

Lee-Yang singularities, series expansions and the critical point

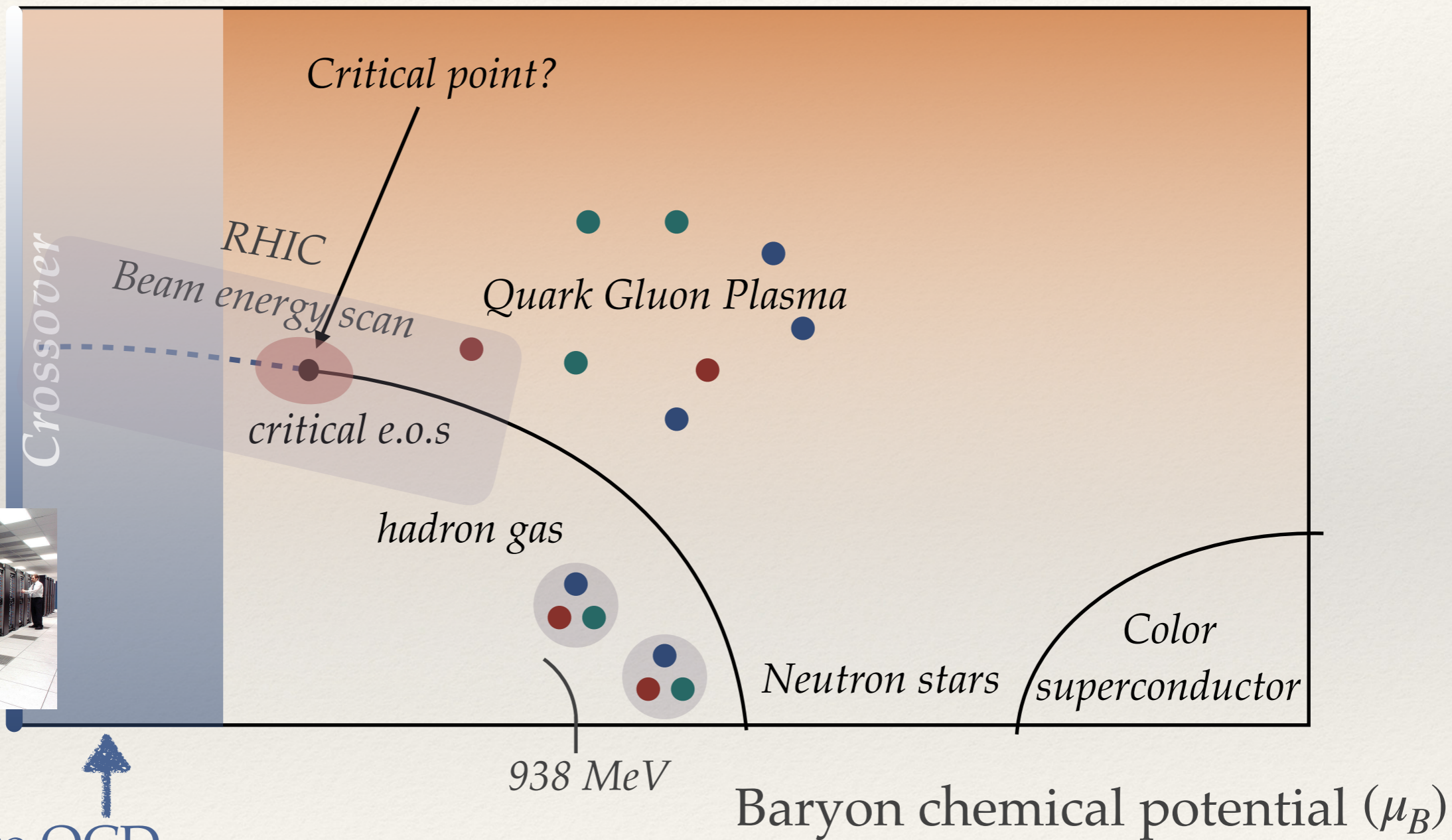
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[arXiv: [2105.08080](https://arxiv.org/abs/2105.08080)]

Lattice 2021
July 26-30, 2021

Motivations



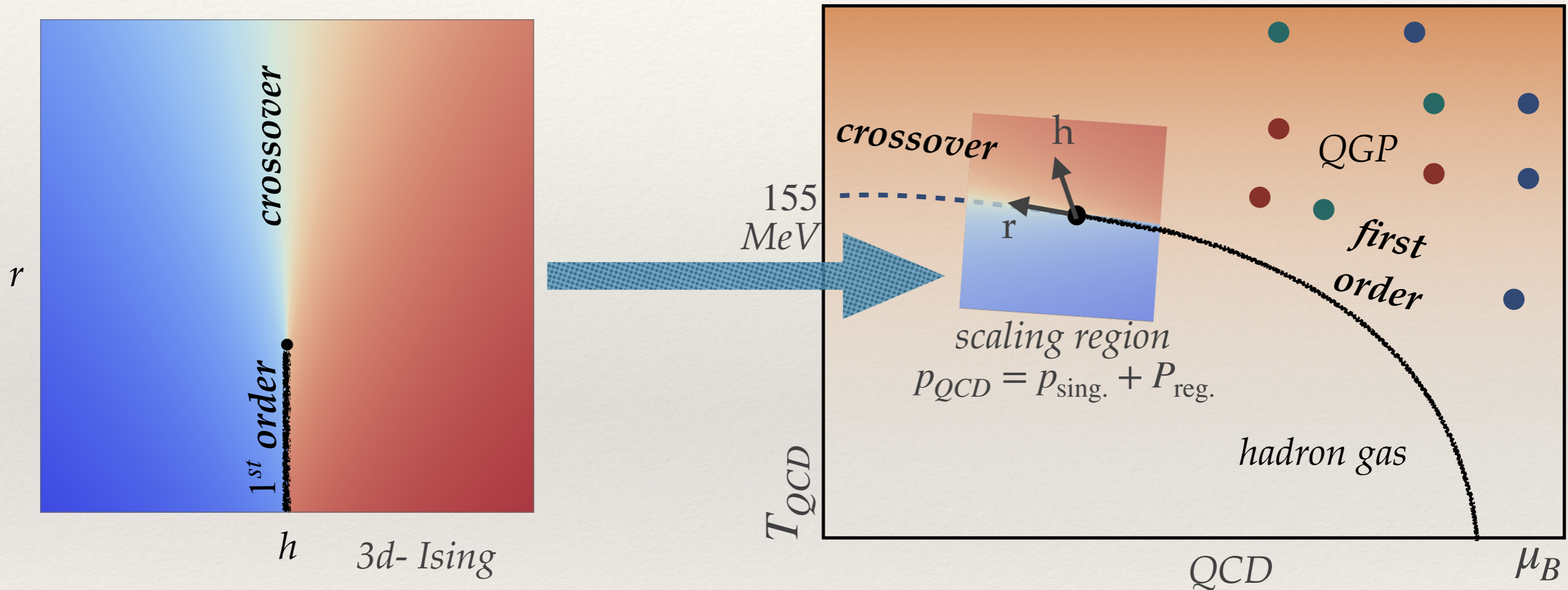
Lattice QCD

Taylor series around $\mu_B = 0$

Motivations



$$\begin{pmatrix} r \\ h \end{pmatrix} = \begin{pmatrix} r_T & r_\mu \\ h_T & h_\mu \end{pmatrix} \begin{pmatrix} T - T_c \\ \mu - \mu_C \end{pmatrix} := M \begin{pmatrix} T - T_c \\ \mu - \mu_C \end{pmatrix}$$



Given the e.o.s. as truncated Taylor series around $\mu=0$, what can we say about *the critical e.o.s* ?

Gross-Neveu Model

$$S = \int d^2x \sum_{a=1}^{N_F} \left(i\bar{\psi}_a(\not{\partial} - m_q)\psi_a + \frac{g^2}{2}(\bar{\psi}_a\psi_a)^2 \right) \quad [\text{Gross, Neveu, '74}]$$

- Solvable in large N_f limit (mean field is exact at $N_f=\infty$)
- Asymptotically free, dimensional transmutation
- Chiral symmetry breaking
- Toy model for QCD

$$\frac{\pi}{Ng^2} = \log \frac{\Lambda}{m}$$

Dimensional transmutation

$$\gamma \equiv \frac{\pi}{Ng^2} \frac{m_q}{m} = \log \frac{m[m_q]}{m[0]}$$

Explicit χ SB parameter

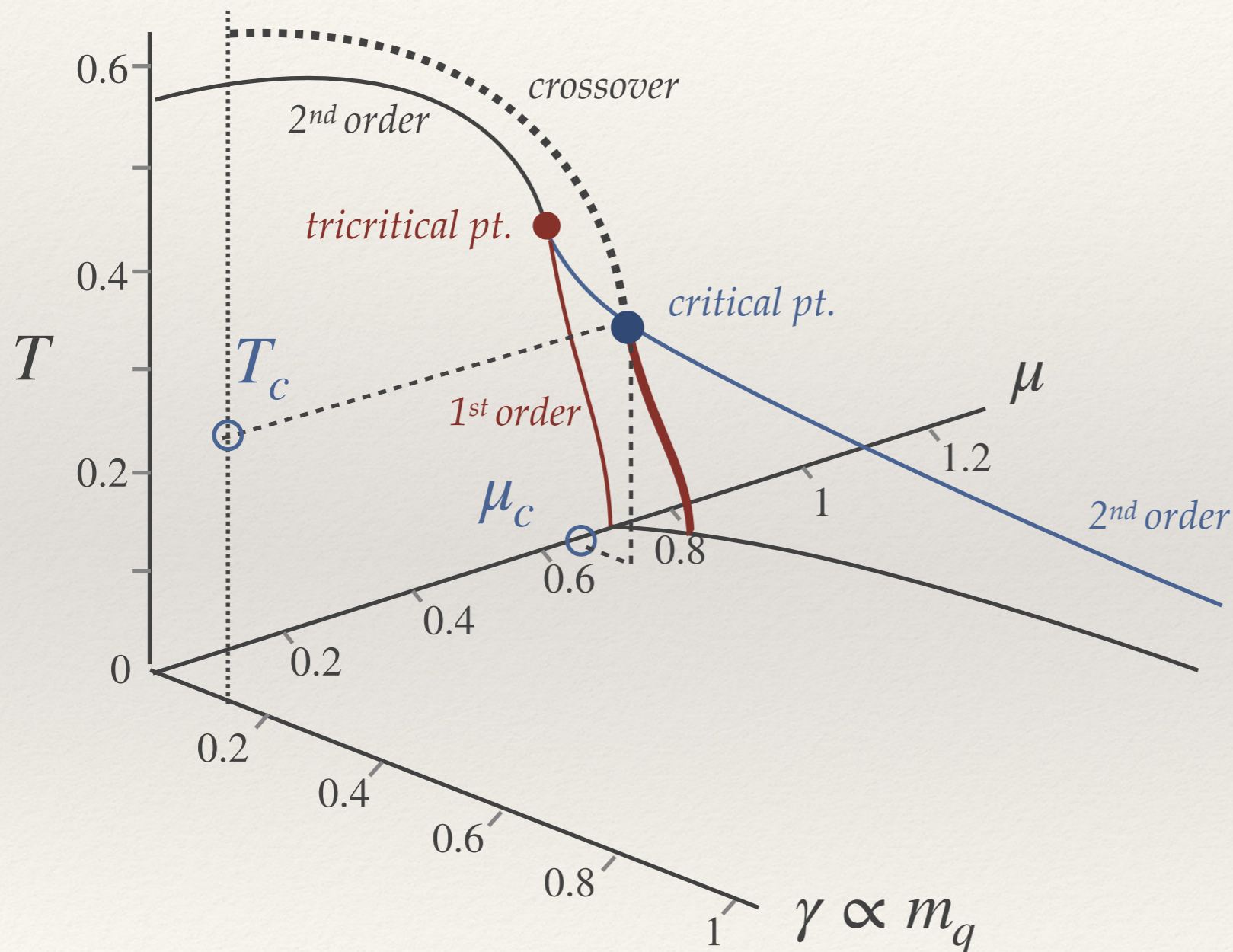
Thermodynamics:

$$\Omega[T, \mu] = \min_{\phi} \left(\frac{\phi^2}{2\pi} \left(\log \phi - \frac{1}{2} + \gamma \right) - \frac{\gamma}{\pi} \phi - T \int \frac{dk}{2\pi} \log \left[\left(1 + e^{-\sqrt{k^2 + \phi^2 - \mu}/T} \right) \left(1 + e^{-\sqrt{k^2 + \phi^2 + \mu}/T} \right) \right] \right)$$

Gross-Neveu Model

“homogeneous”* phase diagram (toy example for QCD)

[Barducci et al. '95]



- Assume $m_q \neq 0$
- Near the critical p.t
 $\Rightarrow \mathbb{Z}_2$ Ising e.o.s

mean field exponents
 $\beta = 1/2, \delta = 3, \sigma_{LY} = 1/2$

- Focus on the crossover

$$T \gtrsim T_c$$

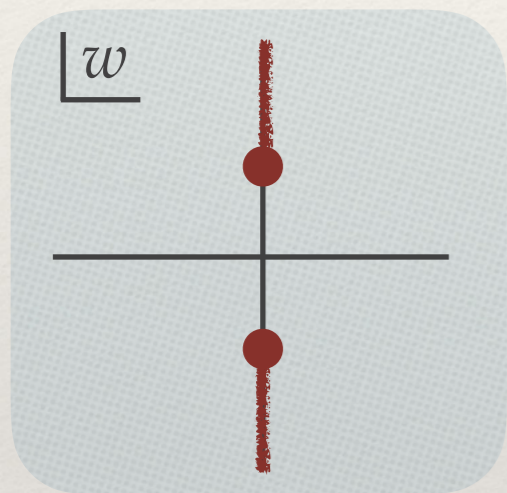
- T, μ normalized to m

*for the phase diagram including crystalline phases see [Schnetz, Thies, Ulrichs '05 ; GB, Dunne, Thies '08]

Lee Yang edge singularity

- The scaling e.o.s, $f_s(w)$, has singularities at $w = \pm iw_{LY}$ ($w := hr^{-\beta\delta}$)

$$\mu_{LY}(T) \approx \mu_c - \frac{h_T}{h_\mu}(T - T_c) \pm iw_{LY} \frac{(\det \mathbb{M})^{\beta\delta}}{h_\mu^{\beta\delta+1}} (T - T_c)^{\beta\delta}$$

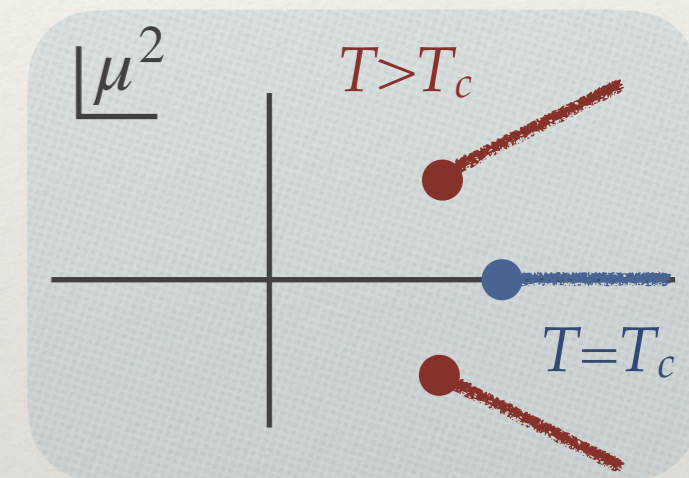


$(\tan \alpha_1)^{-1}$
slope of the
crossover line

$\det \mathbb{M} \propto (\tan \alpha_2 - \tan \alpha_1)$
relative angle
between r, h axes

see e.g

[Pradeep, Stephanov '19]



- The edge singularities pinch the real axis for $T = T_c$
- There is an associated critical exponent, σ , $M(w) \sim (w \pm iw_{LY})^{\sigma_{LY}}$, (M : magnetization)

$$\sigma_{LY,d=3} \approx 0.1, \quad \sigma_{LY,d=6} = 1/2 \text{ (mean field)}$$

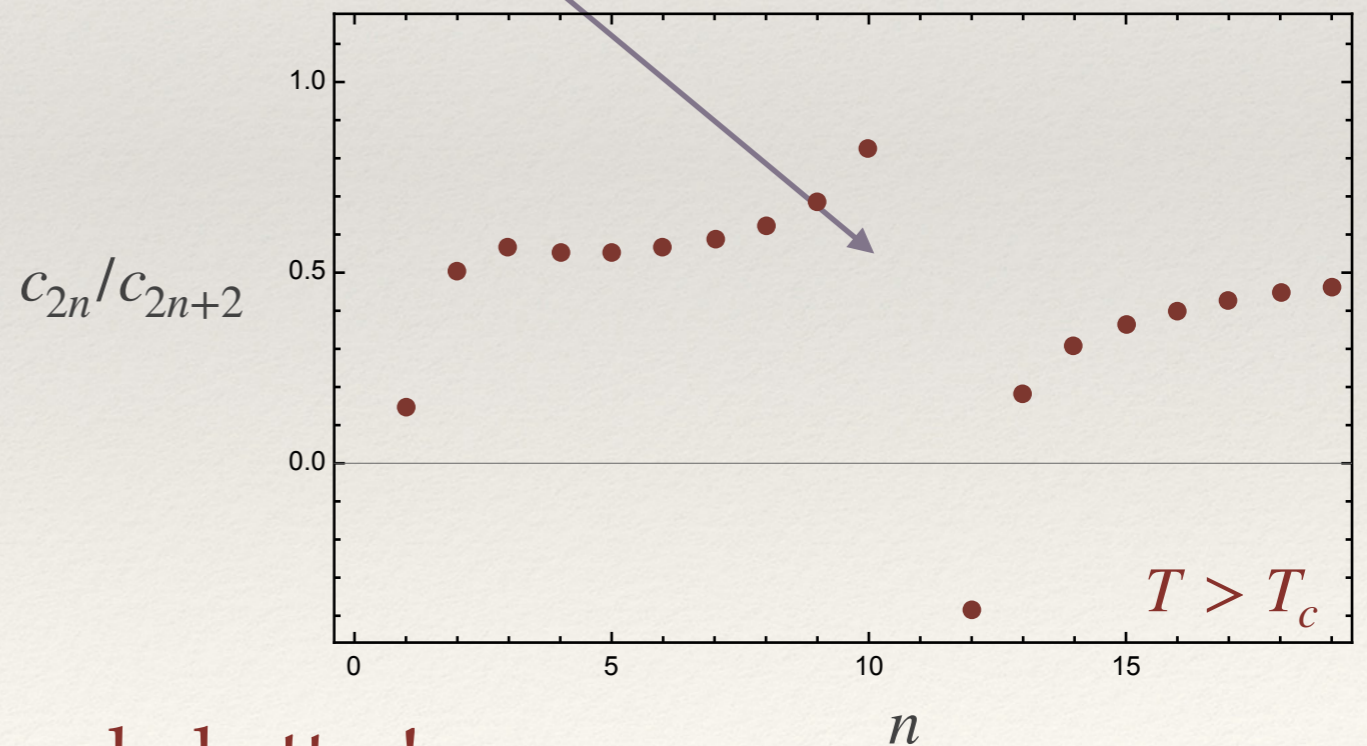
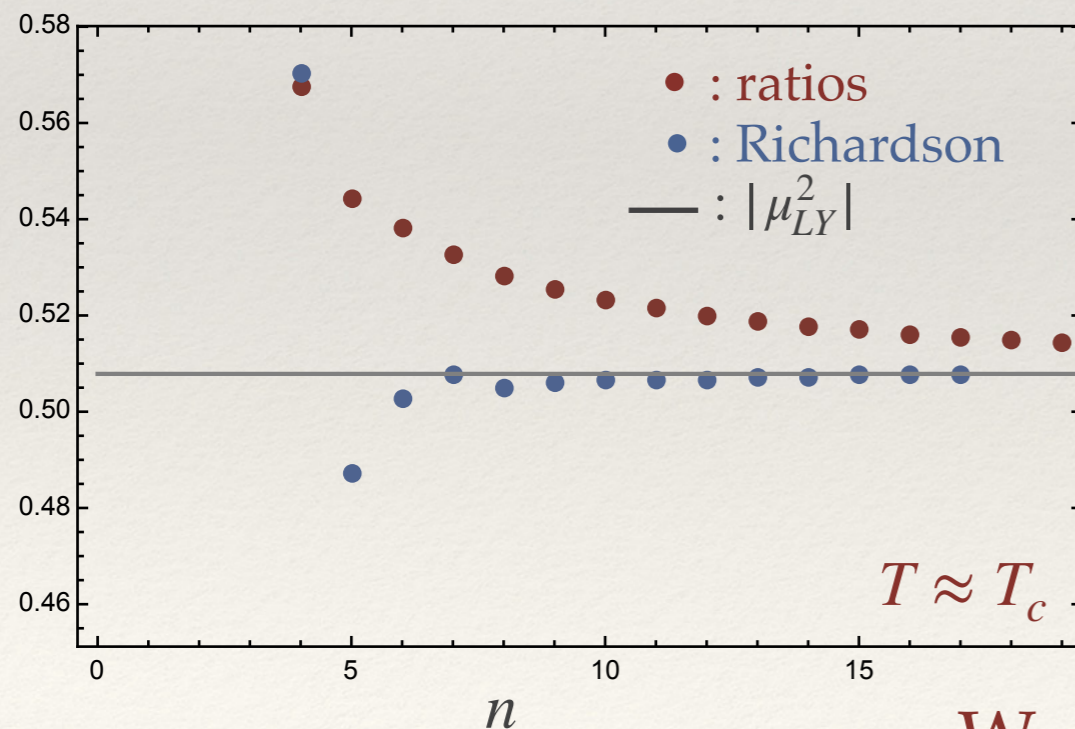
[Fisher, '74; An, Stephanov, Mesterházy '16; Connelly, Johnson, Mukherjee, Skokov '20]

When life gives you Taylor series...

Large order behavior e.g. $\chi(\mu^2) = \sum_n c_{2n}(T)\mu^{2n} \rightarrow c_{2n}/c_{2n+2} \sim |\mu_{LY}^2|, \quad n \rightarrow \infty$

- In the crossover region the Lee-Yang edge singularity is closest to the origin
- The imaginary part of μ_{LY} could make the large order analysis tricky

$$c_{2n} \sim \frac{\Gamma(\sigma + n)}{\Gamma(1 + n) |\mu_{LY}^2|^{n+\sigma}} \cos(\theta(n + \sigma) - \pi\sigma), \quad (\theta := \arg \mu_{LY}^2)$$



We can do better!

When life gives you Taylor series...

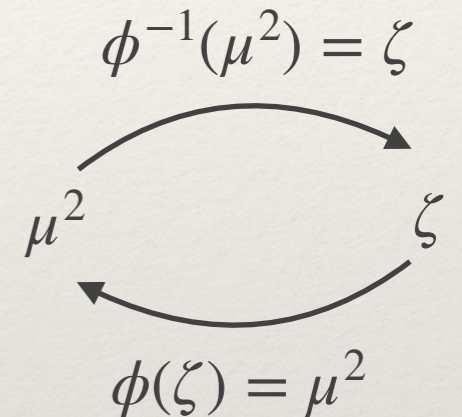
Taylor series: $f(\mu^2) = \sum_{n=0}^N c_{2n} \mu^{2n}$

Padé approximant
(diagonal)

$$P_{[N/2, N/2]} f(\mu^2) = \frac{P_{N/2}(\mu^2)}{Q_{N/2}(\mu^2)}$$

Padé approximant + conformal mapping:
"conformal Padé"

$$P f(\phi(\zeta)) = \frac{\tilde{P}_{N/2}(\zeta)}{\tilde{Q}_{N/2}(\zeta)} \Big|_{\zeta = \phi^{-1}(\mu^2)}$$



e.g. $f(x) = \frac{1}{\sqrt{1-x}} \approx 1 + \frac{x}{2} + \frac{3x^2}{8} + \frac{5x^3}{16} + \frac{35x^4}{128} + \dots$

$$P_{[2,2]} f(x) = \frac{16 - 12x + x^2}{16 - 20x + 5x^2}$$

$$\phi(\zeta) = \frac{4\zeta}{(1+\zeta)^2}$$

$$P_{[n,n]} f(\phi(\zeta)) = \frac{1+\zeta}{1-\zeta}$$

Exact for any number of terms

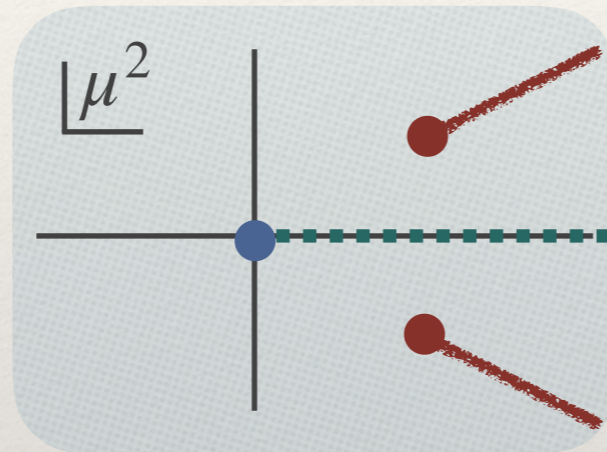
$$\phi^{-1}(x) = \frac{1 - \sqrt{1-x}}{1 + \sqrt{1-x}}$$

(used also in Borel summation of ϵ expansion see eg. [Guida, Zinn-Justin '98])

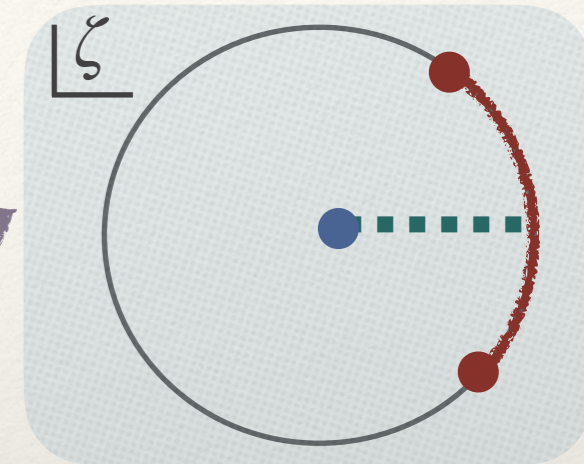
Conformal Maps

- Two cut plane $\mu_{sing}^2 = |\mu_{LY}^2| e^{\pm i\theta}$

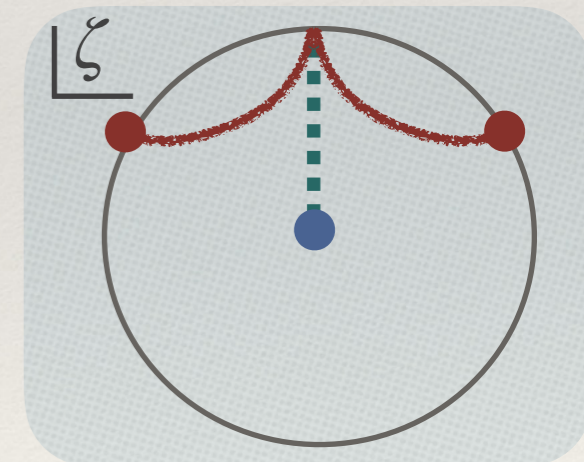
$$\phi(\zeta) = \left(\frac{\theta}{\pi}\right)^{\theta/\pi} \left(1 - \frac{\theta}{\pi}\right)^{1-\theta/\pi} \frac{4\mu_{LY}^2\zeta}{(1+\zeta)^2} \left(\frac{1+\zeta}{1-\zeta}\right)^{2\theta/\pi}$$



Conformal map



Uniformizing map



$$\phi(\zeta) = |\mu_{LY}^2| e^{i\theta} - 2i |\mu_{LY}^2| \sin \theta \lambda \left(i \frac{\mathbb{K}(g(\theta)) - \zeta \mathbb{K}(\bar{g}(\theta))}{\mathbb{K}(\bar{g}(\bar{\theta})) + \zeta \mathbb{K}(g(\theta))} \right)$$

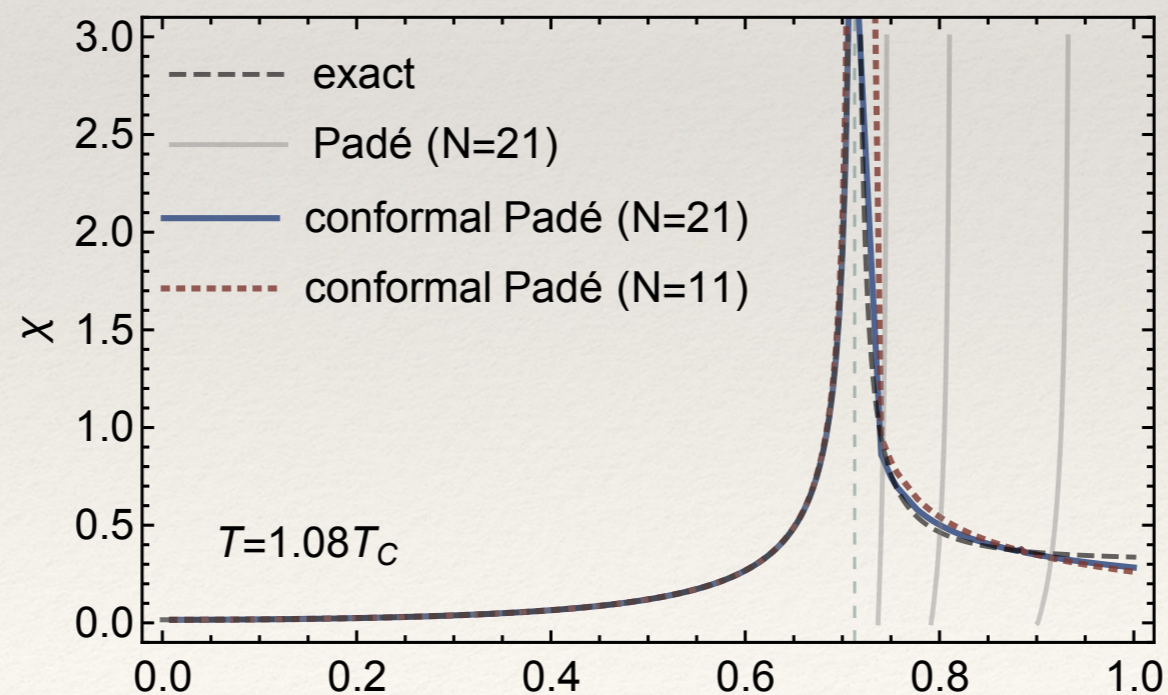
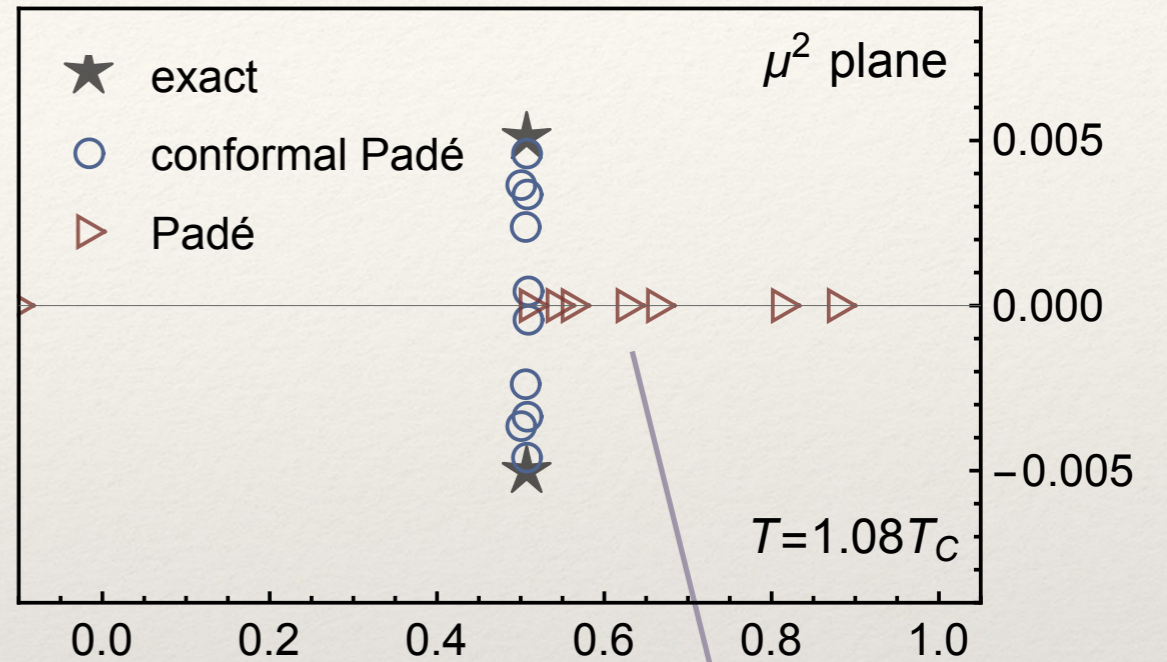
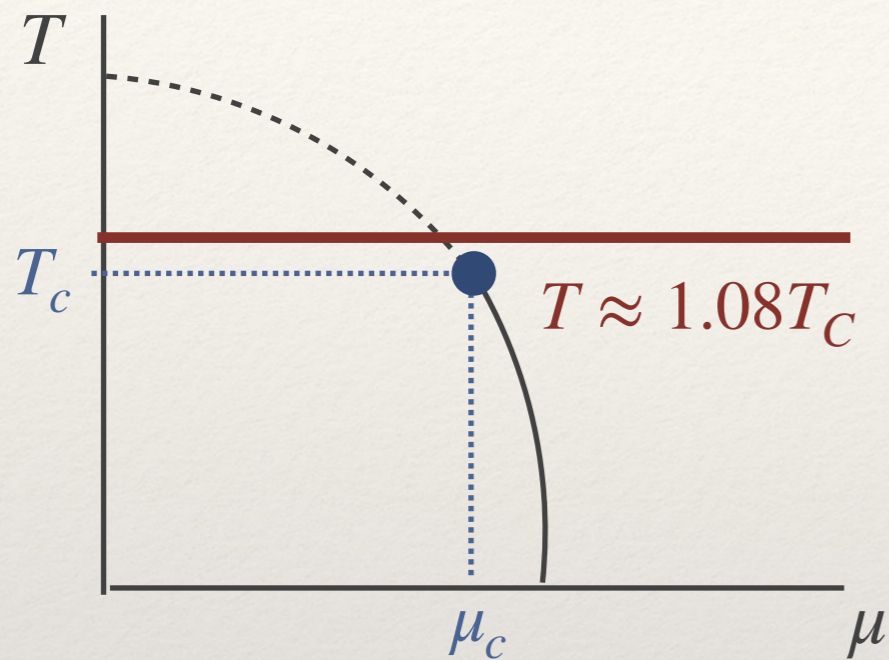
$$g(\theta) := \frac{1}{2} + \frac{i}{2} \cot \theta$$

- One cut plane $\mu_{sing}^2 = \mu_{LY}^2$
“local analysis” around each LY singularity

$$\phi(\zeta) = \frac{4\mu_{LY}^2\zeta}{(1+\zeta)^2}$$

[Costin, Dunne '20]

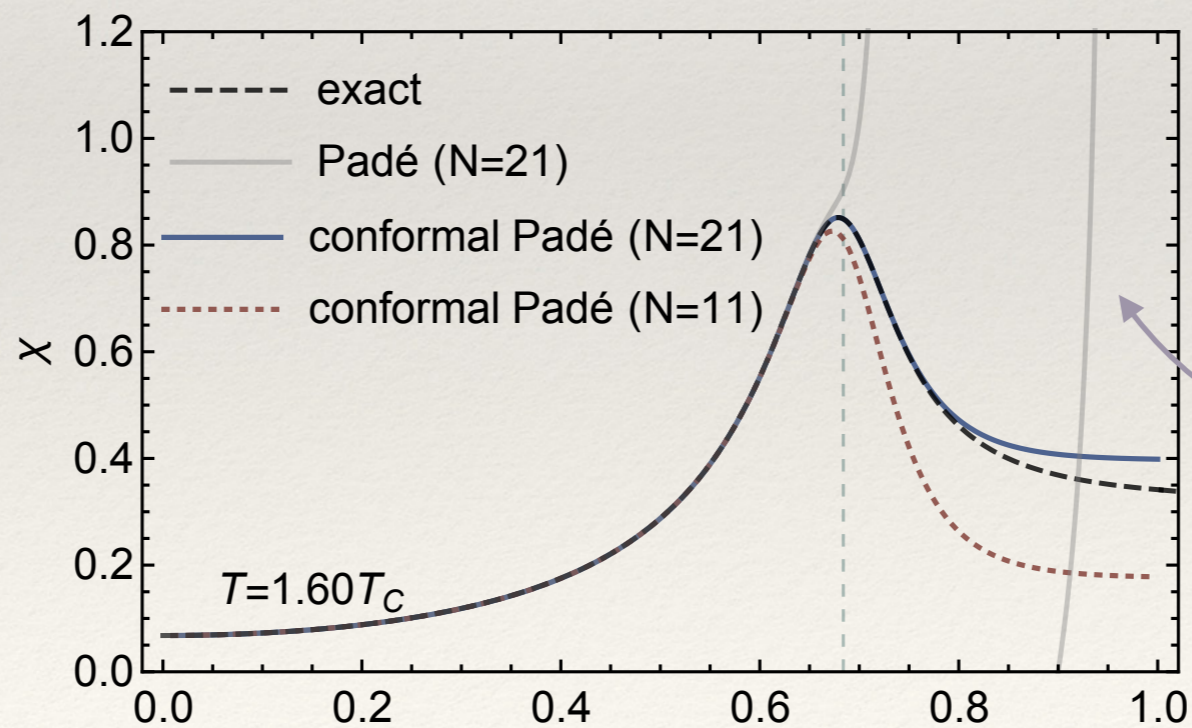
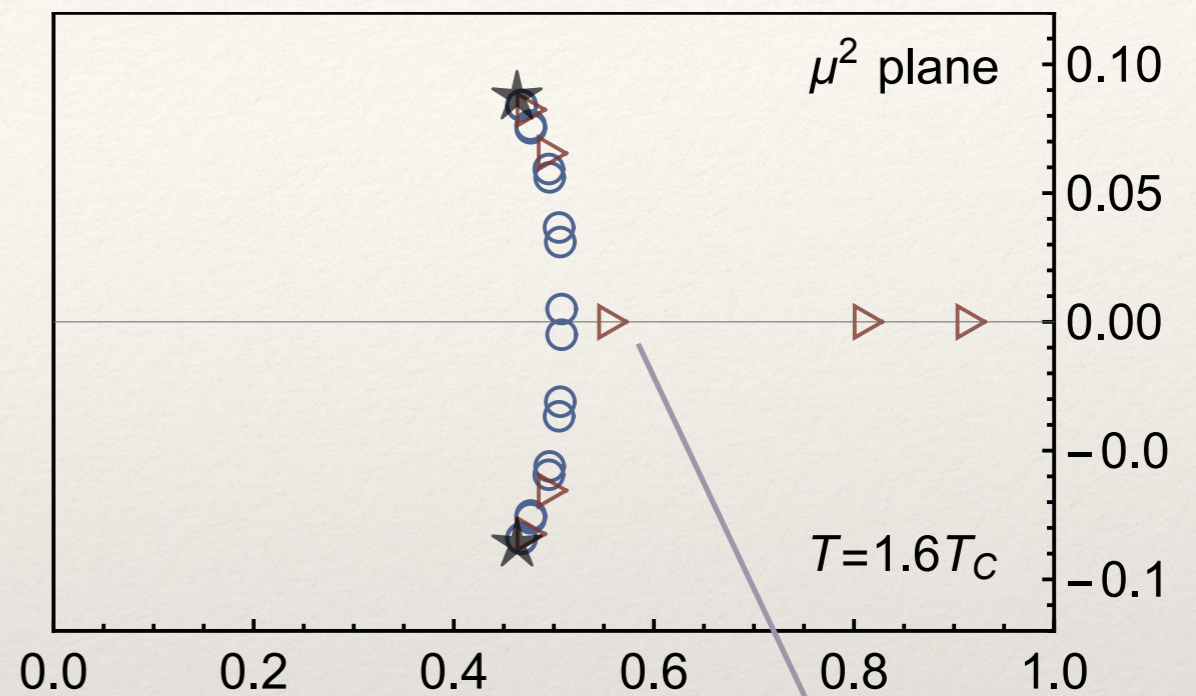
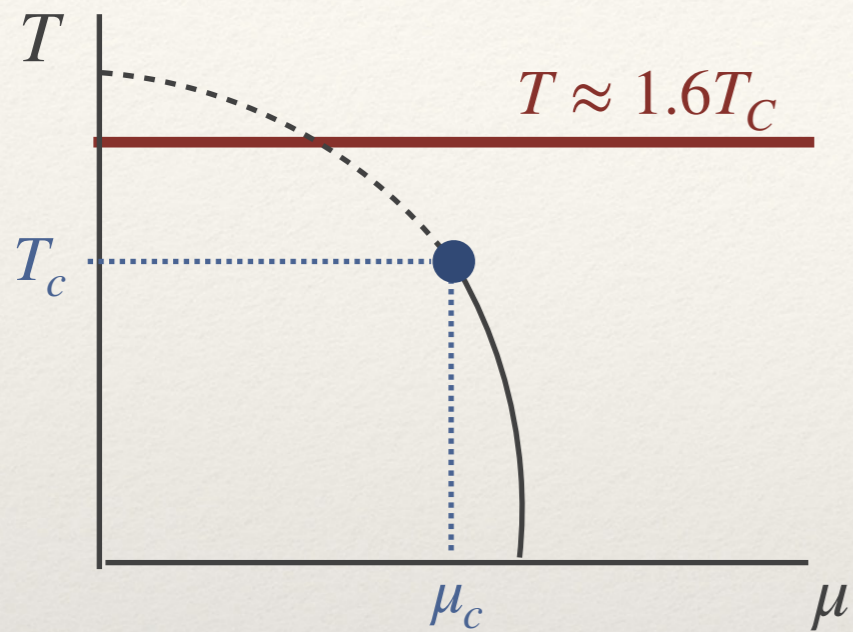
Results: susceptibility



Padé doesn't "see"

$\text{Im}\mu_{LY}^2$

Results: susceptibility

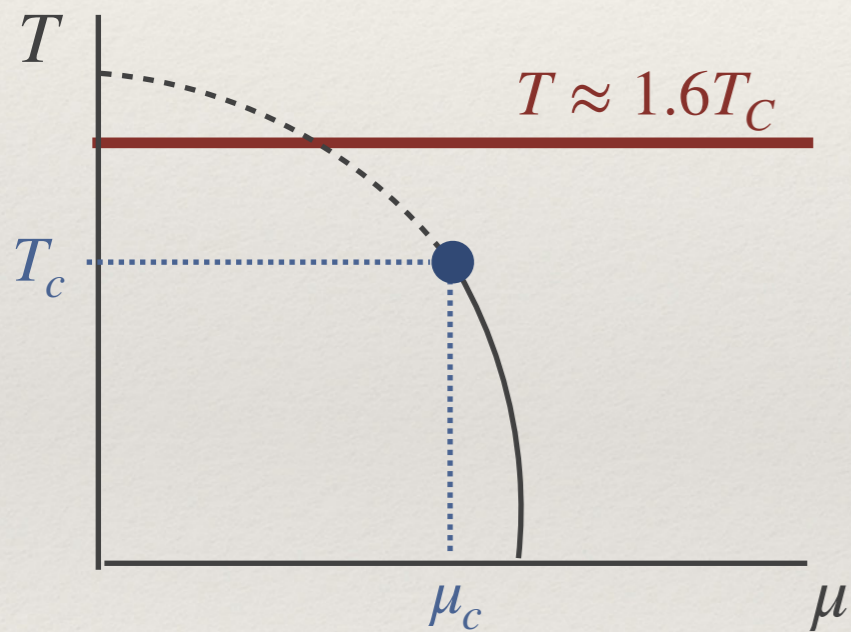


Spurious poles

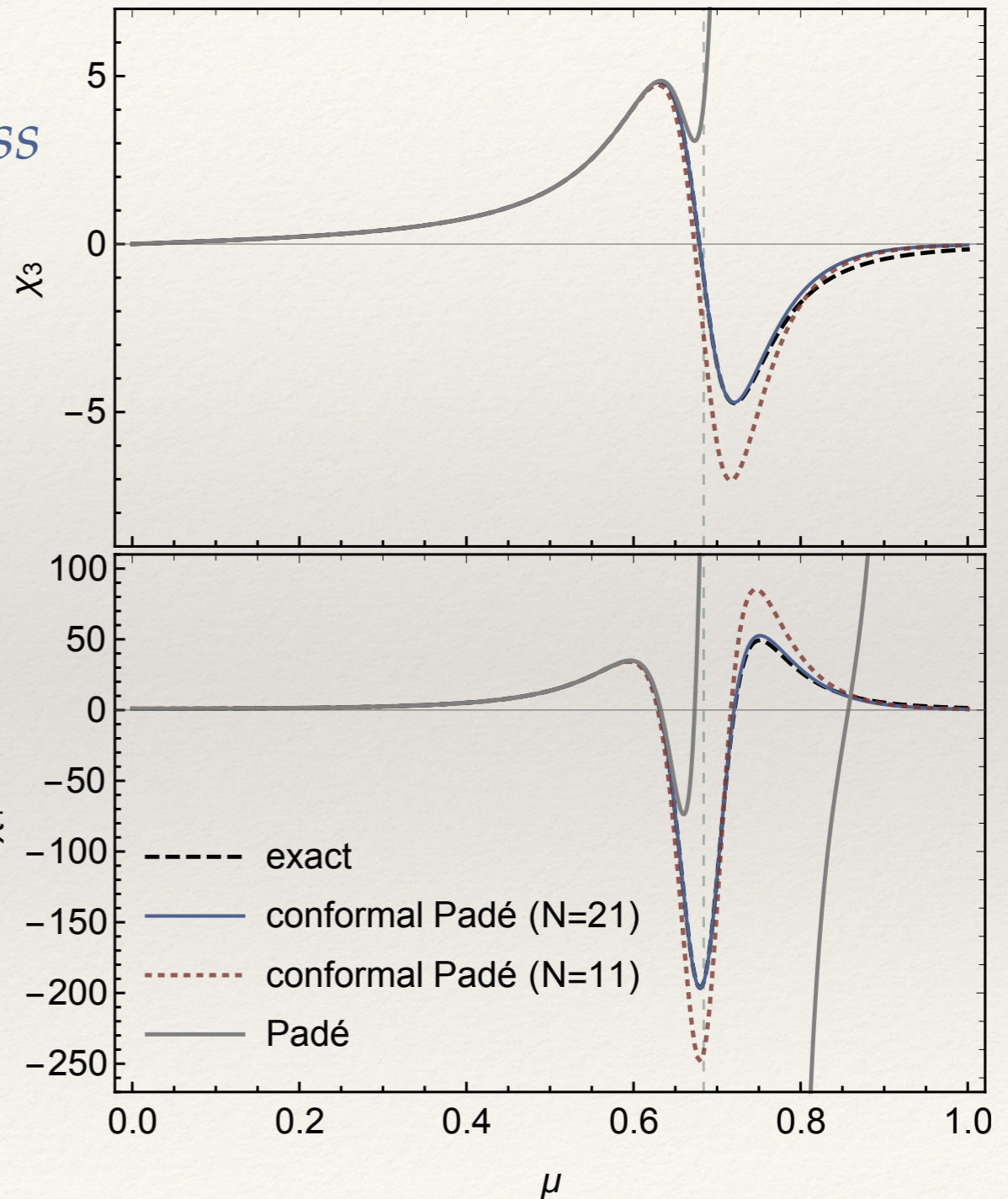
Padé fails beyond

$$\mu^2 \gtrsim \text{Re}\mu_{LY}^2$$

Results: skewness, kurtosis



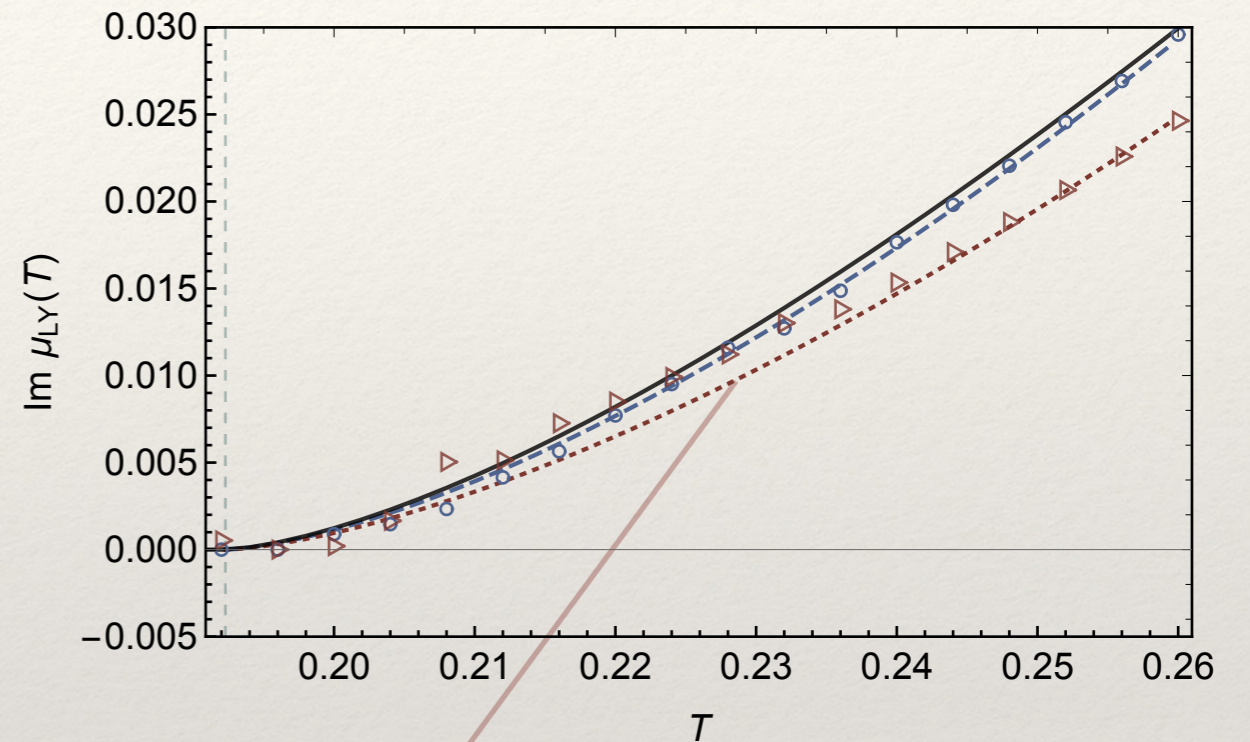
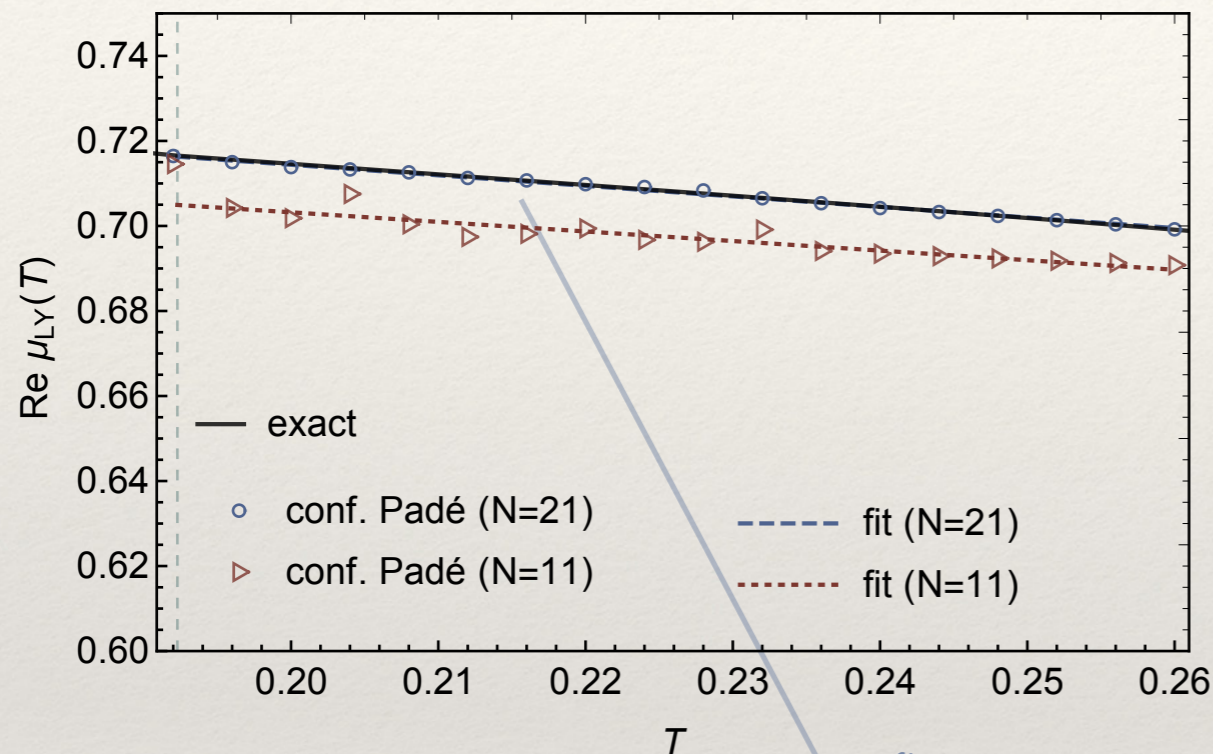
Skewness



Kurtosis

Results

- Find $\mu_{LY}^2(T)$ from poles of the conformal- Padé approximants



$$\mu_{LY}(T) \approx \mu_c - \frac{h_T}{h_\mu}(T - T_c) + iw_{LY} \frac{r_\mu^{3/2}}{h_\mu} \left(\frac{r_T}{r_\mu} - \frac{h_T}{h_\mu} \right)^{3/2} (T - T_c)^{3/2} \quad w_{LY} = \frac{2}{3\sqrt{3}}$$

- Extract μ_c, T_c , crossover slope, $\frac{h_T}{h_\mu}$, and $\frac{r_\mu^{3/2}}{h_\mu} \left(\frac{r_T}{r_\mu} - \frac{h_T}{h_\mu} \right)^{3/2}$

Results

$$\mu_{LY}(T) \approx \mu_c - \frac{h_T}{h_\mu}(T - T_c) + iw_{LY} \frac{r_\mu^{3/2}}{h_\mu} \left(\frac{r_T}{r_\mu} - \frac{h_T}{h_\mu} \right)^{3/2} (T - T_c)^{3/2} \quad w_{LY} = \frac{2}{3\sqrt{3}}$$

	T_c	μ_c	h_T/h_μ	c
<i>exact</i>	0.192	0.717	0.249	4.684
<i>conf. Padé (N=21)</i>	0.195	0.716	0.248	4.323
<i>conf. Padé (N=11)</i>	0.185	0.707	0.225	3.666

Conclusions and Outlook

- Combined with conformal maps, Padé approximants provide a powerful tool to extract information from truncated Taylor series.
- In the crossover region by using this tool it is possible to pin down the location of the *Lee-Yang edge singularity* and also extract information on the *mapping parameters to critical Ising e.o.s.*
- Conformal Padé gives a significantly better approximation to the e.o.s than than the Taylor series.
- All of this comes with no more cost than the Taylor series!