The sign problem, PT symmetry, and exotic phases

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Three key interrelated questions

What is the physical reason finite density QCD is hard?

There is a fundamental change in the system where the transfer matrix no longer has real eigenvalues and orthogonal eigenvectors. This behavior is associated with a generalized PT symmetry, and non-Hermitian behavior. The sign problem is a manifestation of this change, but need not be present.

What new phenomena occur at finite density?

Spectral positivity is lost, and patterned phases may emerge. Computational complexity, e.g. NP-hardness, is associated with the complex structure of equilibrium states.

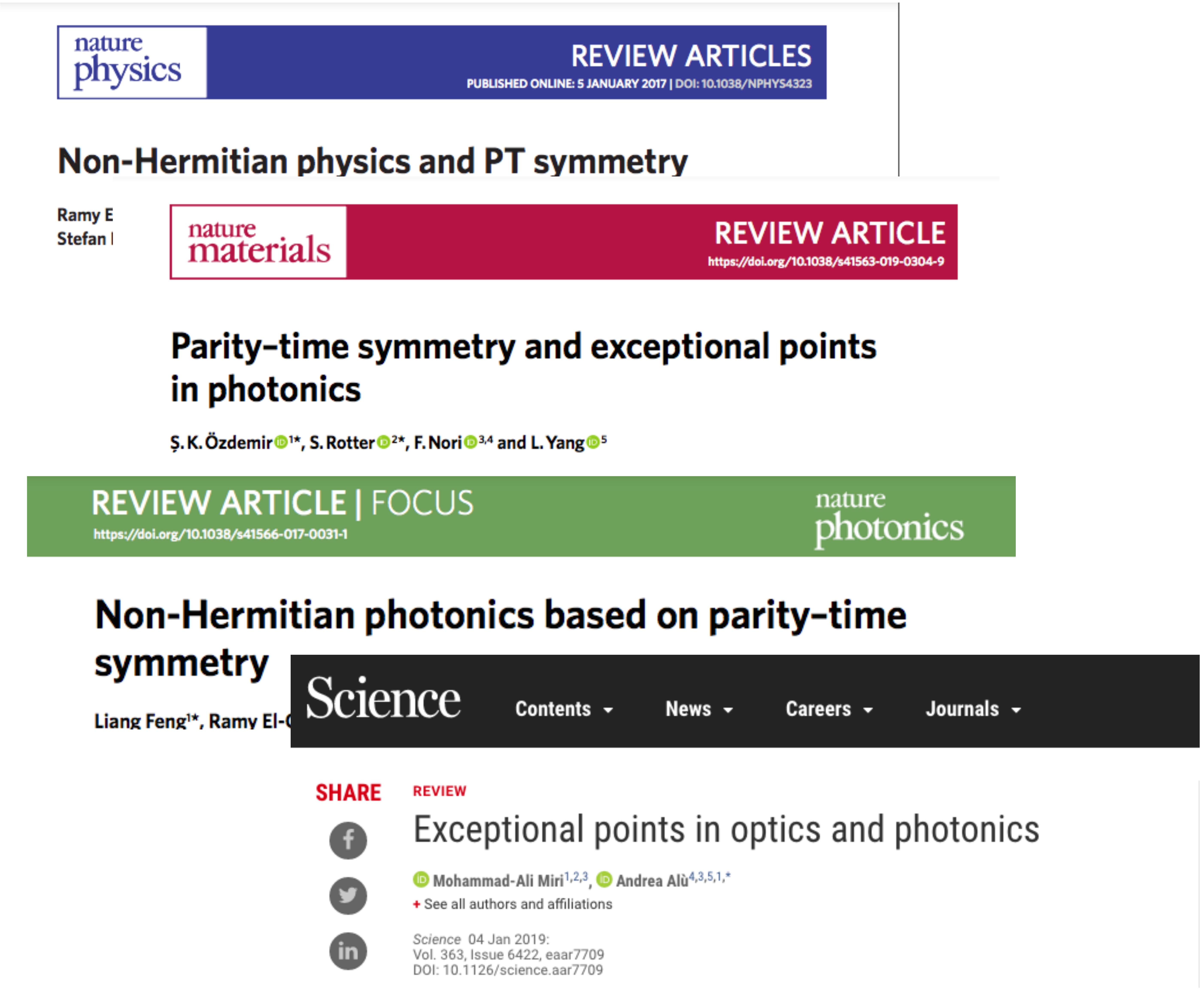
How do we know simulations are getting the physics right?

Tractable models with known properties obtained via simulation and analytical methods provide key tests of proposed new methods.

What is PT symmetry?

Carl M. Bender
Washington University
2017 Heineman Prize for Mathematical Physics

"For developing the *theory of PT symmetry in quantum systems and sustained seminal contributions that have generated profound and creative new mathematics, impacted broad areas of experimental physics, and inspired generations of mathematical physicists.*"



The sign problem and PT symmetry

PT Symmetry and Quantum Mechanics

- PT symmetry is motivated by the $i\phi^3$ field theory associated with the Yang-Lee edge singularity
- In PT-symmetric QM models, $P : x \to -x$ and $p \to -p$ while $T : t \to -t$ and $i \to -i$ so ix^3 is invariant.
- Eigenvalues of H are real or in conjugate pairs:

$$HPT\left|E\right\rangle = PTH\left|E\right\rangle = PTE\left|E\right\rangle = E^*PT\left|E\right\rangle$$

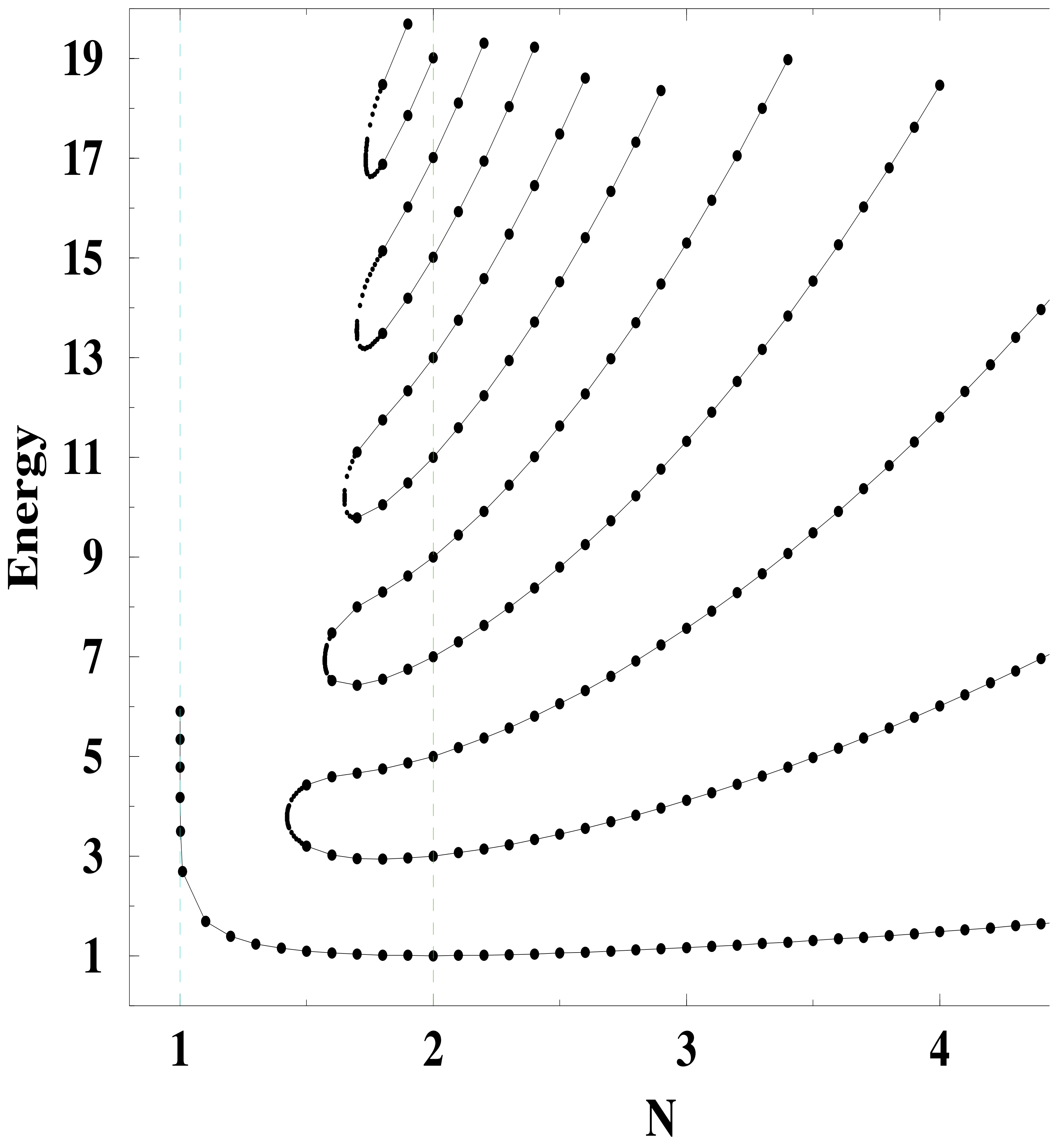
- If all eigenvalues are real, system is equivalent to a Hermitian Hamiltonian

Bender, 0501052

PT Symmetry and Lattice Field Theory

- Eigenvalues of the transfer matrix are real or in conjugate pairs:
- If all eigenvalues of the transfer matrix are real, system is equivalent to a conventional lattice model under a similarity transformation
- Spectral positivity is violated if the transfer matrix has complex eigenvalues.
- Real representations always exist, but are not always positive

Meisinger and Ogilvie, 1208.5077



$$H = p^2 - (ix)^N \quad (N \text{ real})$$

Bender and Boettcher, PRL 80 (1998) 5243-5246

Finite density QCD and related models have a generalized PT symmetry

A prototypical Polyakov loop spin model at finite density

$$S = -J \sum_{\langle jk \rangle} \left(Tr\,P_j\, Tr\,P_k^+ + Tr\,P_j^+\, Tr\,P_k \right) - H \sum_j \left(e^{\beta\mu} Tr\,P_j + e^{-\beta\mu} Tr\,P^+ \right)$$

- Model exhibits Svetitsky-Jaffe universality.
  - The nearest-neighbor interaction induced by plaquettes
  - The complex magnetic term due to heavy quarks at finite density

- Spin model has a CK symmetry, a generalized PT symmetry:
  - Charge conjugation C: $P \rightarrow P^*$
  - Complex conjugation K: $aP \rightarrow a^*P^*$
  - S is invariant under CK symmetry.

An algorithm for simulating PT-symmetric field theories based on duality

$$S(\chi) = \sum_x \left[\frac{1}{2}(\partial_\mu \chi(x))^2 + V(\chi(x)) - ih(x)\chi(x)\right]$$

PT symmetry: $V(\chi)^* = V(-\chi)$

$$\exp\left[-\frac{1}{2}\left(\partial_\mu \chi(x)\right)^2\right] = \int d\pi_\mu(x) \exp\left[\frac{1}{2}\pi_\mu(x)^2 + i\pi_\mu(x)\partial_\mu\chi(x)\right]$$

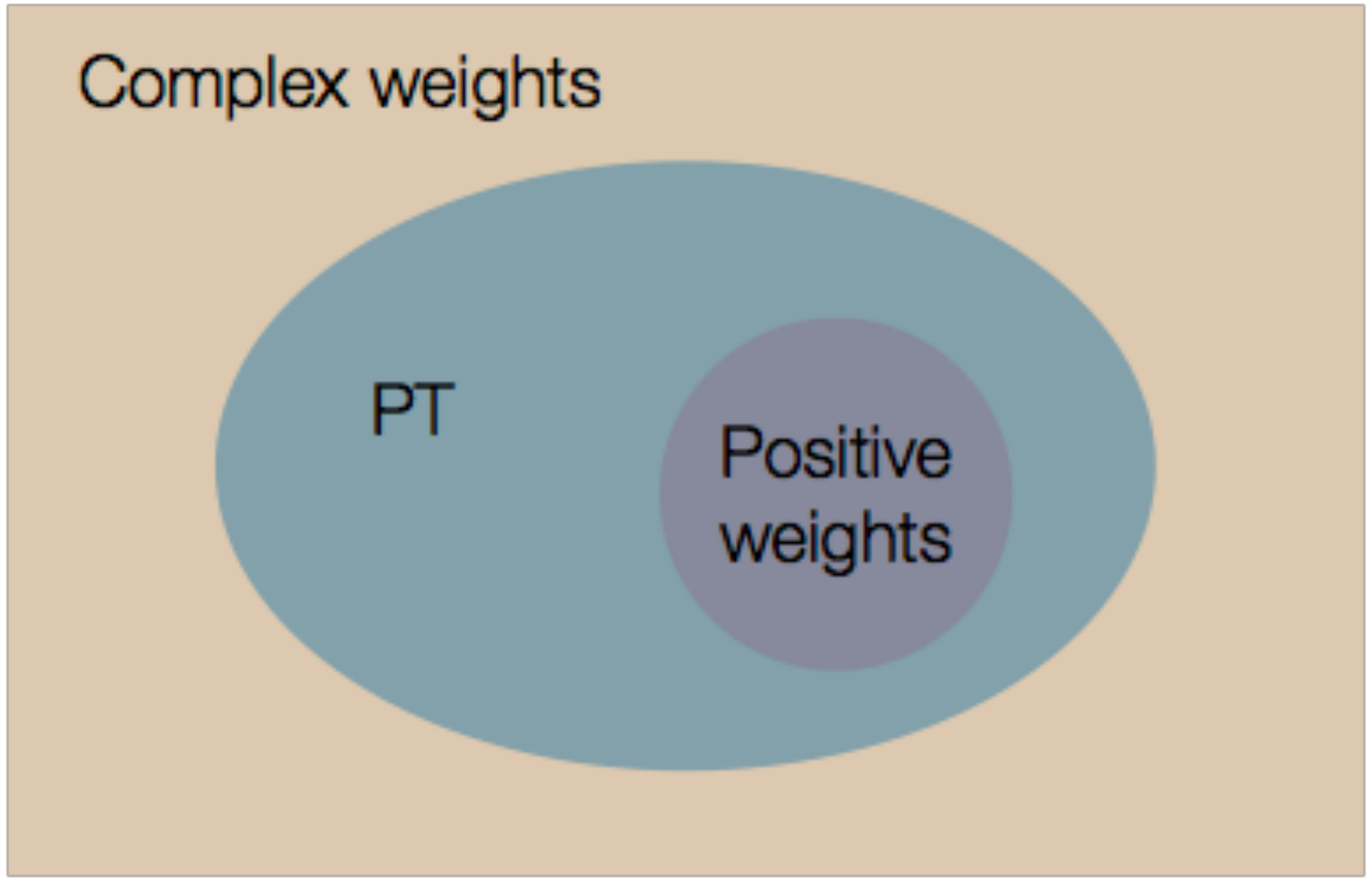
$$\exp\left[-V\left(\chi(x)\right)\right] = \int d\tilde{\chi}(x) \exp\left[-\tilde{V}\left(\tilde{\chi}(x)\right) + i\tilde{\chi}(x)\chi(x)\right]$$

If *dual weight positivity* holds

$$\tilde{w}[\tilde{\chi}(x)] \equiv \exp\left[-\tilde{V}(\tilde{\chi}(x))\right] \geq 0$$

the functional integral is manifestly positive and the dual action $\tilde{S}$ is simulatable by standard methods.

$$\tilde{S} = \sum_x \left[\frac{1}{2}\pi_\mu^2(x) + \tilde{V}(\partial \cdot \pi(x) - h(x))\right]$$



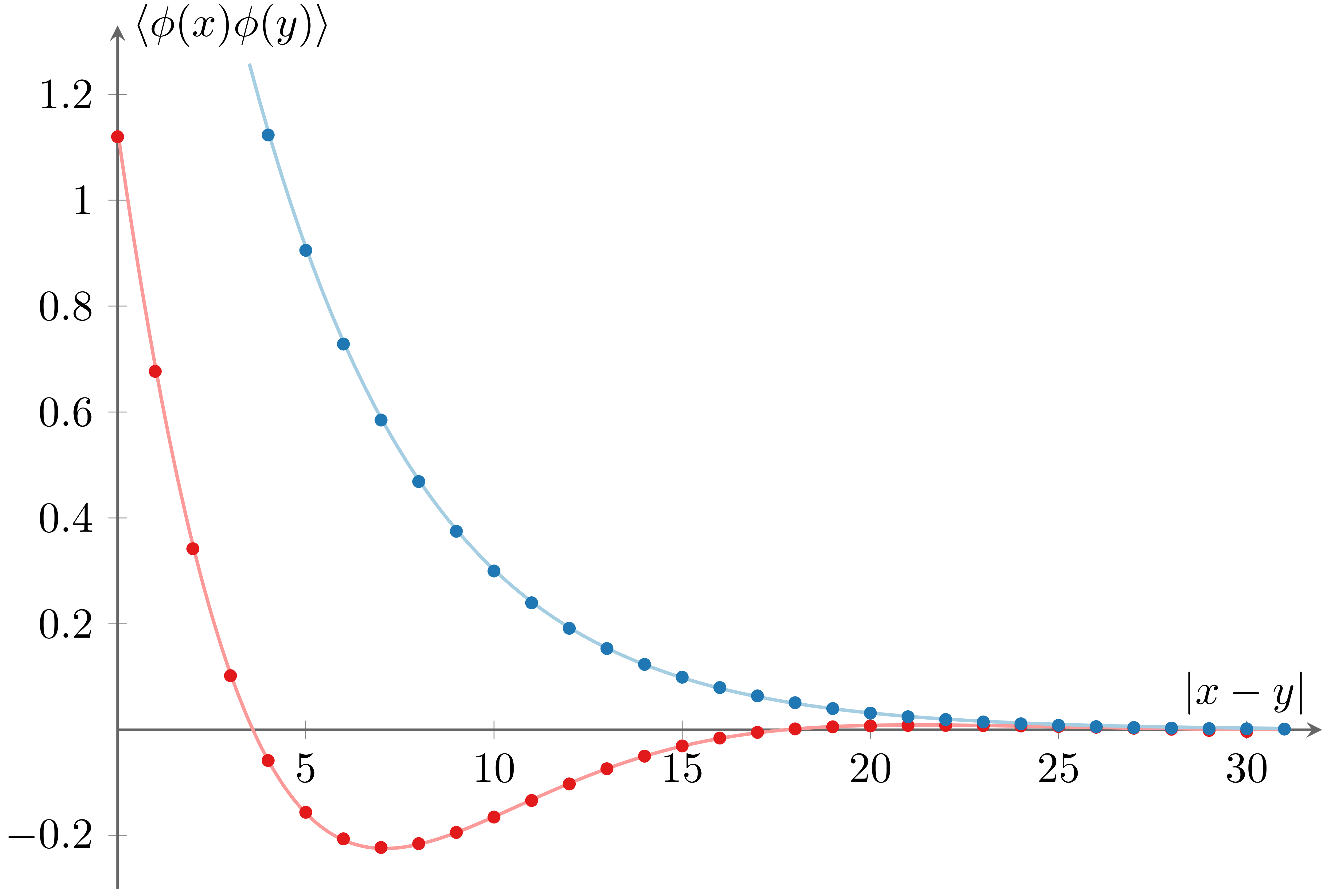
Dual weight positivity implies not only that standard lattice simulation methods can be be applied but also that mean field theory and other analytical methods can be used.

Disorder Lines and Sinusoidal Modulation

Disorder lines mark the boundary between exponential decay of propagators and sinusoidally-modulated exponential decay. The appearance of disorder lines and regions of sinusoidal modulation follows directly in PT-symmetric theories from the existence of conjugate eigenvalue pairs

The first suggestion that oscillatory behavior might be observed in finite density QCD was made by Apoorva Patel (1111.0177, 1210.5907), based on his flux tube model, which is a real form of the complex $Z(3)$ spin model.

Simple mass mixing model



$$L_E(\phi,\chi) = \frac{1}{2}\left(\nabla\phi\right)^2 + \frac{1}{2}\left(\nabla\chi\right)^2 + \frac{1}{2}m_\phi^2\phi^2 + \frac{1}{2}m_\chi^2\chi^2 - ig\phi\chi$$

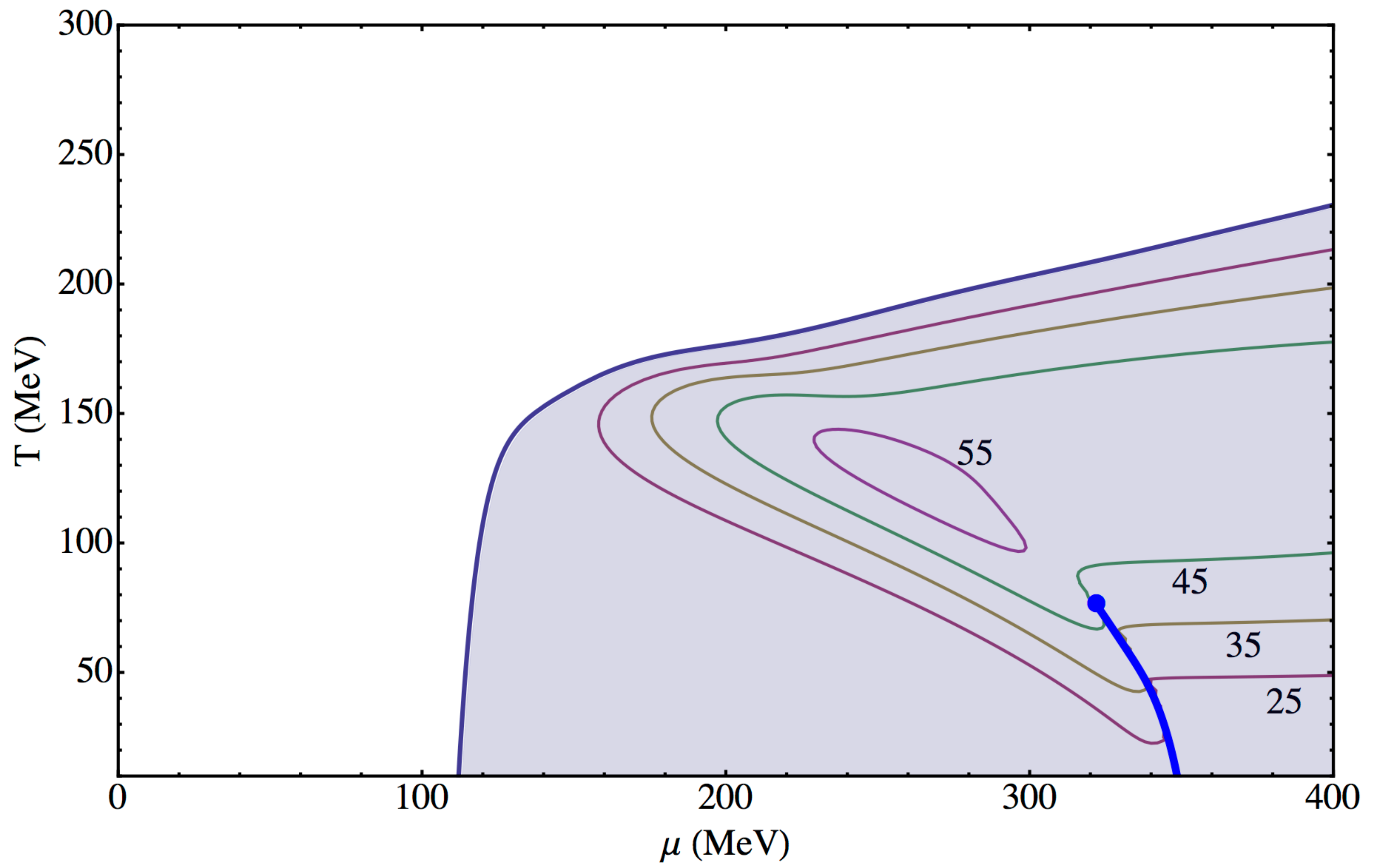
Ogilvie and Medina, 1811.11112

Disorder lines and sinusoidal modulation in finite-density QCD

The evidence for sinusoidally-modulated exponential decay in finite-density QCD is good:

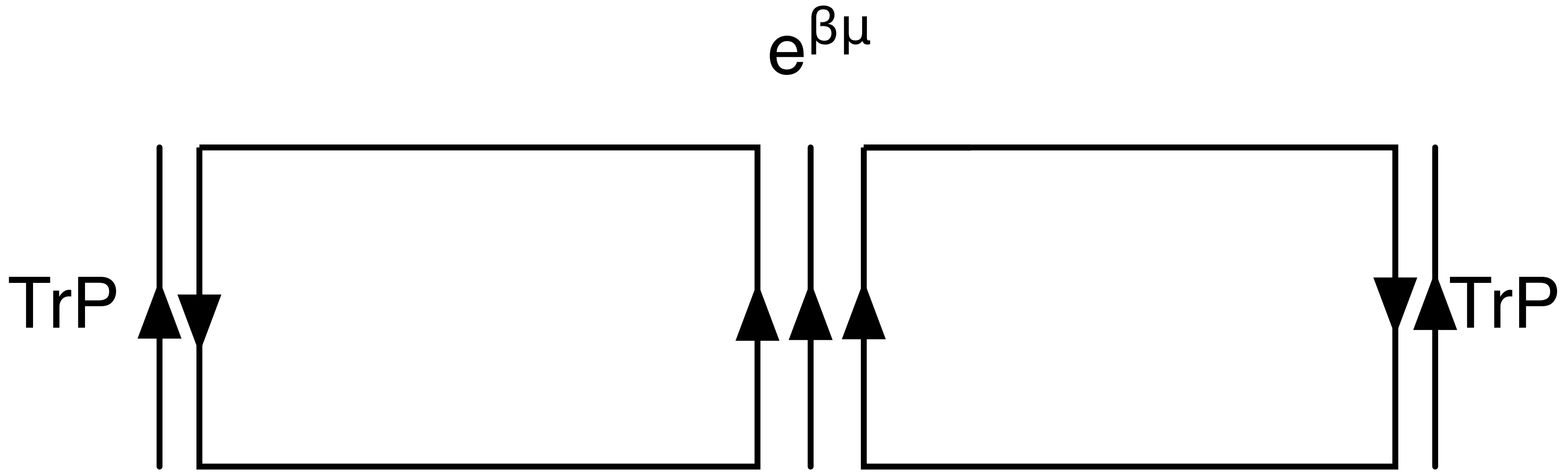
Phenomenological models such as Polyakov-Nambu-Jona Lasinio (PNJL) models

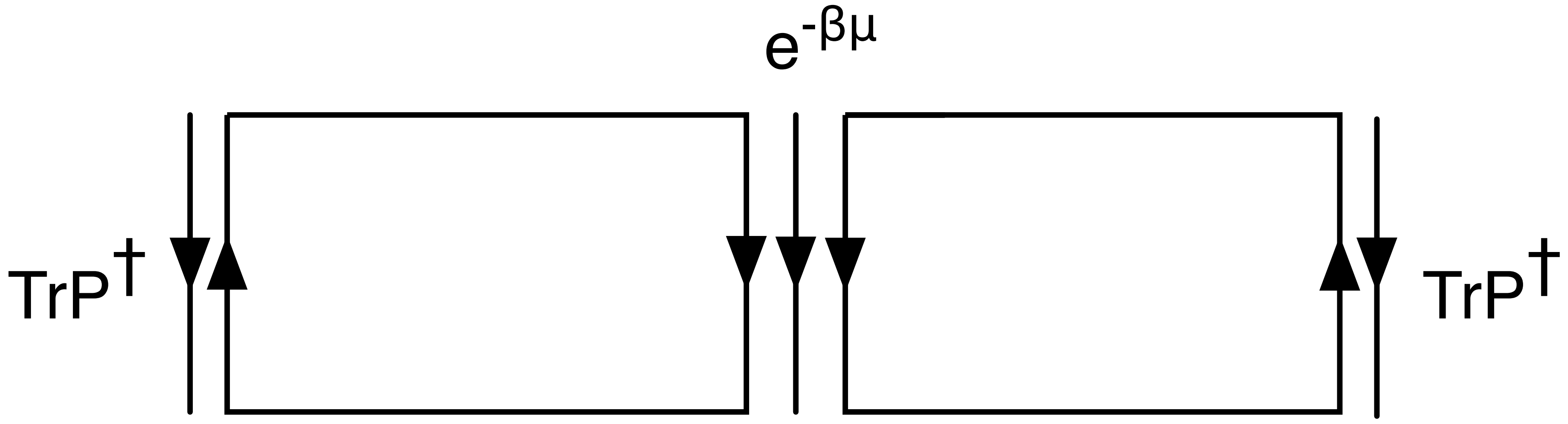
Nishimura, Ogilvie and Pangeni 1401.7982, 1411.4959



Strong-coupling lattice expansions

Nishimura, Ogilvie and Pangeni, 1512.09131





Simulation and mean field theory for $Z(3)$

Akerlund and de Forcrand, 1602.02925

Pattern Formation

(Schindler et al. 1906.07288)

Equivalent forms of the Yukawa-frustrated ϕ^4 action

- **Complex action**

$$L(\phi,\chi) = \frac{1}{2}(\nabla_\mu \phi)^2 + \frac{1}{2}(\nabla_\mu \chi)^2 + \frac{1}{2}m_\chi^2 \chi^2 - ig\phi\chi + \lambda(\phi^2 - v^2)^2 + h\phi$$

- **Nonlocal real action (“attractive vs. repulsive” forces)**

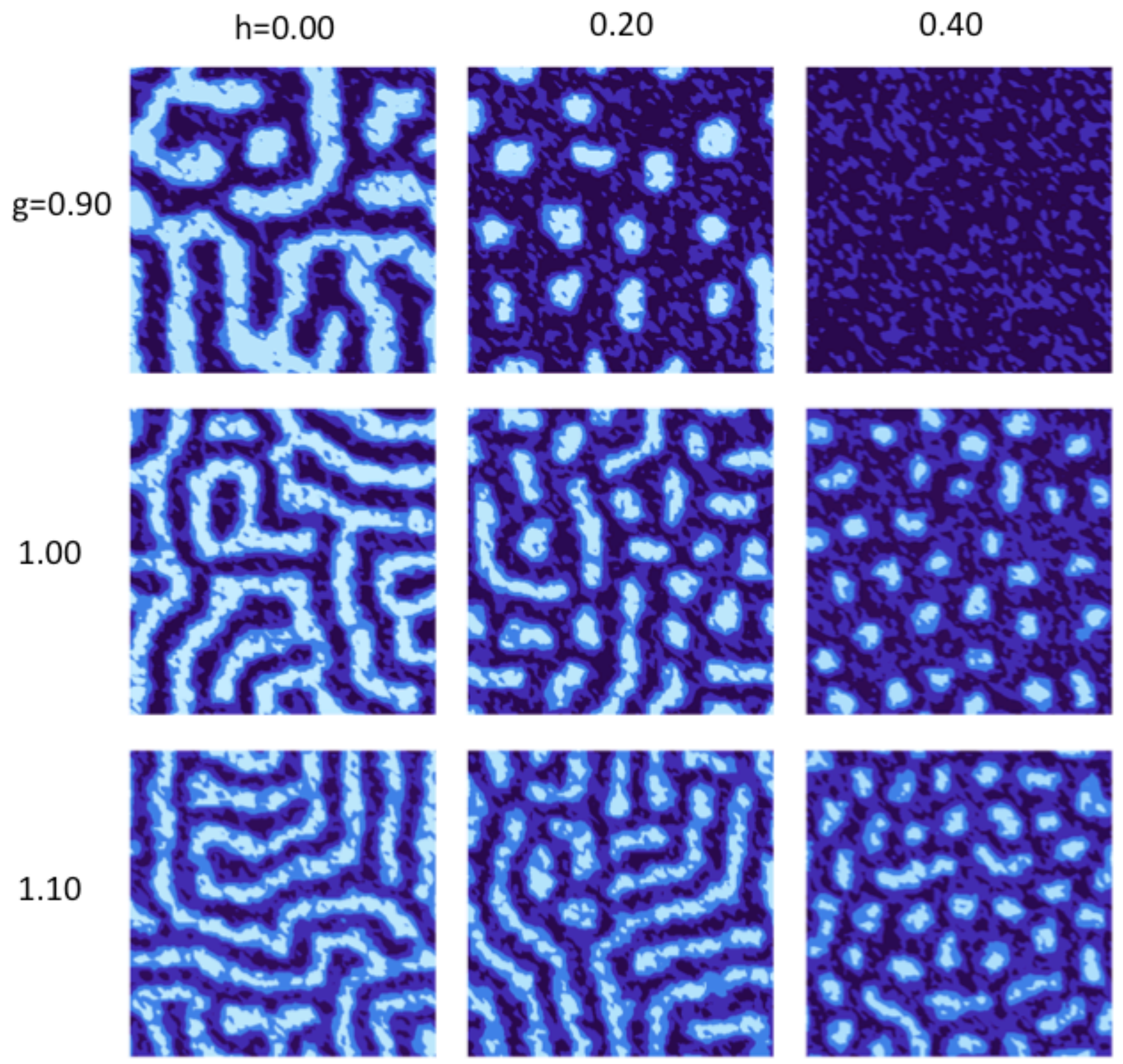
$$S_{\text{eff}} = \sum_x \left[\frac{1}{2}(\partial_\mu \phi(x))^2 + \lambda(\phi^2 - v^2)^2 + h\phi\right] + \frac{g^2}{2}\sum_{x,y} \phi(x)\Delta(x-y)\phi(y)$$

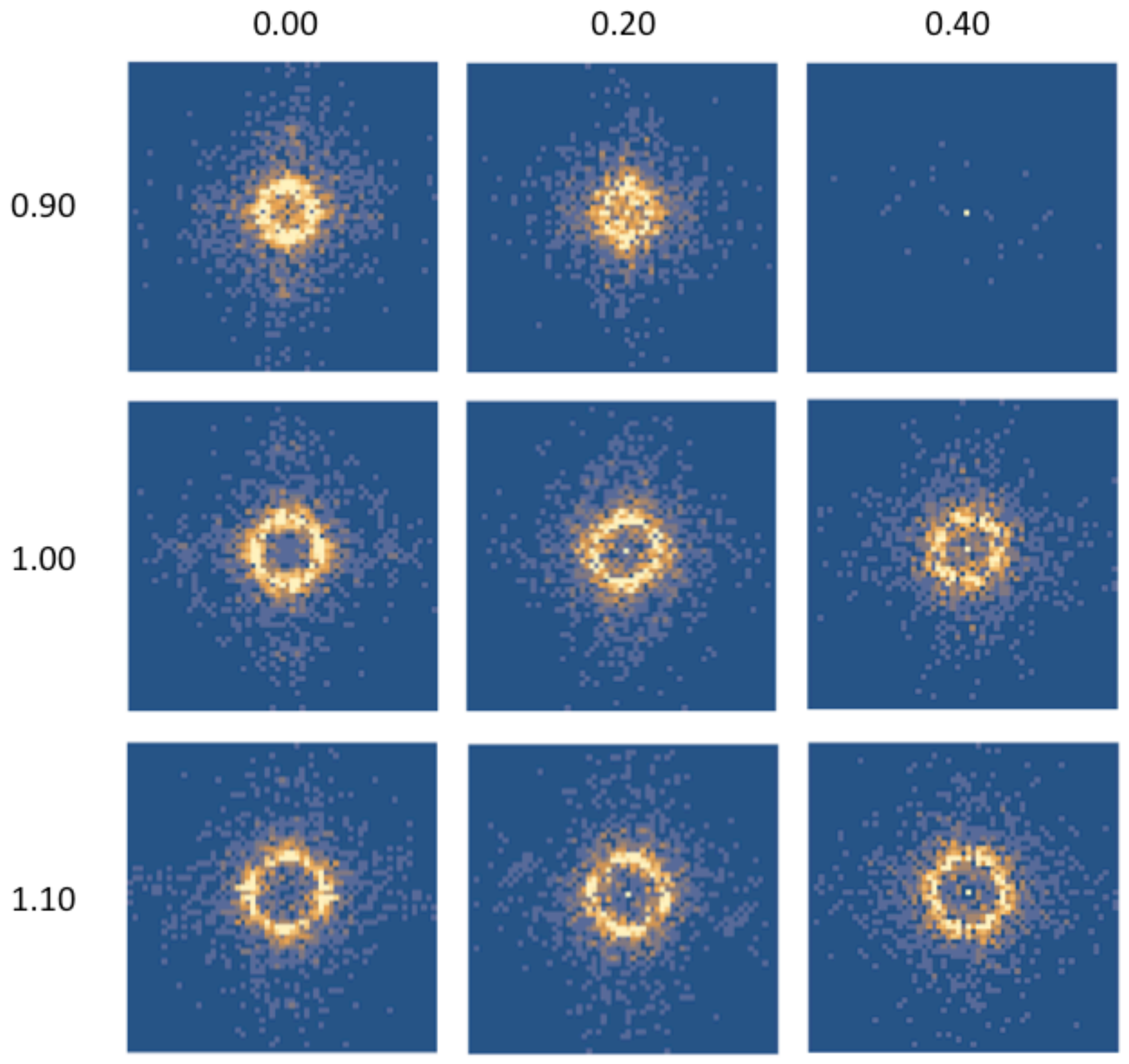
- **Local real action (simulated form)**

$$\tilde{S} = \sum_x \frac{1}{2}[\nabla_\mu \phi(x)]^2 + \frac{1}{2}\pi_\mu^2(x) + \frac{\left(\nabla \cdot \pi - g\phi\right)^2}{2m_\chi^2} + \lambda(\phi^2 - v^2)^2 + h\phi$$

- **Higher derivative expansion (Lifshitz mechanism for pattern formation)**

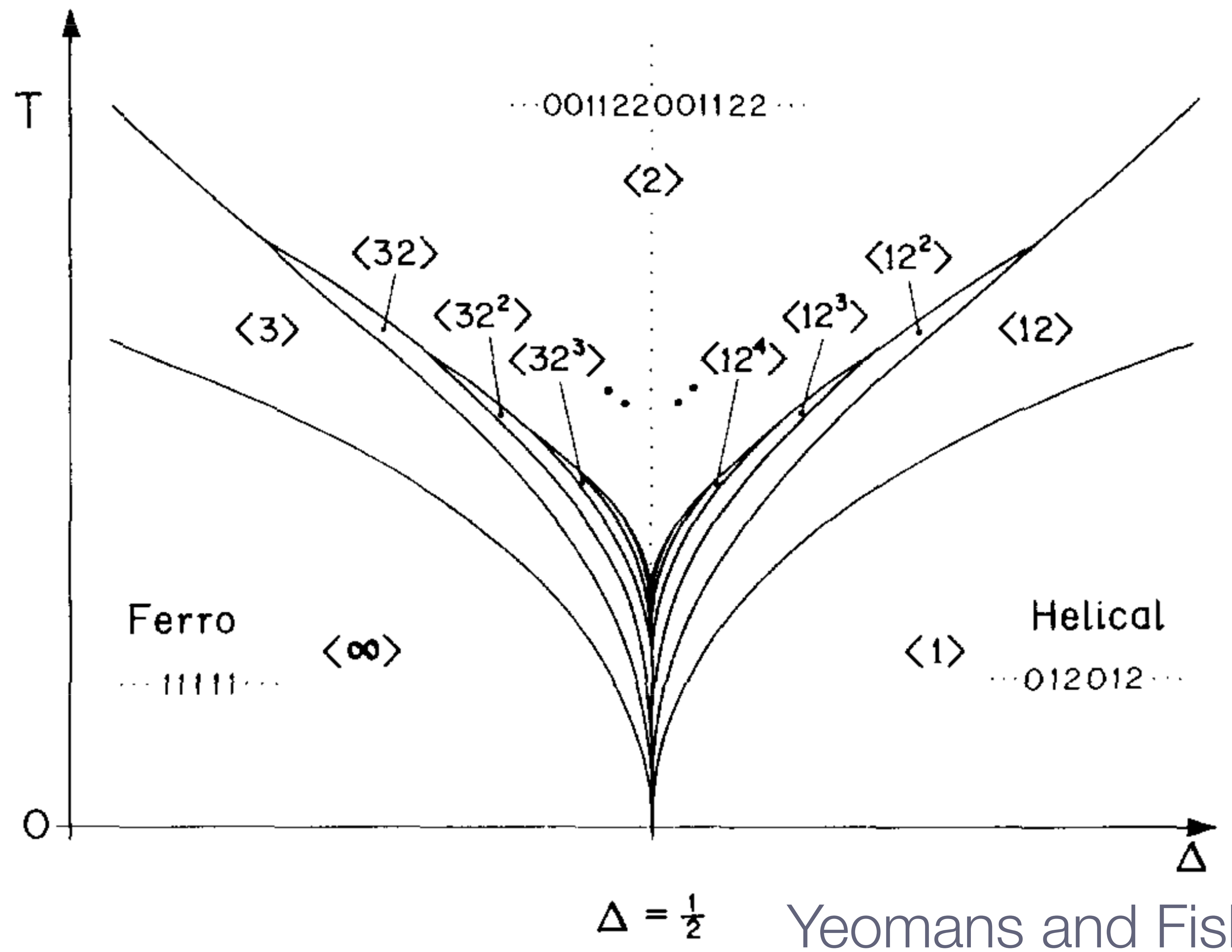
$$S_{\text{eff}} \approx \sum_x \left[\frac{1}{2}(\partial_\mu \phi(x))^2 + \lambda(\phi^2 - v^2)^2 + h\phi\right] + \frac{g^2}{2m_\chi^2}\sum_x \left[\phi(x)^2 - \frac{1}{m_\chi^2}(\partial_\mu \phi(x))^2\right]$$





Z(N) spin models and pattern formation

Chiral $Z(3)$ Devil's Flower



Yeomans and Fisher, 1984

Basic model $$\mathscr{H} = -\frac{J}{2}\sum_{\langle j\nu \rangle}\left(z_j z^*_{j+\hat{\nu}} + z^*_j z_{j+\hat{\nu}}\right)$$

Chemical potential $$\Rightarrow e^{\mu} z_j z^*_{j+\hat{d}} + e^{-\mu} z^*_j z_{j+\hat{d}}$$

Chiral Z(N) model $$\Rightarrow e^{2\pi i \Delta/N} z_j z^*_{j+\hat{d}} + e^{-2\pi i \Delta/N} z^*_j z_{j+\hat{d}}$$

The Villain action Z(N) model has a simple dual form in all d.

$$J \to \tilde{J} = \frac{N^2}{4\pi^2 J} \qquad \mu \to \tilde{\mu} = -\frac{2\pi i J \mu}{N}$$

Meisinger and Ogilvie, 1306.1495, 1311.5515

The chiral model has an intricate low-temperature (large J) structure with patterned phases. These may be commensurate or incommensurate, depending on d. Lattice duality maps between classes of Hamiltonians, complex and real, with non-Hermitian transfer matrices. The 2d case is clear: we are looking at the universality class of 2d Z(N) parafermions and the patterned behavior in the chiral model corresponds to states with nonzero N-ality realized as kinks.

Conclusions

- There is a fundamental change in the system where the transfer matrix no longer has real eigenvalues and orthogonal eigenvectors. This behavior is associated with a generalized PT symmetry, and non-Hermitian behavior. The sign problem is a manifestation of this change, but need not be present.
- Spectral positivity is lost, and patterned phases may emerge. Computational complexity, e.g. NP-hardness, is associated with the complex structure of equilibrium states.
- Tractable models with known properties obtained via simulation and analytical methods provide key tests of proposed new methods.

Finite density QCD? Not yet, but here is a snapshot of our pipeline:

- Mass mixing models (1811.11112)
- Yukawa-frustrated ϕ^4 model (1906.07288, Schindler's talk)
- Heavy-fermion QCD-like models (Schindler's talk)
- Universality class of $i\phi^3$ (Schindler, Schindler and Ogilvie, in progress)
- Non-unitary minimal conformal models (see Dotsenko and Fateev, 1984)
- Affine Toda models (see Hollowood, 1992)
- $SU(N)$ models (see also Fring *et al.* 2004.00723, 2006.02718, 2007.15425, 2103.13519)
- Time-dependent phenomena in heavy-ion physics (1906.07288; in progress)
- Connection to mesonic Lifshitz instabilities and chiral spirals (in progress; see also Pisarski, 2005.00045)

Computational complexity

The well-known work of Troyer and Wiese (PRL 2005) shows that the sign problem of fermionic many-body systems is NP-hard by showing its equivalence to finding the ground state of a random-bond Ising model

It has been proposed that scalar field theory models with long-range interactions (Schmalian and Wolynes, PRL 2001) and higher-derivative interactions (Westfahl et al, Chem. Phys. Lett 2002) can model glassy behavior, a prototypical NP-hard problem.

$$S_{\text{eff}} = \sum_x \left[\frac{1}{2}(\partial_\mu \phi(x))^2 + \lambda(\phi^2 - v^2)^2 + h\phi\right] + \frac{g^2}{2}\sum_{x,y}\phi(x)\Delta(x-y)\phi(y) \qquad \tilde{\Delta}(k) = \frac{1}{k^2}$$

Computational complexity in such systems has its origins in the complexity of the ground states and equilibrium states of the systems, in particular in spatial structure.