

On the three-particle analog of the Lellouch-Lüscher formula

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- Motivation
- Non-relativistic EFT framework
 - Lagrangian
 - Three-particle quantization condition
- Derivation of LL formula
 - Strategy
 - Infinite volume
 - Finite volume
- Conclusion and Outlook

Growing interest in three particle decays:

- $K \rightarrow 3\pi, \eta \rightarrow 3\pi, \omega \rightarrow 3\pi$
- exotica $X(3872)$ and $X(4260)$ largely decay into three particle final states
- Roper resonance
- $\gamma^* \rightarrow 3\pi$ contributing to $g - 2$ of muon

infinite volume:

- bound states, elastic scattering
- resonance matrix elements : $\langle \pi\pi\pi | H_W | K \rangle$

finite volume:

- two- and three-particle energy levels
- matrix elements between eigenstates

How are these two sets related?

In two-particle sector: [Lellouch & Lüscher 2000]

$$\langle \pi\pi | H_W | K \rangle = \Phi_2(E_n; \delta, L) \langle n | H_W | K \rangle$$

Aim:

derive analog of this equation in three-particle sector

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Strategy:

use NREFT and match effective couplings to observables

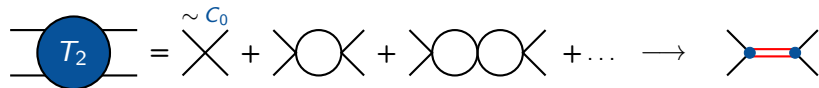
Use NREFT to describe $K \rightarrow 3\pi$ decay: [arXiv:1103.4273]

$$\mathcal{L} = \phi^\dagger \underset{w=\sqrt{m^2-\Delta}}{2w(i\partial_t - w)}\phi + \frac{C_0}{4} \phi^\dagger \phi^\dagger \phi \phi + \frac{D_0}{36} \phi^\dagger \phi^\dagger \phi^\dagger \phi \phi \phi + \frac{G_0}{6} (K^\dagger \phi \phi \phi + \text{h.c.})$$

Introduce dimer field d :

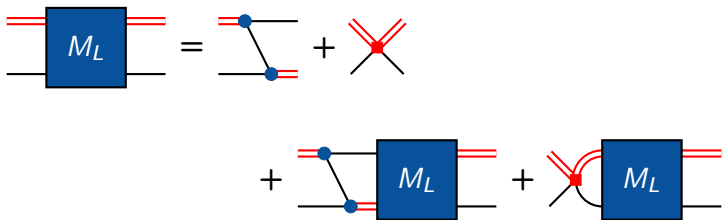
$$\mathcal{L}_{\text{int}} \rightarrow \mathcal{L}_{\text{dim}} = \sigma d^\dagger d + \frac{f_0}{2} (d^\dagger \phi \phi + \text{h.c.}) + h_0 d^\dagger d \phi^\dagger \phi + g_0 (K^\dagger d \phi + \text{h.c.})$$

Two-particle sector:



$$C_0 = -\sigma f_0^2 \text{ matched to } p \cot \delta(p) = -1/a + \dots$$

Three particle quantization condition

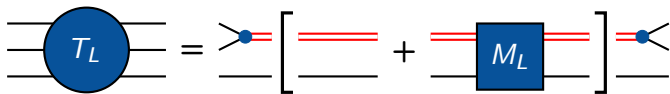


$$M_L(\mathbf{k}, \mathbf{p}; P_0) = Z(\mathbf{k}, \mathbf{p}; P_0) + \frac{1}{L^3} \sum_{\mathbf{q}}^{\Lambda} Z(\mathbf{k}, \mathbf{q}; P_0) \frac{\tau_L(\mathbf{q}; P_0)}{2w(\mathbf{q})} M_L(\mathbf{q}, \mathbf{p}; P_0)$$

$$Z(\mathbf{k}, \mathbf{p}; P_0) = \left[\frac{1}{2w(\mathbf{p} + \mathbf{k})(w(\mathbf{p} + \mathbf{k}) + w(\mathbf{k}) + w(\mathbf{p}) - P_0)} + \frac{h_0}{f_0^2} \right]$$

$$\tau_L(\mathbf{q}; P_0) = \frac{16\pi\sqrt{s}}{-16\pi\sqrt{s}\sigma f_0^{-2} + p^* \cot \phi^d(s)}, \quad s = (P_0 - w(\mathbf{q}))^2 - \mathbf{q}^2$$

Three particle quantization condition



$$T_L = \tau_L + \tau_L M_L \tau_L = (\tau_L^{-1} - Z)^{-1}$$

Quantization condition

Poles in amplitude, i.e. $\det(\tau_L^{-1} - Z) = 0$



finite volume energy spectrum

Strategy:

- C_0 in τ_L from fit in two-particle sector
- D_0 in Z extracted from fit to three-particle energies

Derivation of three-particle LL equation

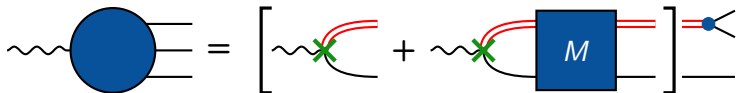
Observation:

At LO $K \rightarrow 3\pi$ matrix element $\propto G_0$, coupling **unknown** but equal (up to $\exp(-mL)$) in infinite and finite volume

Strategy:

- calculate ratio of matrix elements in finite in infinite volume
- result will only depend on C_0 and D_0 , known from **two-** and **three-particle-energies**

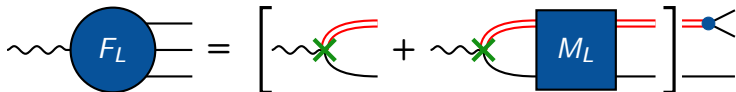
Infinite volume



$$\Gamma J_K^\dagger = g_0 d^\dagger \phi^\dagger$$

$$\langle \pi(\mathbf{k}_1) \pi(\mathbf{k}_2) \pi(\mathbf{k}_3); \text{out} | J_K^\dagger(0) | 0 \rangle$$

$$= \frac{g_0}{f_0} \sum_{\alpha=1}^3 \tau(-\mathbf{k}_\alpha; P_0) \left[1 + \int^\Lambda \frac{d^3 \mathbf{q}}{(2\pi)^3} M(-\mathbf{k}_\alpha, -\mathbf{q}; P_0) \frac{1}{2w(\mathbf{q})} \tau(-\mathbf{q}; P_0) \right]$$



On the other hand, in perturbation theory:

$$\langle 0 | \mathcal{O}(x_0; \{\mathbf{k}\}) J_K^\dagger(0) | 0 \rangle = \int \frac{dP_0}{2\pi i} \frac{e^{-iP_0 x_0} F_L(\{\mathbf{k}\}; P_0)}{8w(\mathbf{k}_1)w(\mathbf{k}_2)w(\mathbf{k}_3)(w(\mathbf{k}_1) + w(\mathbf{k}_2) + w(\mathbf{k}_3) - P_0 - i\epsilon)}$$

$$F_L(\{\mathbf{k}\}; P_0) = \frac{g_0}{f_0} \sum_{\alpha=1}^3 \tau_L(-\mathbf{k}_\alpha; P_0) \left[1 + \frac{1}{L^3} \sum_{\mathbf{q}}^{\Lambda} M_L(-\mathbf{k}_\alpha, -\mathbf{q}; P_0) \frac{1}{2w(\mathbf{q})} \tau_L(-\mathbf{q}; P_0) \right]$$

$$\Rightarrow L^{3/2} |\langle n | J_K^\dagger(0) | 0 \rangle| = \left| \frac{g_0}{f_0} \frac{1}{L^3} \sum_{\mathbf{q}}^{\Lambda} \psi_L^{(n)}(-\mathbf{q}) \frac{1}{2w(\mathbf{q})} \tau_L(-\mathbf{q}; E_n) \right|$$

$$\langle \pi(\mathbf{k}_1)\pi(\mathbf{k}_2)\pi(\mathbf{k}_3); \text{out} | J_K^\dagger(0) | 0 \rangle = \Phi_3(\{\mathbf{k}\}) \cdot L^{3/2} \langle n | J_K^\dagger(0) | 0 \rangle$$

$$\Phi_3(\{\mathbf{k}\}) = \pm \frac{\sum_{\alpha=1}^3 \tau(-\mathbf{k}_\alpha; P_0) \left[1 + \int^\Lambda \frac{d^3\mathbf{q}}{(2\pi)^3} M(-\mathbf{k}_\alpha, -\mathbf{q}; P_0) \frac{1}{2w(\mathbf{q})} \tau(-\mathbf{q}; P_0) \right]}{\frac{1}{L^3} \sum_{\mathbf{q}} \psi_L^{(n)}(-\mathbf{q}) \frac{1}{2w(\mathbf{q})} \tau_L(-\mathbf{q}; E_n)}$$

To obtain infinite volume matrix element:

- "measure" $\langle n | J_K^\dagger(0) | 0 \rangle$ on the lattice
- determine C_0 from fit to two-particle spectrum
→ fixes τ, τ_L
- determine D_0 from fit to three-particle spectrum
→ fixes M and $\psi_L^{(n)}$ by solving the integral equations

Conclusion:

derived the three-particle analogue of the LL formula at LO

- only multiplicative factor
- only depends on final particle interactions, which can be measured in the same lattice setup
- Lorentz-invariant framework for lowest order

Outlook:

- two-particle decay: one independent parameter $\sim g_0 K^\dagger \phi \phi$
for three-particle decay: tower of parameters G_0, G_1, \dots
 \Rightarrow at higher orders LL factor will be matrix
- add spin and isospin, include higher partial waves and moving frames