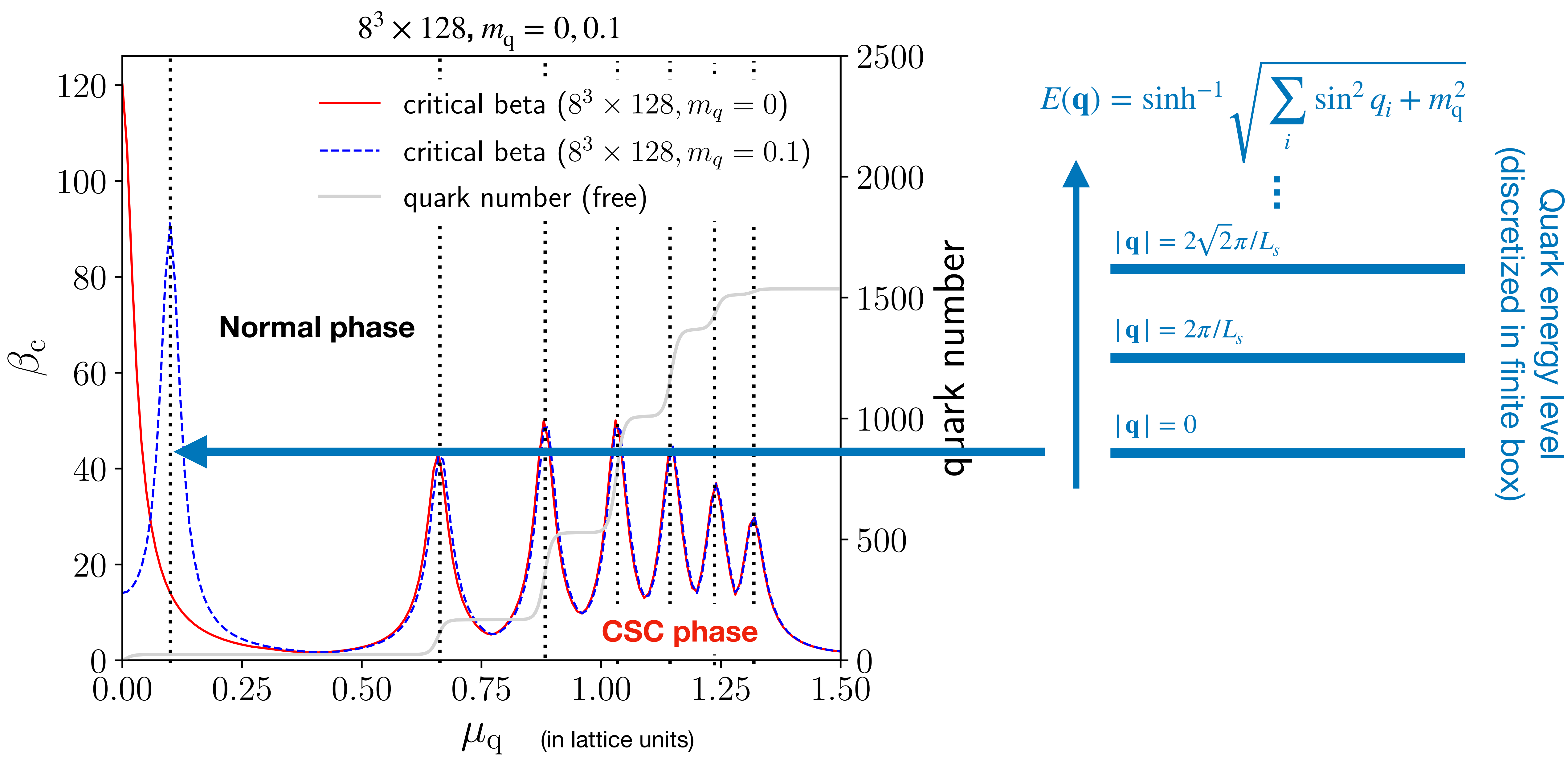

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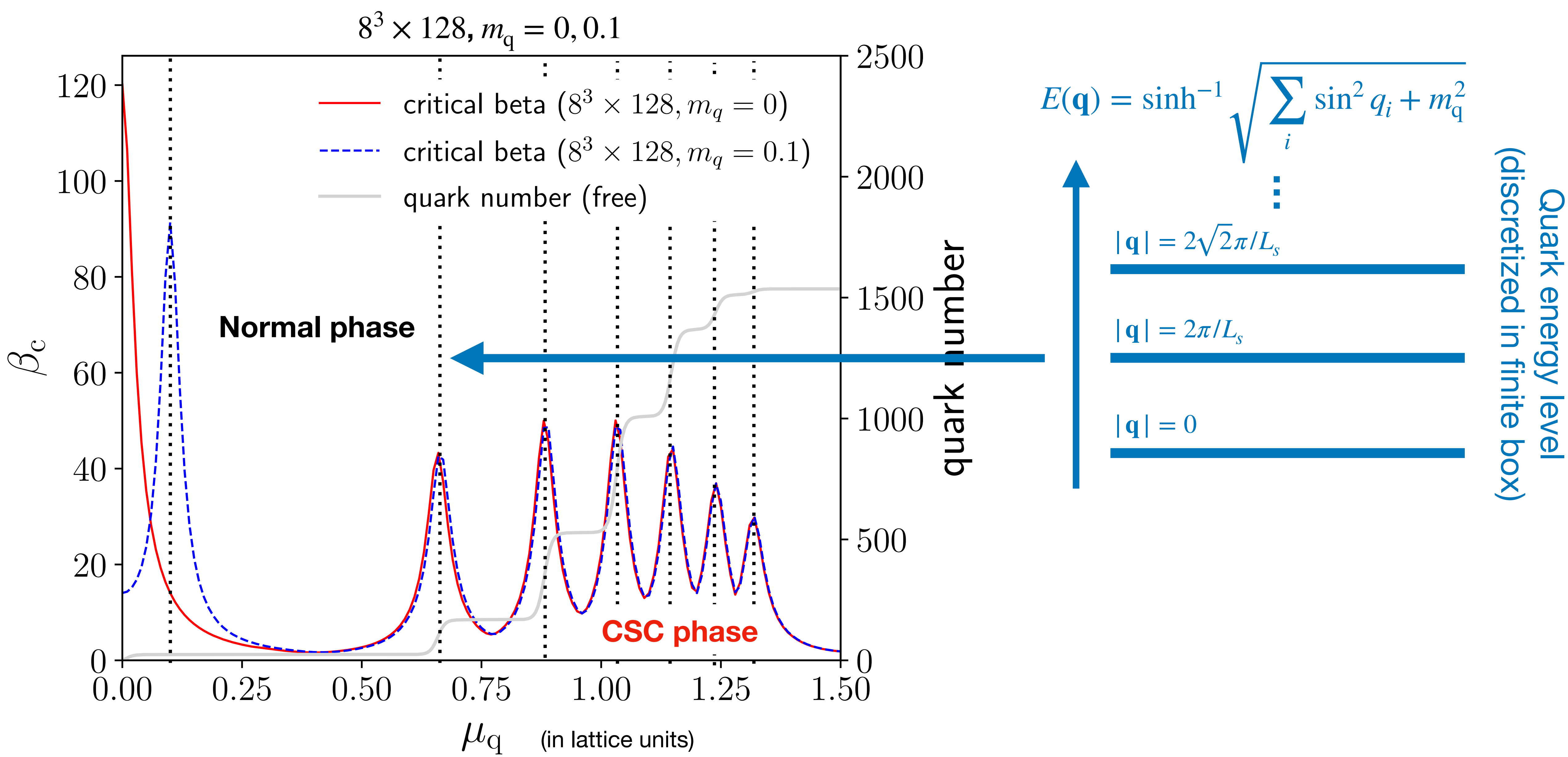

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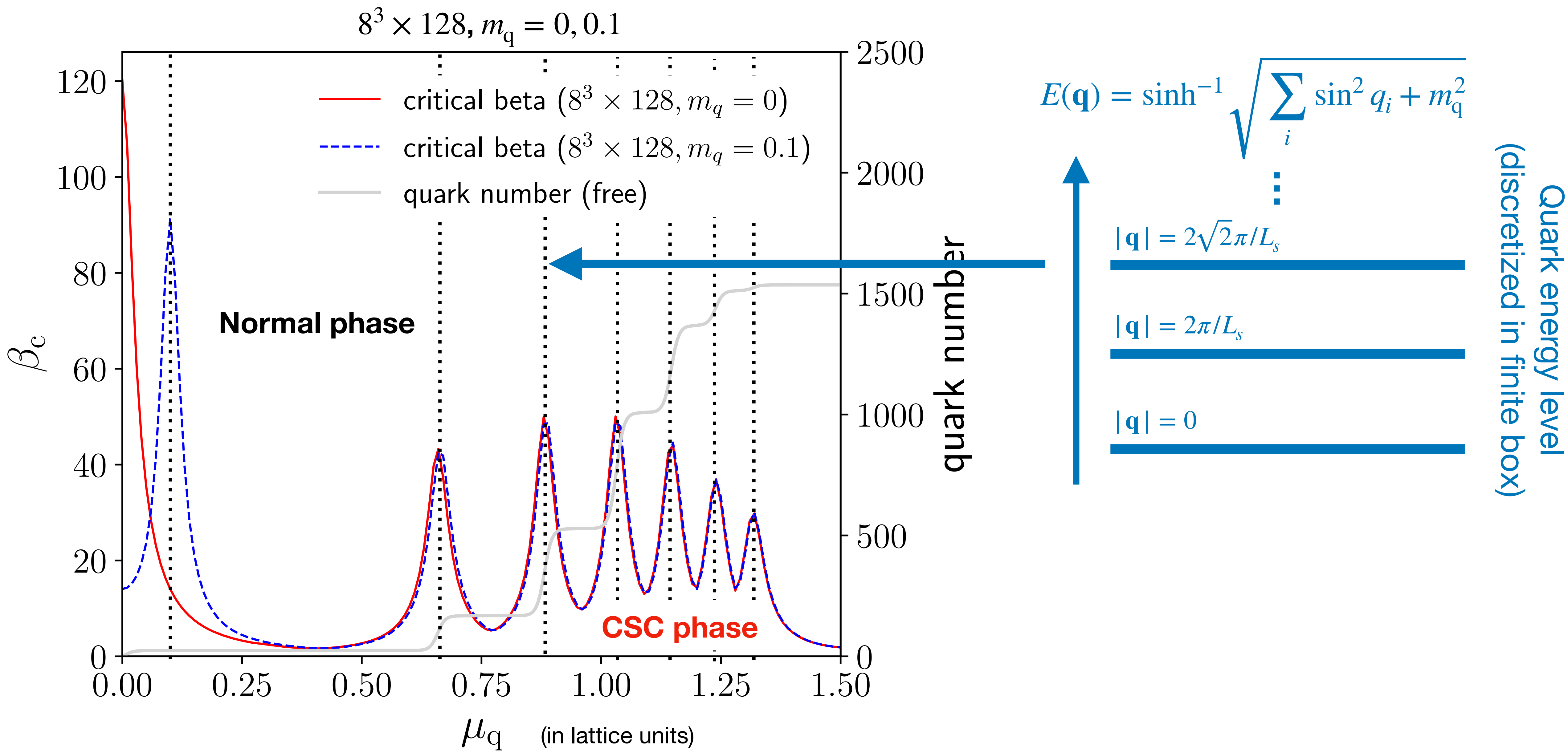

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Flavor structure of Cooper pairs

Color-antisymmetric Lorentz-scalar condensate

obtained from Σ_{12} (up to overall factor)

charge-conjugation operator

$$S^{(-)}_{12,a_1a_2}(N) = \epsilon_{\alpha\beta 3} \left\langle \psi^{\alpha \mathrm{t}}_{a_1}(N) C \psi^{\beta}_{a_2}(0) \right\rangle$$

color indices

flavor indices

Flavor structure of Cooper pairs

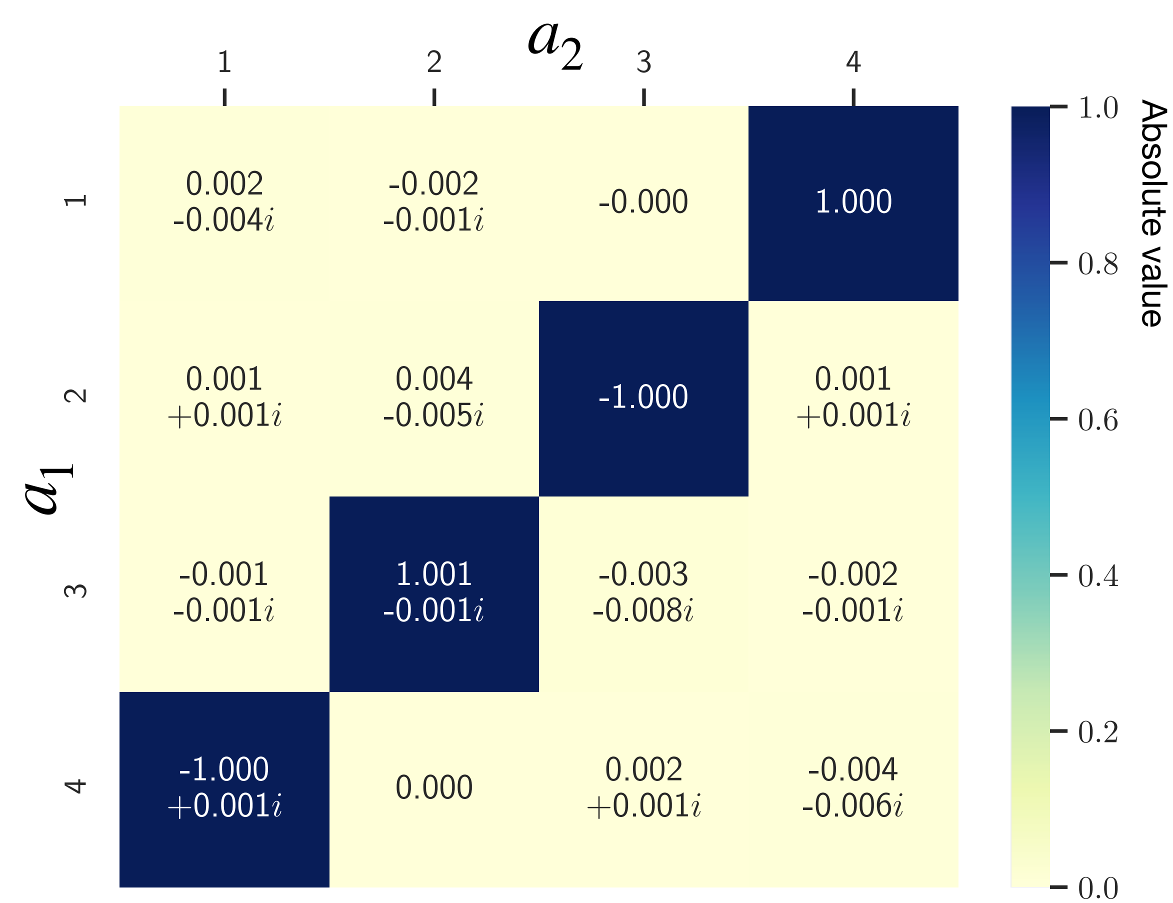
Color-antisymmetric Lorentz-scalar condensate

obtained from Σ_{12} (up to overall factor)

$$S^{(-)}_{12,a_1a_2}(N) = \epsilon_{\alpha\beta 3}\left\langle \psi^{\alpha\mathrm{t}}_{a_1}(N)\, C\, \psi^{\beta}_{a_2}(0)\right\rangle$$

charge-conjugation operator: C; color indices: $\alpha\beta3$; flavor indices: a_1, a_2

Flavor dependence of $S^{(-)}_{12,a_1a_2}(0)$ (normalized by $S^{(-)}_{12,14}(0)$)



$8^3 \times 128$, $m_\mathrm{q} = 0$, $\mu_\mathrm{q} = 0.66$, at the critical β

Standard representation for Euclidian γ matrices is used:

$$\gamma_i = \begin{pmatrix} 0 & -i\sigma^i \\ i\sigma^i & 0 \end{pmatrix} \quad \gamma_4 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$$

Flavor structure of the condensate:

$$S^{(-)}_{12,a_1a_2}(N) = \kappa(N)\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}_{a_1a_2} = \kappa(N)(t_2t_4)_{a_1a_2}$$

($t_\mu = \gamma^{\mathrm{t}}_\mu$ acts on the flavor space)

Consistent with the following symmetries:

- Antisymmetry for flavor indices
- $U(1)$ symmetry of the staggered fermions:

$$\psi(N) \to e^{i\theta\gamma_5\otimes t_5}\psi(N)$$

- Result does not change with different initial cond. for power iteration
  → No degeneracy.
- Small but nonzero values of other components may be caused by the numerical error of the Fourier transform.

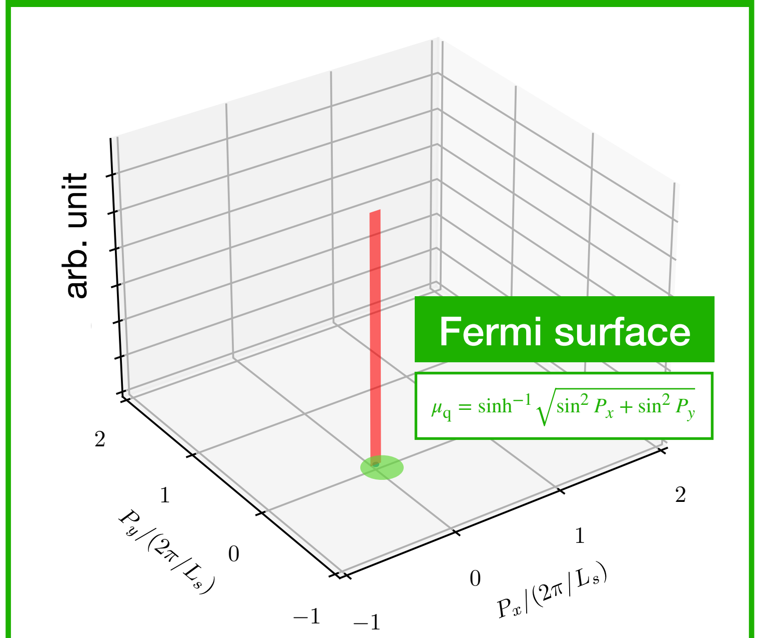
Cooper pairs in momentum space (Brillouin zone)

P_x and P_y-dependence of $\tilde{\kappa}(P_x, P_y, 0, \pi/L_t)$

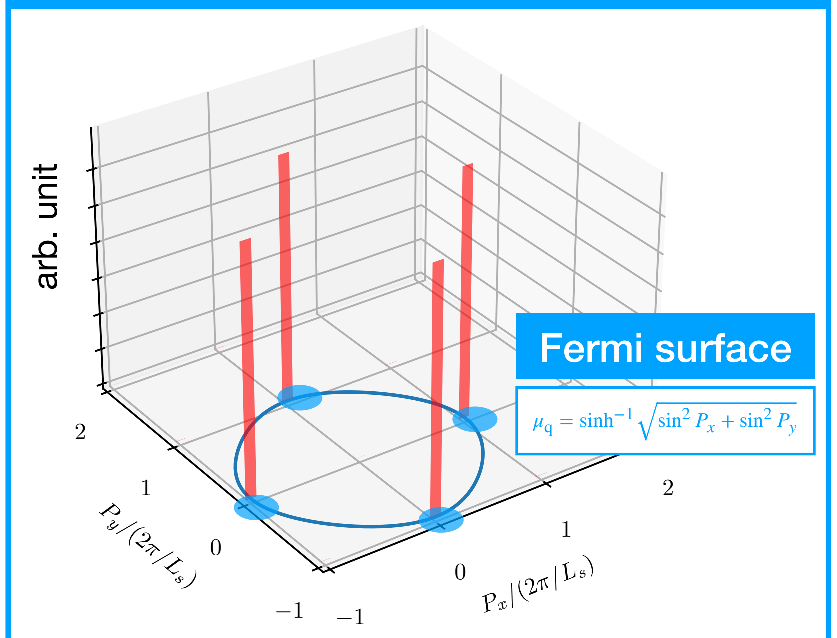
$$\tilde{S}^{(-)}_{12,a_1a_2}(P) = \tilde{\kappa}(P)(t_2t_4)_{a_1a_2}$$

$$8^3 \times 128\,(=: L_s^3 \times L_t),\ m_q = 0$$

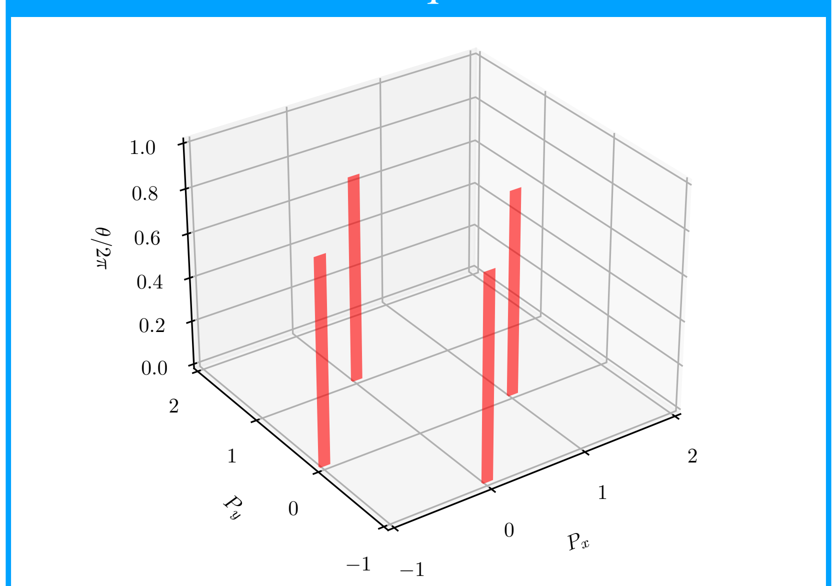
Absolute value ($\mu_q = 0.01$)

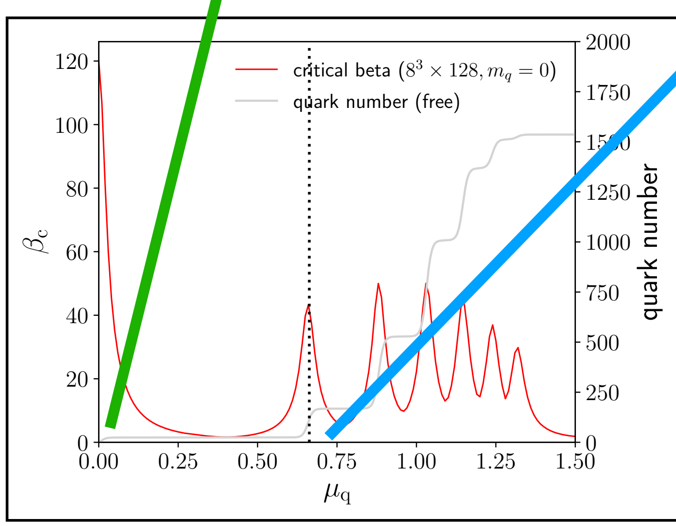


Absolute value ($\mu_q = 0.66$)



Phase ($\mu_q = 0.66$)

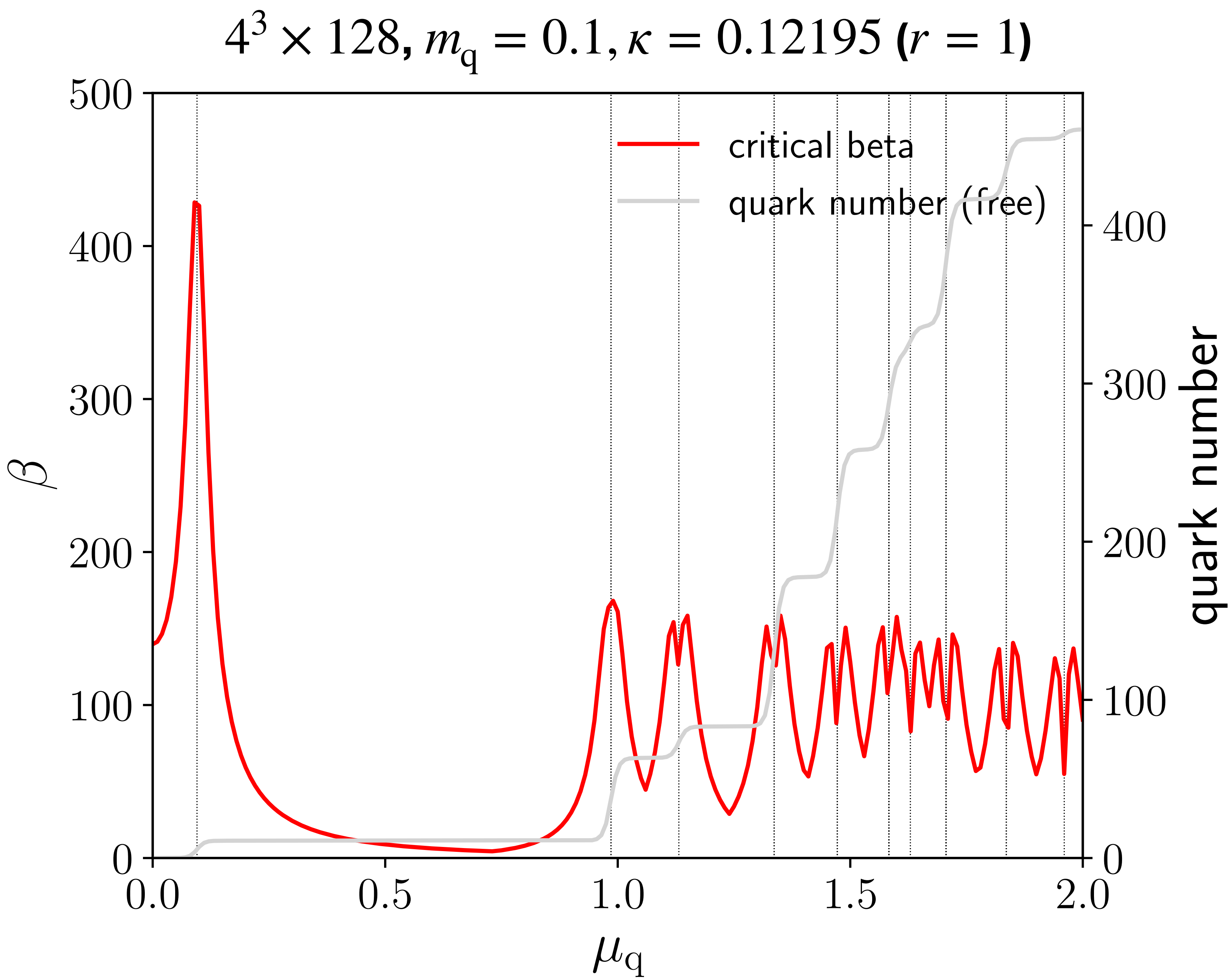




- **Cooper pairs are actually formed by the quarks on the Fermi surface.**
- **All modes have the same phase.**

These are obtained from calc. without any ansatz.

Critical β with Wilson fermion ($N_\mathrm{f} \geq 2$)



Vertical dotted lines: $\mu_\mathrm{q} = E(\mathbf{q})$

Energy levels of free Wilson fermions

$$E(\mathbf{q}) = 2\sinh^{-1}\sqrt{\frac{\sum_i \sin^2 q_i + \left(m_\mathrm{q} + 2\sum_i \sin^2 \frac{q_i}{2}\right)^2}{4\left(1 + m_\mathrm{q} + 2\sum_i \sin^2 \frac{q_i}{2}\right)}}$$

- **Peaks corresponding to $E(\mathbf{q})$ appear as in the case of staggered fermions.**
- **Splitting of peaks is observed at high μ_q.**

  Related to structure of Cooper pairs? (under investigation)

Conclusion

Study of color superconductivity (CSC) based on lattice QCD perturbation theory

A new method to analyze CSC quantitatively

- First application of (generalized) Thouless criterion to QCD
- Numerical method without any ansatz on the form of Cooper pairs

Quantitative prediction for critical lattice β

- $N_\mathrm{f} = 4$ staggered and Wilson fermions ($N_\mathrm{f} \geq 2$)
- Peak structure in (μ_q, β) plane (effect of discretized energy levels of quarks)

Prediction of the structure of Cooper pairs

- Symmetry in flavor space
- Momentum modes composing Cooper pairs

Outlook

- Analysis of Cooper pair with Wilson fermions
- Continuum limit, scaling
- Complex Langevin analysis of CSC in the parameter region given by our perturbative analysis (talk by Tsutsui)