Particle density probability distribution function and center symmetry breaking in finite density lattice gauge theories

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2021.7.27

Fluctuations in a heat bath generated by Heavy Ion C.
indicator for finding the QCD critical point at finite density

- Particle density probability distribution function $W(N)$
  - N : number of particles generated in a heat bath:

$$Z_{GC}(T,\mu) = \sum_N Z_C(T,N)\exp(N\mu/T) \equiv \sum_N W(N)$$

Grand partition function　Canonical partition function　Fugacity　Provability distribution

- Symmetry that breaks in the confinement phase transition : center symmetry
  - Canonical partition function of QCD $Z_C(T,N)$ is zero except N is a multiple of 3
  - In the case of U(1) lattice gauge theory, strictly $Z_C = 0$ except $N = 0$

$$Z_C(N,T) \Rightarrow e^{iN\theta} Z_C(N,T)$$

- In this talk, we focus on U(1) lattice gauge theory.
- Calculate Z_C by adding a small external field to break the center symmetry.
- We propose a method to avoid the sign problem using the U(1) symmetry.

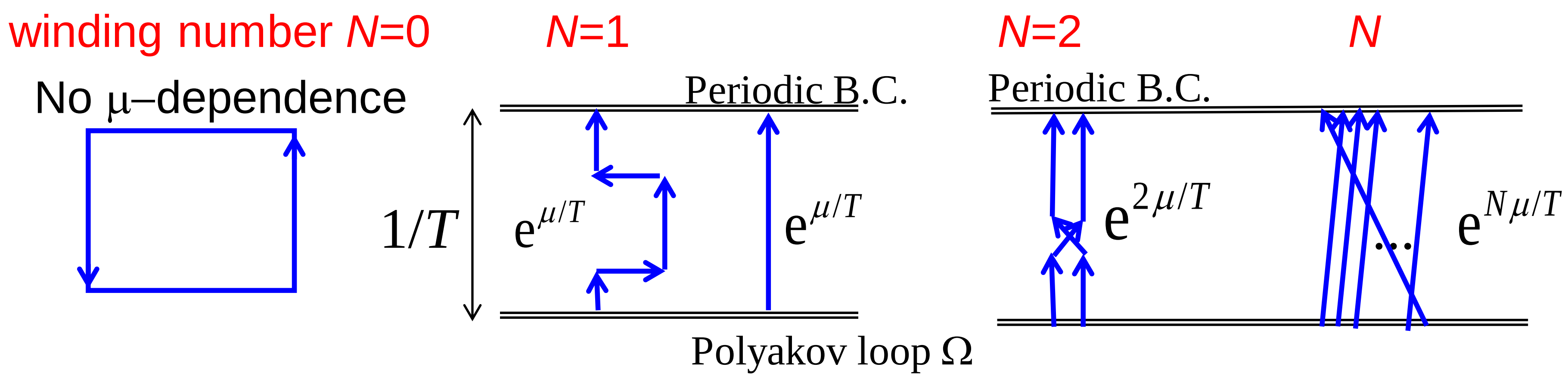
Distribution function $$-\ln W(N) = -\ln Z_C(T,N) - N\frac{\mu}{T}$$

Hopping parameter expansion, Fugacity expansion

- Grand partition function $Z_{GC} = \int \prod_{x,\mu} dU_\mu(x)\ (\det M)^{N_f}\ e^{-S_g}$ ← fermion matrix
- Hopping parameter expansion [K~1/(fermion mass)]

$$\ln(\det M(K)) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[\frac{\partial^n (\ln \det M)}{\partial K^n} \right]_{K=0} K^n = \sum_{n=1}^{\infty} \frac{1}{n!} D_n K^n$$

- D_n: Sum of all n-step (connected) Wilson loops



- Classify the Wilson loops by the winding number.
- Fugacity expansion: expansion with the winding number N.

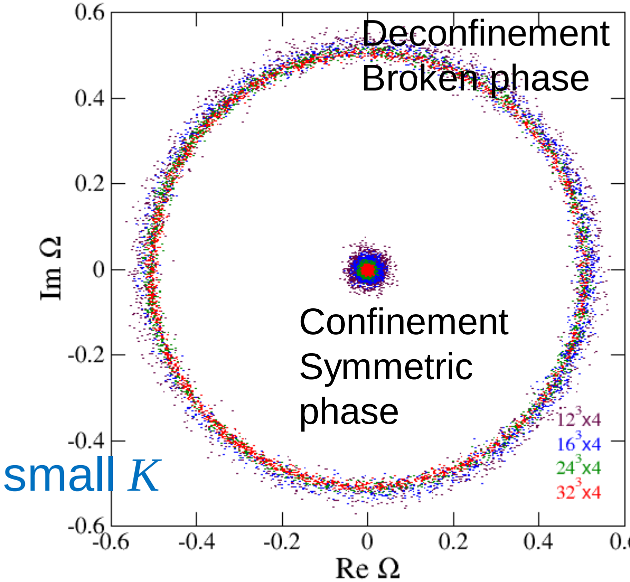
$$Z_{GC}(T,\mu) = \sum_N Z_C(N,T)\exp(N\mu/T)$$

e.g. $Z_C(1,T) \propto K^{N_t} \langle \Omega \rangle_{\text{quench}}$ for small K

Center symmetry in U(1) gauge theory

- Expectation value of Polyakov loop $\langle \Omega \rangle = 0$
- The distribution is always U(1) symmetry.
- Under the center transformation $\langle \Omega \rangle \Rightarrow e^{i\theta} \langle \Omega \rangle$
- Canonical partition function

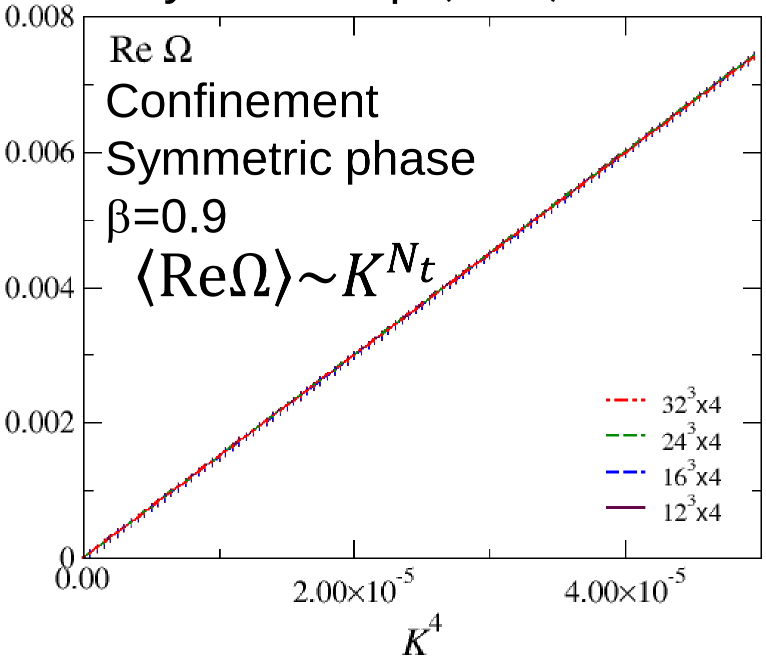
$$Z_C(N,T) \Rightarrow e^{iN\theta} Z_C(N,T)$$

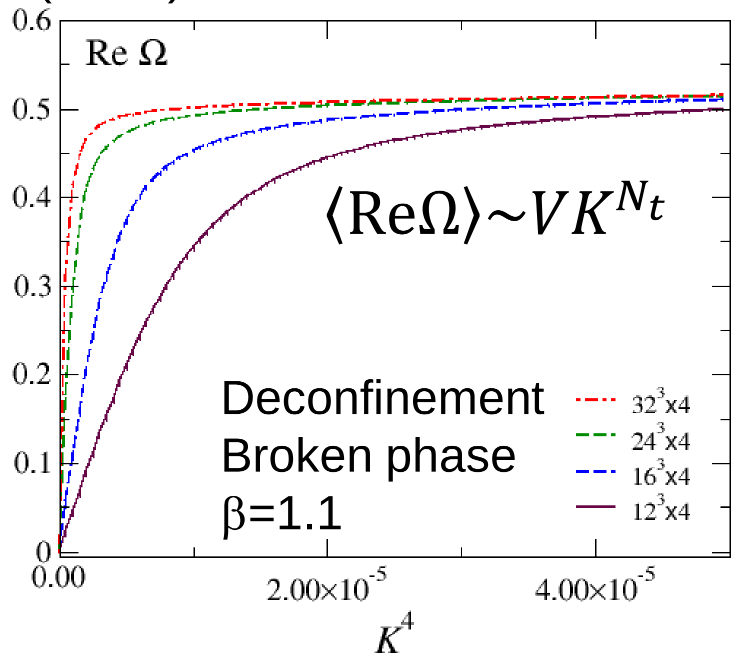


Polyakov loop Ω distribution

Break the symmetry adding dynamical fermion with a small K

Polyakov loop $\langle \mathrm{Re}\Omega \rangle$ K~1/(mass): small

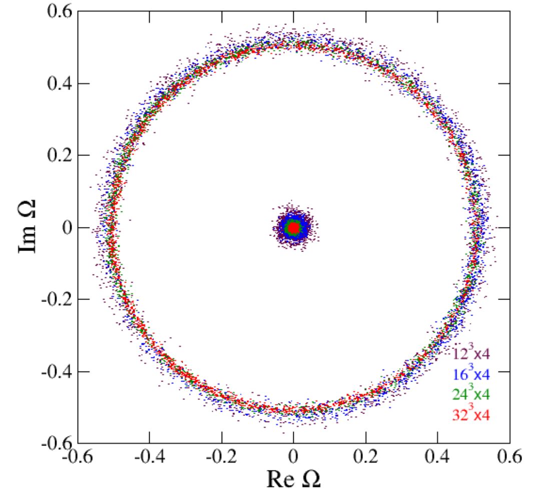




Lattice
Nt=4
Ns=12
Ns=16
Ns=24
Ns=32

$V \to \infty$, $K = 0$ limit, Symmetric phase: $\langle \mathrm{Re}\Omega \rangle = 0$, Broken phase: $\langle \mathrm{Re}\Omega \rangle \sim V K^{N_t}$ (finite)

Integrate over the complex phase for small K



- The distribution function $W(|\Omega|)$

  U(1) symmetry

  → A function of only $|\Omega|$, independent of θ

$$\ln \det M = 6 \times 2^{N_t} N_s^3 K^{N_t} (\Omega + \Omega^*) + \cdots$$

$$\langle \mathrm{Re}\Omega \rangle = \frac{1}{Z} \int DU \, \mathrm{Re}\Omega e^{\varepsilon V \mathrm{Re}\Omega} e^{-S_g} = \int |\Omega| \cos\theta e^{\varepsilon V |\Omega| \cos\theta} W(|\Omega|) d\theta d|\Omega|$$

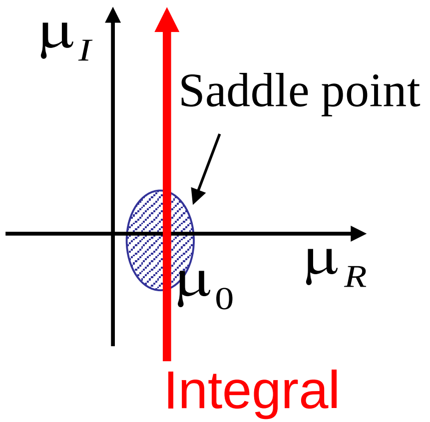
$$= \varepsilon V \pi \int |\Omega|^2 W(|\Omega|) \, d|\Omega| + \cdots \qquad [\ \varepsilon \sim O K^{N_t}\ ] \qquad V = N_s^3$$

$$\langle \mathrm{Re}\Omega^n \rangle = \frac{1}{Z} \int DU \, \mathrm{Re}\Omega^n e^{\varepsilon V \mathrm{Re}\Omega} e^{-S_g} = \frac{(\varepsilon V)^n \pi}{n! \, 2^{n-1}} \int |\Omega|^{2n} W(|\Omega|) d|\Omega| + \cdots$$

No complex phase, No sign problem

$\dfrac{\langle \mathrm{Re}\Omega^4 \rangle}{\langle \mathrm{Re}\Omega^2 \rangle^2}$ The leading term does not depend on εV.

Canonical partition function by Saddle point approximation (S.E., Phys. Rev. D78, 074507 (2008))



- Inverse Laplace transformation

$$Z_C(T,N) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d(\mu_I/T)\, e^{-N(\mu_0/T+i\mu_I/T)} Z_{GC}(T,\mu_0+i\mu_I)$$

- Saddle point: z_0 $\quad D'(z_0) = \left(\frac{N_f}{V}\frac{\partial(\ln\det M)}{\partial(\mu/T)}\right)_{\frac{\mu}{T}=z_0} = \rho$ density

Arbitrary μ_0

- Canonical partition function in a saddle point approximation

$$\frac{-1}{V}\frac{\partial \ln Z_C(T,\rho V)}{\partial \rho} \approx \frac{\langle z_0\, e^{F+i\varphi}\rangle_{\text{quench}}}{\langle e^{F+i\varphi}\rangle_{\text{quench}}}$$

Similar to the reweighting method
(sign problem & overlap problem)

saddle point reweighting factor

- At $W(N)$ maximum ($-\ln W(N)$ minimum),

$$-\frac{\partial \ln W(N)}{\partial N} = -\frac{1}{V}\frac{\partial \ln Z_C(T,\rho V)}{\partial \rho} - \frac{\mu}{T} = 0$$

μ/T where the number of particles with the maximum generation probability is N

Heavy fermions (K small) U(1) theory

Approximation: $\ln \det M \approx 6 \times 2^{N_t} N_s^3 K^{N_t} \left(e^{\mu/T} \Omega + e^{-\mu/T} \Omega^* \right)$

$$\frac{-1}{V} \frac{\partial \ln Z_C(T,\rho)}{\partial \rho} \approx \frac{\langle z_0 \, e^{F+i\varphi} \rangle_{\text{quench}}}{\langle e^{F+i\varphi} \rangle_{\text{quench}}} \approx \frac{\varepsilon^N \int x_0 \, e^{-V_{\text{eff}}} d|\Omega|}{\varepsilon^N \int e^{-V_{\text{eff}}} d|\Omega|} \qquad z_0 = x_0 + i y_0$$

U(1) symmetric

$\varphi = -N\theta$ $\quad (\Omega = |\Omega| e^{i\theta})$

Phase of Ω

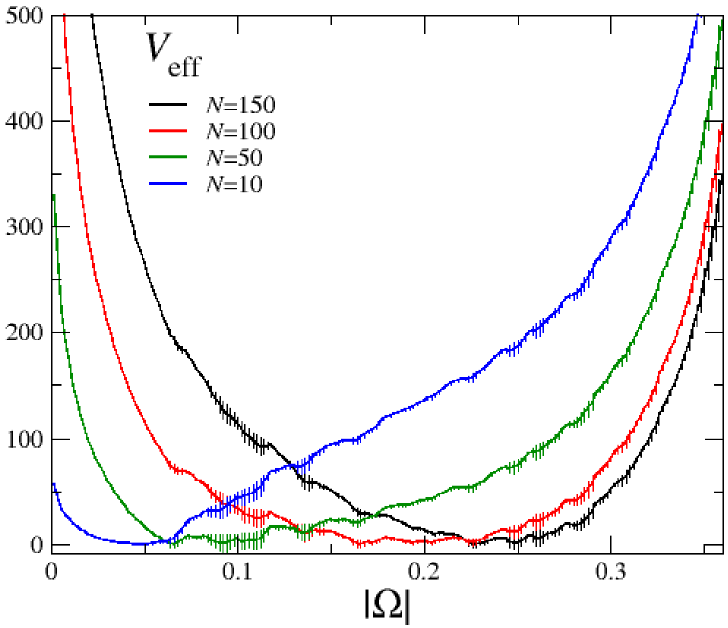
$$\langle e^{F+i\varphi} \rangle = \int e^F \cos(N\theta) \, e^{\varepsilon V |\Omega| \cos\theta} W(|\Omega|) \, d\theta \, d|\Omega|$$

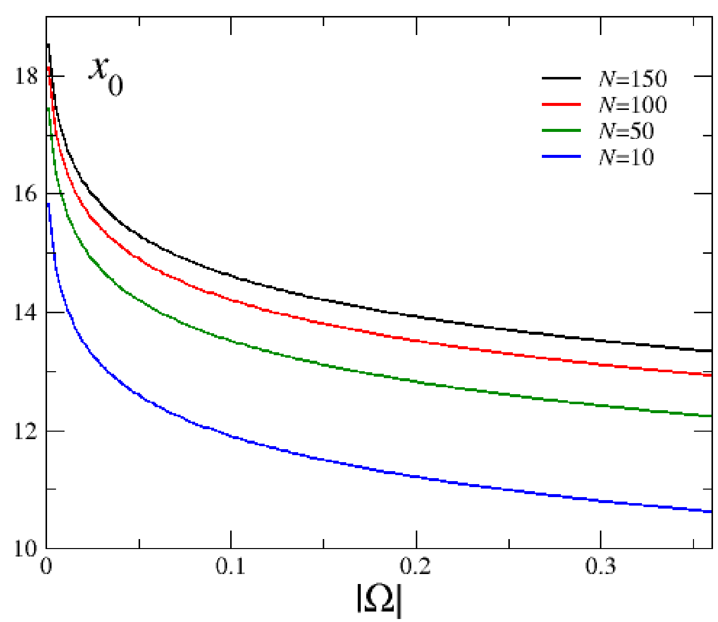
External source

$N = \rho V$

$(N_f = 2, K = 0.02) \quad (24^3 \times 6 \text{ lattice})$

Solving the sign problem





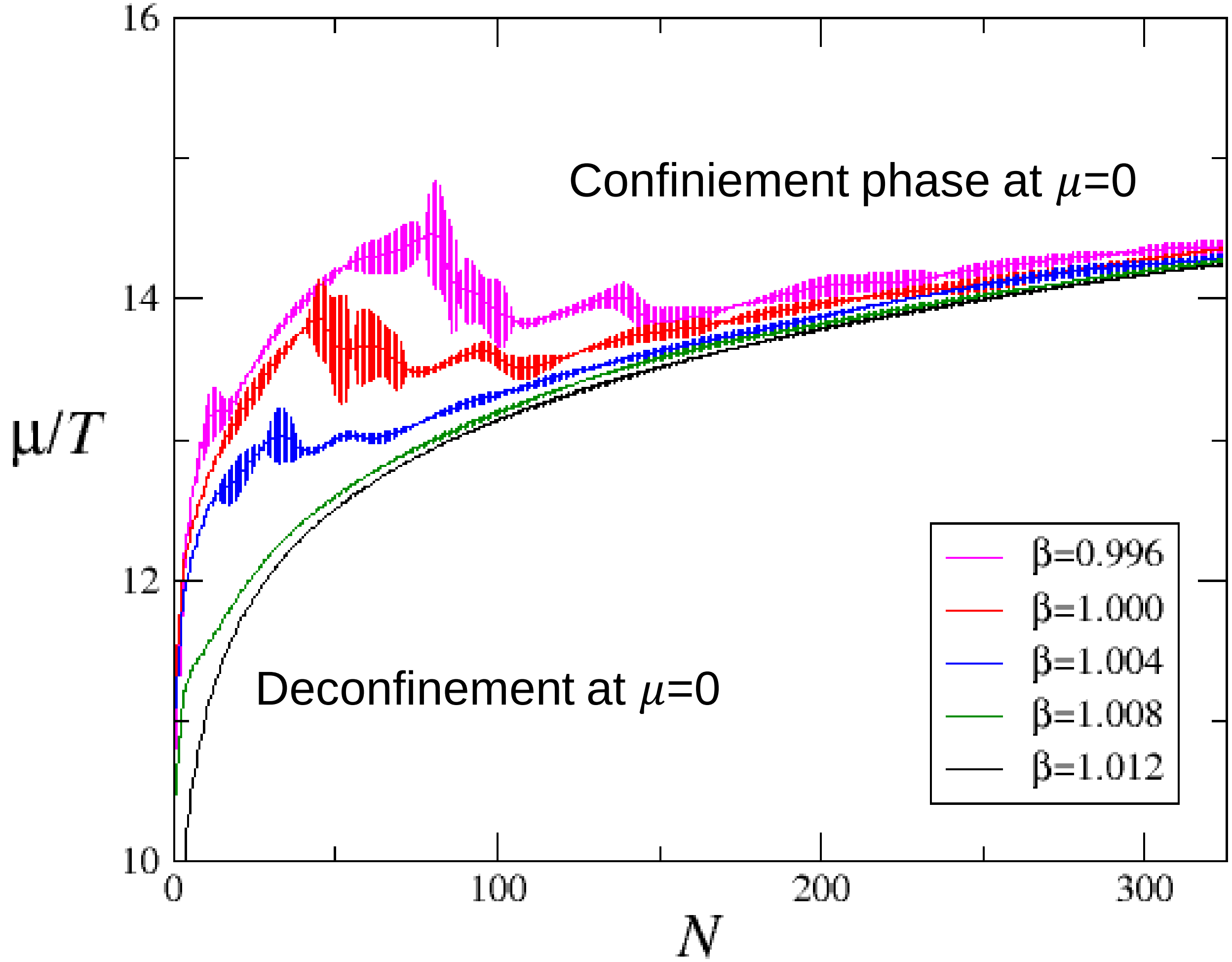
Quark number N vs. $\frac{\mu}{T}$

Thermodynamic limit,

$$\frac{-1}{V}\frac{\partial \ln Z_C(T,\rho)}{\partial \rho} = \frac{\mu}{T}$$

μ/T where the probability is maximum at N

$$\frac{\partial V_{\rm eff}(N)}{\partial N} = -\frac{\partial \ln Z_C(T,N)}{\partial N} - \frac{\mu}{T} = 0$$



Critical β at μ=0

$\beta_c = 1.0096$

$(N_{\rm f} = 2, K = 0.02)$

$(24^3 \times 6$ lattice)

$N = \rho V$

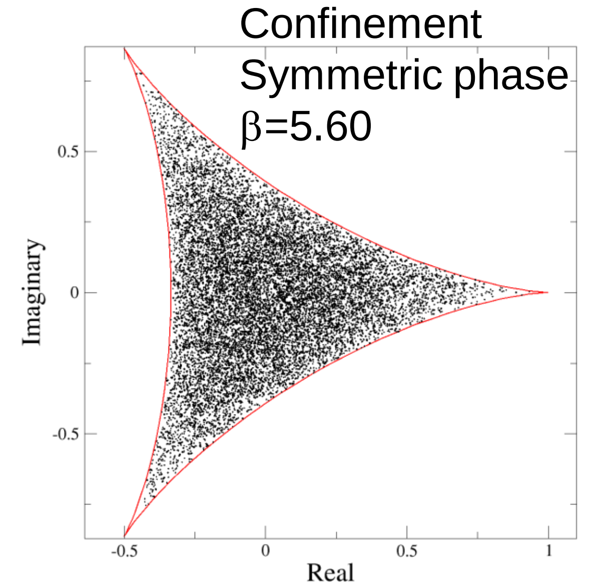
Application to SU(3) gauge theory

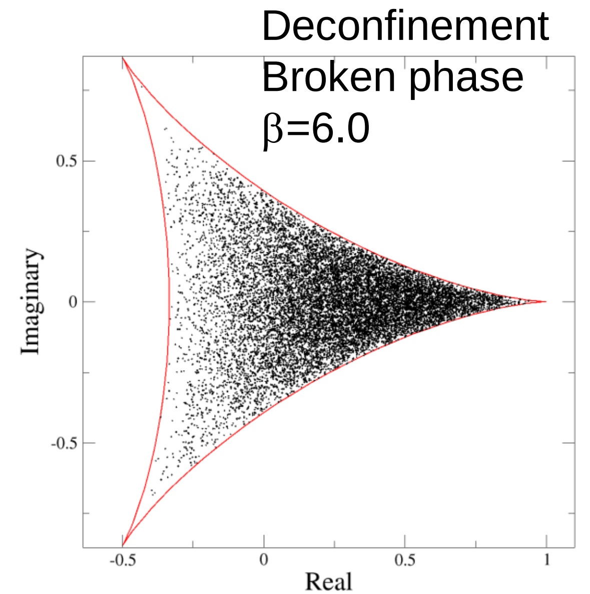
- <u>In the low temperature phase of SU(3)</u>
  - Local Polyakov loop has Z(3) center symmetry.
    (No spatial average)
  - The distribution of the spatial average of Ω has U(1) (not Z(3)) in the large volume limit, if the Z(3) center symmetry is unbroken (low T).
  - Central limit theorem: Gaussian distribution
  - The variances for the real and imaginary directions, $\overline{x^2}$ and $\overline{y^2}$, are the same.

  → U(1) symmetric Gaussian distribution:

$$W(|\Omega|, \theta) \sim e^{-a(x^2+y^2)} = e^{-a|\Omega|^2}$$

  - The same method as for U(1) can be used to solve the sign problem.
- <u>In the high temperature phase of SU(3)</u>
  - Local Polyakov loop does not have Z(3) center symmetry.
  - However, the sign problem is not serious.





$(24^3 \times 4 \text{ lattice})$

Summary

- Quark number distribution function $W(N)$

$$Z_{GC}(T,\mu) = \sum_N Z_C(T,N)\exp(N\mu/T) \equiv \sum_N W(N)$$

- Considering the Center symmetry,
  - Canonical partition function of QCD is zero for $N \neq 3n \ \ (n\text{: integer})$.
  - For U(1) gauge theory, $Z_C = 0$ for $N \neq 0$.
- It is important to break the center symmetry adding an external field term.

U(1) gauge theory

- Using the U(1) center symmetry, complex phase can be removed.
- When the particles are heavy, we calculated the canonical partition function and illustrated how to avoid the $Z_C = 0$ problem and the sign problem.

Application to SU(3) gauge theory

- Confinement phase: the same method as for U(1) can be used to solve the sign problem.
- Deconfinement phase: the sign problem is not serious.