Heavy-dense QCD at fixed baryon number without a sign problem

Urs Wenger

Albert Einstein Center for Fundamental Physics
University of Bern

b
UNIVERSITÄT
BERN

in collaboration with Patrick Bühlmann

LATTICE 2021, 26 July 2021
ZOOM/GATHER@MIT Boston, USA

Motivation for the canonical formulation

- Consider the grand-canonical partition function at finite μ:

$$Z_{\mathrm{GC}}(\mu) = \mathrm{Tr}\left[e^{-\mathcal{H}(\mu)/T}\right] = \mathrm{Tr}\prod_t \mathcal{T}_t(\mu)$$

- The sign problem at finite density is a manifestation of huge cancellations between different states:
  - all states are present for any μ and T
  - different states need to cancel out at different μ and T

- In the canonical formulation:

$$Z_{\mathrm{C}}(N_q) = \mathrm{Tr}_{N_q}\left[e^{-\mathcal{H}/T}\right] = \mathrm{Tr}\prod_t \mathcal{T}_t^{(N_q)}$$

  - dimension of Fock space tremendously reduced
  - less cancellations necessary:
  - e.g. $Z_{\mathrm{C}}^{SU(N_c)}(N_q) = 0$ for $N_q \neq 0 \bmod N_c$ in QCD

Motivation for the canonical formulation

- Consider the grand-canonical partition function at finite μ:

$$Z_{\mathrm{GC}}(\mu) = \mathrm{Tr}\left[e^{-\mathcal{H}(\mu)/T}\right] = \mathrm{Tr}\prod_t \mathcal{T}_t(\mu)$$

- The sign problem at finite density is a manifestation of huge cancellations between different states:
  - all states are present for any μ and T
  - different states need to cancel out at different μ and T

- In the canonical formulation:

$$Z_{\mathrm{C}}(N_q) = \mathrm{Tr}_{N_q}\left[e^{-\mathcal{H}/T}\right] = \mathrm{Tr}\prod_t \mathcal{T}_t^{(N_q)}$$

  - dimension of Fock space tremendously reduced
  - less cancellations necessary:
  - e.g. $Z_{\mathrm{C}}^{U(1)}(N_q) = 0$ for $N_q \neq 0$ in the Schwinger model

[cf. P. Bühlmann, Tue 13:15, (Hadron Spectroscopy)]

Motivation for the canonical formulation

- Consider the grand-canonical partition function at finite μ:

$$Z_{\mathrm{GC}}(\mu) = \mathrm{Tr}\left[e^{-\mathcal{H}(\mu)/T}\right] = \mathrm{Tr}\prod_t \mathcal{T}_t(\mu)$$

- The sign problem at finite density is a manifestation of huge cancellations between different states:
  - all states are present for any μ and T
  - different states need to cancel out at different μ and T

- In the canonical formulation:

$$Z_{\mathrm{C}}(N_q) = \mathrm{Tr}_{N_q}\left[e^{-\mathcal{H}/T}\right] = \mathrm{Tr}\prod_t \mathcal{T}_t^{(N_q)}$$

  - dimension of Fock space tremendously reduced
  - less cancellations necessary:
  - e.g. "Silver Blaze" phenomenon realised automatically

Fermion matrix and canonical determinants

Canonical transfer matrices are known explicitly! [Steinhauer, UW '15]

- Fugacity expansion

$$\det M[\mathcal{U};\mu] = \sum_{N_q} e^{-N_q\,\mu_q/T} \cdot \det{}_{N_q} M[\mathcal{U}]$$

yields the canonical determinants

$$\det{}_{N_q} M[\mathcal{U}] = \sum_{J} \det \mathcal{T}^{JJ}[\mathcal{U}] = \mathrm{Tr} \prod_{t} \mathcal{T}_t^{(N_q)} = \mathrm{Tr}\, \mathcal{M}_{N_q}$$

as the trace over the minor matrix $\mathcal{M}_{N_q}$ of order N_q of $M[\mathcal{U}]$.

Heavy-dense limit of QCD

- The heavy-dense approximation in general consists of taking the limit $\kappa \equiv (2m+8)^{-1} \to 0$, $\mu \to \infty$ while keeping $\kappa e^{+\mu}$ fixed.

- Better: just drop the spatial hopping terms, but keep forward and backward hopping in time:
  - system of static quarks and antiquarks

- 3-dim. eff. ferm. action in terms of Polyakov loops P and $P^\dagger$:

$$\det M_{GC}^{HD} = \prod_{\bar{x}} \det \left[\mathbb{I} - (2\kappa e^{+\mu})^{N_t} P_{\bar{x}} \right]^2 \det \left[\mathbb{I} - (2\kappa e^{-\mu})^{N_t} P_{\bar{x}}^\dagger \right]^2$$

Heavy-dense limit of canonical QCD

- The canonical determinants are given by the trace over the minor matrix $\mathcal{M}_{N_q}$,

$$\det{}_{N_q} M^{HD} \propto \mathrm{Tr}\,\mathcal{M}_{N_q}\left[\left((2\kappa)^{+N_t}\cdot P_+\mathcal{P} + (2\kappa)^{-N_t}\cdot P_-\mathcal{P}\right)\right]$$

where $\mathcal{P}$ denotes the Polyakov loops $\mathcal{P}_{\bar{x},\bar{y}} = \mathbb{I}_{4\times 4}\otimes P_{\bar{x}}\cdot\delta_{\bar{x},\bar{y}}$.

- Structure of $\mathcal{M}_{N_q}$ directly gives $\det_{N_q} M^{HD} \rightarrow z_k^{-N_q}\cdot\det_{N_q} M^{HD}$ under global $z_k \in \mathbb{Z}_{N_c}$, i.e.,

$$\det{}_{N_q} M = 0 \;\text{ for } N_q \neq 0 \mod N_c \quad \Leftrightarrow \quad \text{cancellations!}$$

Heavy-dense limit of canonical QCD

- Canonical determinants in heavy-dense limit:

$$\det_{N_q} M^{HD} \quad \sim \quad \mathrm{Tr}\, P_{\bar{x}}^{\dagger}\, \mathrm{Tr}\, P_{\bar{y}} \quad \text{and} \quad \mathrm{Tr}\, P_{\bar{x}}\, \mathrm{Tr}\, P_{\bar{y}}\, \mathrm{Tr}\, P_{\bar{z}}$$

  - invariant under global $\mathbb{Z}_3$-transformations
  - describes the propagation of mesons and baryons
  - description in terms of (anti-)quark occupation numbers $n_{\bar{x}}$

- System suffers from a severe sign problem, unless
  - all $P_{\bar{x}}$ align $\iff$ deconfined phase
  - global $\mathbb{Z}_3$ is promoted to a local one $\iff$ strong coupling

The heavy-dense strong coupling limit $\beta \to 0$

- In the strong coupling limit the global $\mathbb{Z}(N_c)$-transformations are promoted to local ones
- Partition function becomes a summation over all baryon configurations $n_B(\bar{x})$ with (essentially) positive contributions:

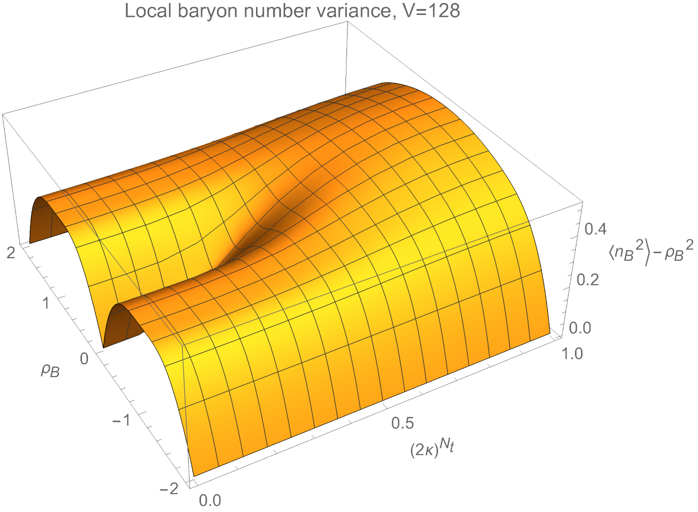
$$Z_C(N_B) = (2\kappa)^{2N_c N_t L_s^3} \cdot \sum_{\{n_B\},|n_B|=N_B} \int \mathcal{D}U \prod_{\bar{x}} \det \mathcal{M}^{HDSS}_{n_B(\bar{x})}[\mathrm{Tr}\, P_{\bar{x}}]$$

  - single site weights are polynomials of $\mathrm{Tr}\, P_{\bar{x}}$ and $\mathrm{Tr}\, P^\dagger_{\bar{x}}$
  - $\mathcal{D}U$ can of course be integrated analytically,
  - but also possible to simulate by Monte Carlo

Sign problem is solved in the strong coupling limit!

Heavy-dense strong coupling without a sign problem

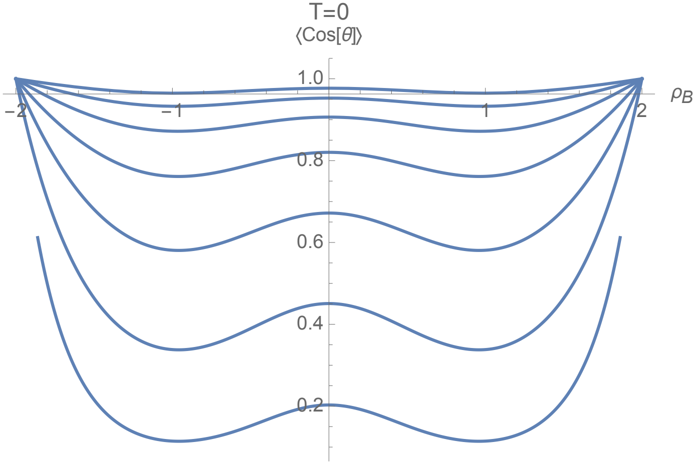
- E.g. baryon number fluctuations (or susceptibility):



Anatomy of the heavy-dense sign problem

- Average sign for canonical ensemble for various volumes V:

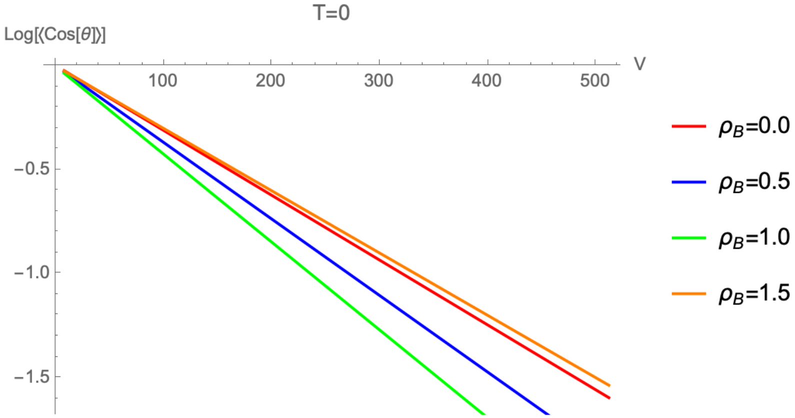
$$Z_C(N_B)_{|.|}/Z_C(N_B) = \langle \cos\theta \rangle$$



Anatomy of the heavy-dense sign problem

- Severity of the sign problem for canonical ensemble:

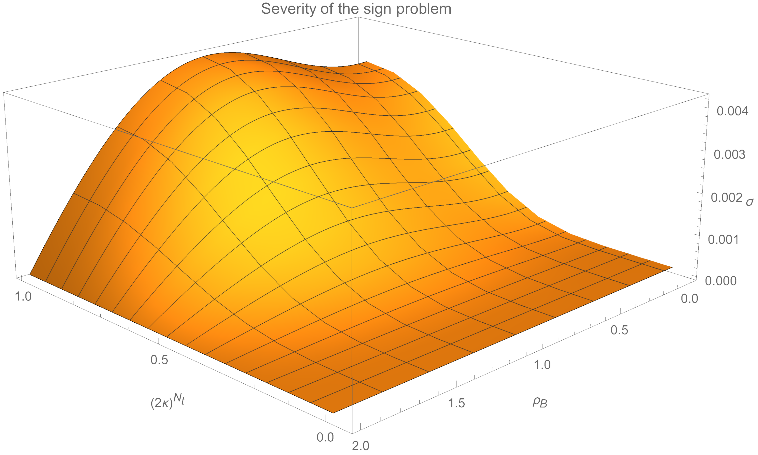
$$Z_C(N_B)_{|.|}/Z_C(N_B) = \exp[-\sigma \cdot V] \quad \text{with} \quad \sigma = \Delta f/T$$



Anatomy of the heavy-dense sign problem

- Severity of the sign problem for canonical ensemble:

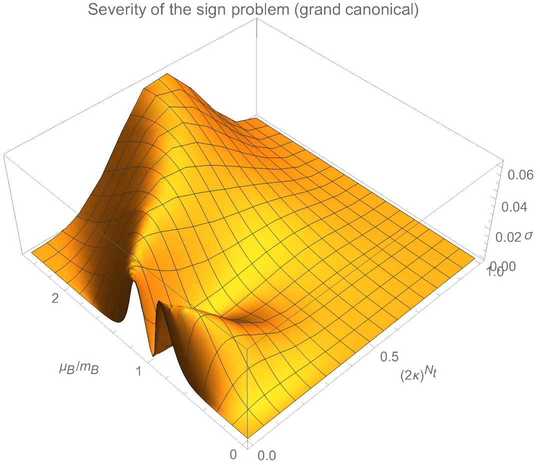
$$Z_C(N_B)_{|.|}/Z_C(N_B) = \exp[-\sigma \cdot V] \quad \text{with} \quad \sigma = \Delta f/T$$



Anatomy of the heavy-dense sign problem

- Severity of the sign problem for grandcanonical ensemble:

$$Z_{GC}(N_B)_{|.|}/Z_{GC}(N_B) = \exp[-\sigma \cdot V] \quad \text{with} \quad \sigma = \Delta f / T$$



Beyond strong coupling

- The effective gauge action [Fromm et al. '12]

$$\exp\left(-S_{\text{eff}}[\mathcal{U}]\right) = \prod_{<\bar{x}\bar{y}>} \left(1 + 2\,\lambda\, \text{Re Tr}\, P_{\bar{x}}\, \text{Tr}\, P_{\bar{y}}^{\dagger}\right)$$

induces flux $0, \pm 1 \propto \lambda$ on bonds.

- Description in terms of (anti-)quark and bond occupation numbers $n_{\bar{x}}$ and n_b
  - quark and antiquarks can now separate,
  - only contributions with triality-0 survive:

$\Rightarrow$ local flux conserved modulo 3 and $n_{\bar{x}}$

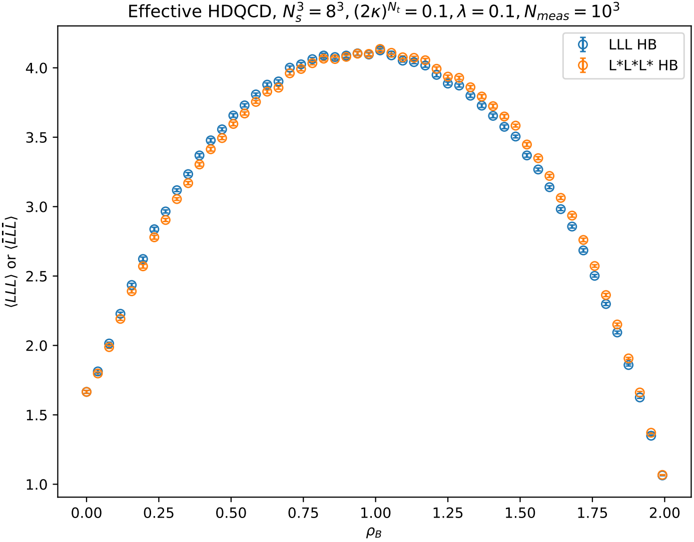
  - $\mathcal{DU}$ can be integrated analytically:

Sign problem is solved beyond strong coupling

  - MC simulations in terms of integer occupation numbers

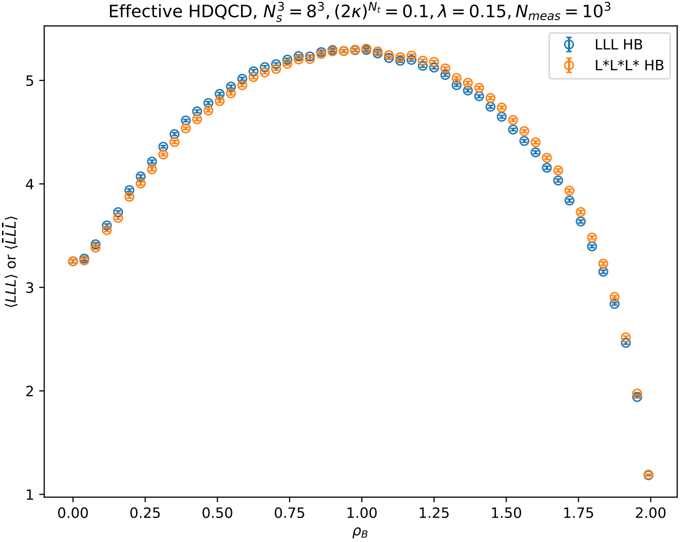
Beyond strong coupling

- Static (anti-)baryon in finite baryon density background:



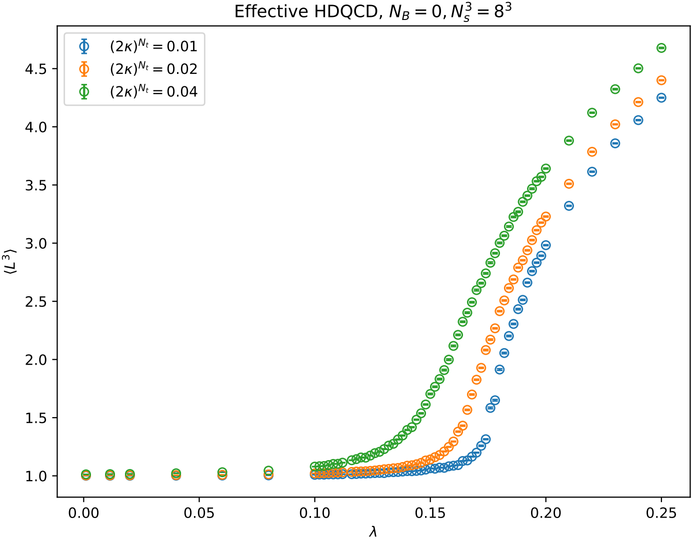
Beyond strong coupling

- Static (anti-)baryon in finite baryon density background:



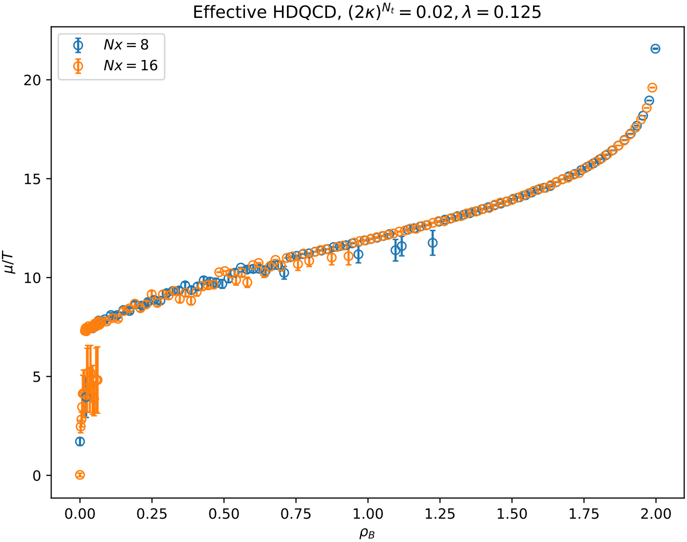
Beyond strong coupling

- Static (anti-)baryon at zero density for various couplings:



Beyond strong coupling

- Quark chemical potential vs. baryon density:



Summary and outlook

- Canonical formulation approach to finite density:
  - alternative views on interesting physics
  - for heavy-dense QCD at large gauge coupling:

$\Rightarrow$ fermion sign problem can be solved

- The solution provides an appealing physical picture:
  - quarks confined in clusters
  - similar mechanism as in $\mathbb{Z}_3$ Potts model
  - $\mathbb{Z}_3$ cluster algorithm can be extended to heavy-dense QCD:

$\Rightarrow$ $\mathbb{Z}_3$ clusters for gauge fields at any coupling