# Three-particle quantization condition for nondegenerate particles

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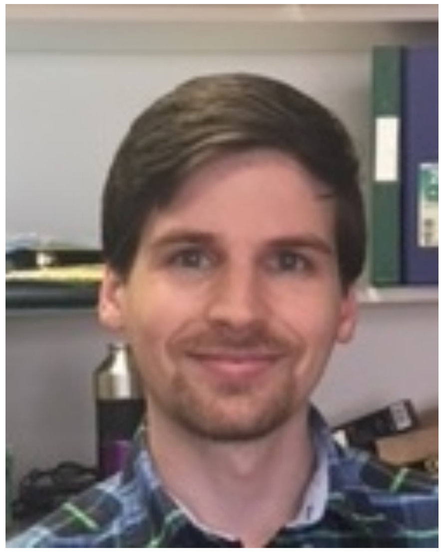
Based on work with Tyler Blanton:
2011.05520 [hep-lat] (PRD)



See also later talk by Tyler Blanton:
"Three-particle finite-volume formalism for $\pi^+\pi^+K^+$ and related systems"

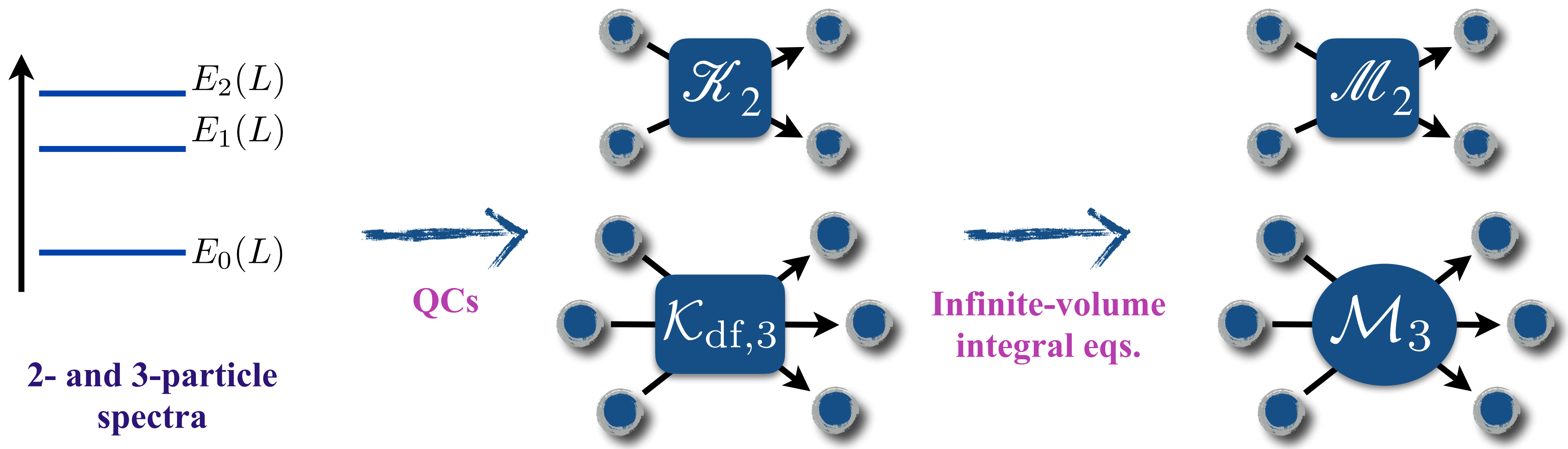
# Aims

- Generalize the three-particle quantization condition (QC3) beyond degenerate scalars (e.g. $3\pi^+$, $3K^+$)
  - Work in generic relativistic field theory (RFT) approach, and (as usual) make no restrictions on relative angular momenta
  - Applications of nondegenerate QC3 to QCD are limited: examples are $D_s^+ D^0 \pi^-$, $D_s^+ D^0 D^+$, …
- Provide a simplified method of derivation that facilitates future generalizations
  - Generalization to "2+1" case (e.g. $\pi^+\pi^+K^+$) already completed (see Tyler's talk)
  - Next up: multiple channels (of all types); particles with spin

N.B. Prior work in NREFT approach for DDK system

[34] J.-Y. Pang, J.-J. Wu, and L.-S. Geng, Phys. Rev. D **102**, 114515 (2020), 2008.13014.

# Workflow




- $\mathcal{K}_{\mathrm{df},3}$ is a real, infinite-volume (but scheme-dependent) K matrix that is smooth aside from possible 3-particle resonance poles; integral equations ensure unitarity of $\mathcal{M}_3$

N.B. I only display QCs in this talk, but the integral equations are known in all cases

# History of RFT Methods

- Original RFT derivation based on all-orders analysis using Feynman diagrams [Hansen & SS, 2014 & 2015]
    - Applies to identical scalars
    - Systematic but complicated derivation yields QC3: $\det[F_3^{-1} + \mathscr{K}_{\mathrm{df},3}] = 0$
    - Nontrivial that $\mathscr{K}_{\mathrm{df},3}$ is symmetric under particle exchange and Lorentz invariant
    - $\mathscr{K}_{\mathrm{df},3}$ is known only implicitly; $F_3 = \tilde{F}\left[\frac{1}{3} - \frac{1}{\tilde{F} + \tilde{G} + 1/(2\omega L^3 \mathscr{K}_2)}\tilde{F}\right]$
- Generalized to degenerate but distinguishable particles (e.g. three pions of any isospin) [Hansen, Romero-López & SS, 2020]
    - QC3 has same form, but with additional matrix index, whose dimension is given by number of independent two-particle subchannels
    - E.g. there are two independent subchannels for $3\pi(I=2)$: $\pi(\pi\pi)_{I=2}$ and $\pi\rho$

# History of RFT Methods

- Alternative derivation using time-ordered PT (TOPT) leads to second form of QC3 for identical scalars [Blanton & SS, 2020]
  - Much simpler derivation involving summing a geometric series
  - Yields QC3: $\det[1 + (2\omega L^3 \mathcal{K}_2 + \mathcal{K}_{\mathrm{df},3}^{(u,u)})(\tilde{F} + \tilde{G})] = 0$
  - $\mathcal{K}_{\mathrm{df},3}^{(u,u)}$ is known explicitly, but is not symmetric under particle exchange or Lorentz inv.
- Asymmetrization identities lead to third form of QC3
  - $\det[F_3^{-1} + \mathcal{K}_{\mathrm{df},3}] = 0 \;\Rightarrow\; \det[1 + (2\omega L^3 \mathcal{K}_2 + \mathcal{K}_{\mathrm{df},3}^{\prime(u,u)})(\tilde{F} + \tilde{G})] = 0$

    Notice the prime!
  - Same form as above, but $\mathcal{K}_{\mathrm{df},3}^{\prime(u,u)}$ is Lorentz invariant (though still asymmetric)
  - $\mathcal{K}_{\mathrm{df},3}^{\prime(u,u)}$ is related to the contact term in FVU approach [Mai & Döring, 2017]

# New results (1)

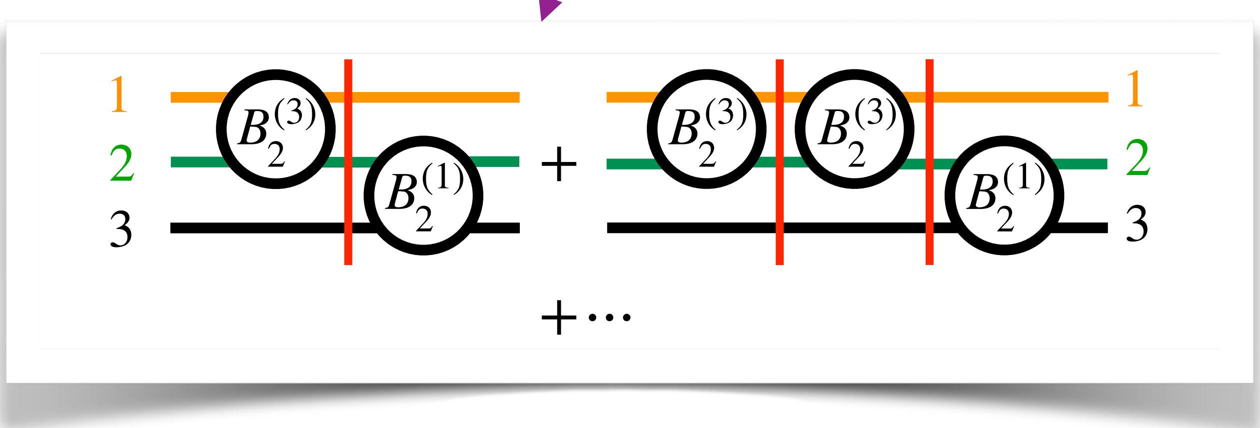
- Generalized TOPT method to nondegenerate scalars
  - Naturally written using a 3d matrix notation, with indices corresponding to which of three particles is the "spectator"

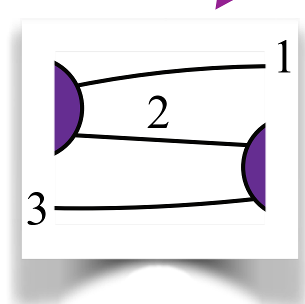
- $\det[1 + (2\omega L^3 \mathcal{K}_2 + \mathcal{K}_{\text{df},3}^{(u,u)})(\tilde{F} + \tilde{G})] = 0 \longrightarrow \det[1 + (\widehat{\overline{\mathcal{K}}_{2,L}} + \widehat{\mathcal{K}}_{\text{df},3}^{(u,u)})\, \widehat{F}_G]$

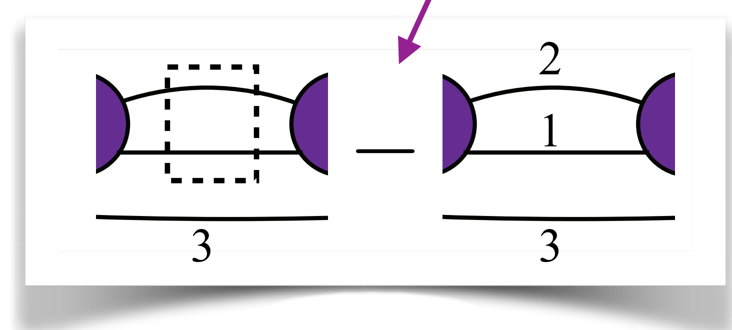
$$\widehat{\overline{\mathcal{K}}_{2,L}} = \text{diag}(2\omega_1 L^3 \mathcal{K}_2^{(23)},\, 2\omega_2 L^3 \mathcal{K}_2^{(31)},\, 2\omega_3 L^3 \mathcal{K}_2^{(12)})$$

$$\widehat{F}_G = \begin{pmatrix} \widetilde{F}^{(1)} & \widetilde{G}^{(12)} P_L & P_L \widetilde{G}^{(13)} \\ P_L \widetilde{G}^{(21)} & \widetilde{F}^{(2)} & \widetilde{G}^{(23)} P_L \\ \widetilde{G}^{(31)} P_L & P_L \widetilde{G}^{(32)} & \widetilde{F}^{(3)} \end{pmatrix}$$

$$\widehat{\mathcal{K}}_{\text{df},3}^{(u,u)} = \begin{pmatrix} [\mathcal{K}_{\text{df},3}^{(1,1)}]_{k_1\ell m, p_1\ell' m'} & [\mathcal{K}_{\text{df},3}^{(1,2)}]_{k_1\ell m, p_2\ell' m'} & [\mathcal{K}_{\text{df},3}^{(1,3)}]_{k_1\ell m, p_3\ell' m'} \\ [\mathcal{K}_{\text{df},3}^{(2,1)}]_{k_2\ell m, p_1\ell' m'} & [\mathcal{K}_{\text{df},3}^{(2,2)}]_{k_2\ell m, p_2\ell' m'} & [\mathcal{K}_{\text{df},3}^{(2,3)}]_{k_2\ell m, p_3\ell' m'} \\ [\mathcal{K}_{\text{df},3}^{(3,1)}]_{k_3\ell m, p_1\ell' m'} & [\mathcal{K}_{\text{df},3}^{(3,2)}]_{k_3\ell m, p_2\ell' m'} & [\mathcal{K}_{\text{df},3}^{(3,3)}]_{k_3\ell m, p_3\ell' m'} \end{pmatrix}$$



# New results (1)

- Generalized TOPT method to nondegenerate scalars
  - Naturally written in terms of a 3d matrix notation, with indices corresponding to which of three particles is the "spectator"
  - $\det[1 + (2\omega L^3 \mathcal{K}_2 + \mathcal{K}_{\mathrm{df},3}^{(u,u)})(\tilde{F} + \tilde{G})] = 0 \longrightarrow \det[1 + (\widehat{\overline{\mathcal{K}}_{2,L}} + \widehat{\mathcal{K}}_{\mathrm{df},3}^{(u,u)})\widehat{F}_G]$

$$\widehat{\overline{\mathcal{K}}_{2,L}} = \mathrm{diag}(2\omega_1 L^3 \mathcal{K}_2^{(23)},\, 2\omega_2 L^3 \mathcal{K}_2^{(31)},\, 2\omega_3 L^3 \mathcal{K}_2^{(12)})$$

3 different two-particle Lorentz-invariant K matrices

$$\widehat{F}_G = \begin{pmatrix} \widetilde{F}^{(1)} & \widetilde{G}^{(12)} P_L & P_L \widetilde{G}^{(13)} \\ P_L \widetilde{G}^{(21)} & \widetilde{F}^{(2)} & \widetilde{G}^{(23)} P_L \\ \widetilde{G}^{(31)} P_L & P_L \widetilde{G}^{(32)} & \widetilde{F}^{(3)} \end{pmatrix}$$

Known kinematic functions

$$\widehat{\mathcal{K}}_{\mathrm{df},3}^{(u,u)} = \begin{pmatrix} [\mathcal{K}_{\mathrm{df},3}^{(1,1)}]_{k_1\ell m, p_1\ell' m'} & [\mathcal{K}_{\mathrm{df},3}^{(1,2)}]_{k_1\ell m, p_2\ell' m'} & [\mathcal{K}_{\mathrm{df},3}^{(1,3)}]_{k_1\ell m, p_3\ell' m'} \\ [\mathcal{K}_{\mathrm{df},3}^{(2,1)}]_{k_2\ell m, p_1\ell' m'} & [\mathcal{K}_{\mathrm{df},3}^{(2,2)}]_{k_2\ell m, p_2\ell' m'} & [\mathcal{K}_{\mathrm{df},3}^{(2,3)}]_{k_2\ell m, p_3\ell' m'} \\ [\mathcal{K}_{\mathrm{df},3}^{(3,1)}]_{k_3\ell m, p_1\ell' m'} & [\mathcal{K}_{\mathrm{df},3}^{(3,2)}]_{k_3\ell m, p_2\ell' m'} & [\mathcal{K}_{\mathrm{df},3}^{(3,3)}]_{k_3\ell m, p_3\ell' m'} \end{pmatrix}$$

9 different "asymmetric" amplitudes (expressed in 9 different coordinate systems)
Not Lorentz invariant

# New results (2)

- New Feynman-diagram-based analysis that mimics structure of TOPT approach and allows explicit all-orders expressions
  - Leads to identical form of QC3 to TOPT approach, but with Lorentz-invariant $\widehat{\mathcal{K}}'^{(u,u)}_{\rm df,3}$
  - $\det[1 + (\overline{\widehat{\mathcal{K}}_{2,L}} + \widehat{\mathcal{K}}^{(u,u)}_{\rm df,3})\widehat{F}_G] \longrightarrow \det[1 + (\overline{\widehat{\mathcal{K}}_{2,L}} + \widehat{\mathcal{K}}'^{(u,u)}_{\rm df,3})\widehat{F}_G]$

Notice the prime!

$$\widehat{\mathcal{K}}'^{(u,u)}_{\rm df,3} = \begin{pmatrix} [\mathcal{K}'^{(1,1)}_{\rm df,3}]_{k_1\ell m,p_1\ell'm'} & [\mathcal{K}'^{(1,2)}_{\rm df,3}]_{k_1\ell m,p_2\ell'm'} & [\mathcal{K}'^{(1,3)}_{\rm df,3}]_{k_1\ell m,p_3\ell'm'} \\ [\mathcal{K}'^{(2,1)}_{\rm df,3}]_{k_2\ell m,p_1\ell'm'} & [\mathcal{K}'^{(2,2)}_{\rm df,3}]_{k_2\ell m,p_2\ell'm'} & [\mathcal{K}'^{(2,3)}_{\rm df,3}]_{k_2\ell m,p_3\ell'm'} \\ [\mathcal{K}'^{(3,1)}_{\rm df,3}]_{k_3\ell m,p_1\ell'm'} & [\mathcal{K}'^{(3,2)}_{\rm df,3}]_{k_3\ell m,p_2\ell'm'} & [\mathcal{K}'^{(3,3)}_{\rm df,3}]_{k_3\ell m,p_3\ell'm'} \end{pmatrix}$$

9 different "asymmetric" amplitudes (expressed in 9 different coordinate systems)

**Lorentz invariant**

  - This form can presumably related to a nondegenerate generalization of the FVU formalism, using the methods of [Blanton & SS, 2007.16190]

# New results (3)

- We apply generalized symmetrization identities to obtain symmetric forms of the previous two QC3s
  - **Both** lead to the same symmetric form of the QC3, containing the **same**, Lorentz-invariant and symmetric $\widehat{\mathcal{K}}_{\mathrm{df},3}$
  - $\det[1 + (\widehat{\overline{\mathcal{K}}_{2,L}} + \widehat{\mathcal{K}}^{(')(u,u)}_{\mathrm{df},3}) \widehat{F}_G] \longrightarrow \det[\widehat{F}_3^{-1} + \widehat{\mathcal{K}}_{\mathrm{df},3}]$

Matrix version of original QC3 of [Hansen & SS, 14]

$$\widehat{F}_3 = \frac{1}{3}\widehat{F} - \widehat{F}\frac{1}{\widehat{\overline{\mathcal{K}}}_{2,L}^{-1} + \widehat{F}_G}\widehat{F}$$

$$\widehat{F}_G = \begin{pmatrix} \widetilde{F}^{(1)} & \widetilde{G}^{(12)}P_L & P_L\widetilde{G}^{(13)} \\ P_L\widetilde{G}^{(21)} & \widetilde{F}^{(2)} & \widetilde{G}^{(23)}P_L \\ \widetilde{G}^{(31)}P_L & P_L\widetilde{G}^{(32)} & \widetilde{F}^{(3)} \end{pmatrix} = \widehat{F} + \widehat{G}$$

$$\widehat{\mathcal{K}}_{\mathrm{df},3} = \begin{pmatrix} [\mathcal{K}_{\mathrm{df},3}]_{k_1\ell m,p_1\ell'm'} & [\mathcal{K}_{\mathrm{df},3}]_{k_1\ell m,p_2\ell'm'} & [\mathcal{K}_{\mathrm{df},3}]_{k_1\ell m,p_3\ell'm'} \\ [\mathcal{K}_{\mathrm{df},3}]_{k_2\ell m,p_1\ell'm'} & [\mathcal{K}_{\mathrm{df},3}]_{k_2\ell m,p_2\ell'm'} & [\mathcal{K}_{\mathrm{df},3}]_{k_2\ell m,p_3\ell'm'} \\ [\mathcal{K}_{\mathrm{df},3}]_{k_3\ell m,p_1\ell'm'} & [\mathcal{K}_{\mathrm{df},3}]_{k_3\ell m,p_2\ell'm'} & [\mathcal{K}_{\mathrm{df},3}]_{k_3\ell m,p_3\ell'm'} \end{pmatrix}$$

**Single**, Lorentz-invariant amplitude expressed in 9 different coordinate systems
**Much simpler to parametrize**

# Summary of preferred method

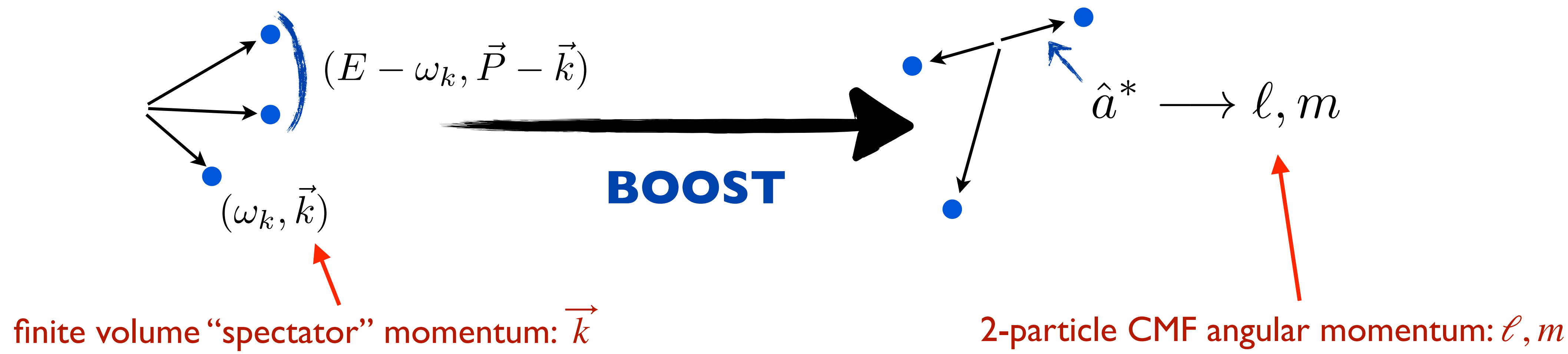
- Use TOPT to derive asymmetric QC3 involving non-Lorentz invariant $\mathcal{K}_{\rm df,3}^{(u,u)}$
  - All-orders result follows from summing simple geometric series
  - Need to introduce appropriate matrix structure (often several choices)
- Use symmetrization identities to convert to symmetric form of QC3, which automatically involves Lorentz-invariant $\mathcal{K}_{\rm df,3}$
  - Actually apply symmetrization to finite-volume scattering amplitude $\mathcal{M}_{3,L}$
  - Obtain integral equations relating $\mathcal{K}_{\rm df,3}$ to infinite-volume amplitude $\mathcal{M}_3$ with little extra effort
  - Symmetrization identities nontrivial to derive, and algebra required to apply them is nasty, but the same approach can be recycled from one application to the next
- Method avoids Feynman-diagram-based intermediate step

# Thanks
# Any questions?

# Backup slides

# Scope & Notation

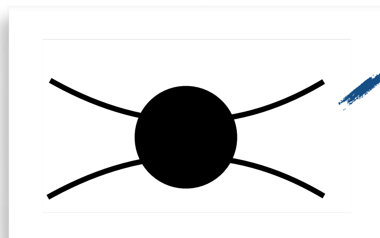
- Identical spinless particles of mass m (e.g. $3\pi^+$)
- Z$_2$ symmetry — no $2 \to 3$ transitions
- All quantities in QC3 are infinite-dimensional matrices with indices $\{\vec{k}, \ell, m\}$ describing 3 on-shell particles with total energy-momentum $(E, \vec{P})$

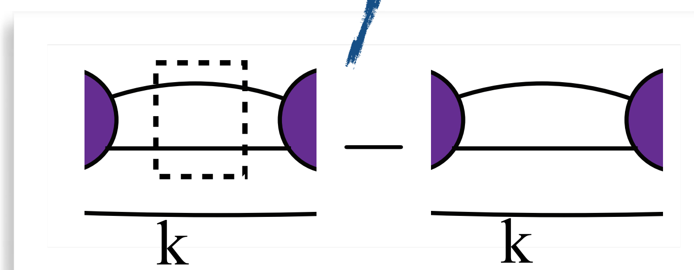


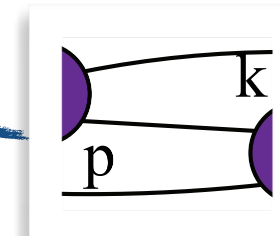

e.g. $\left[\mathscr{K}^{(u,u)}_{\rm df,3}\right]_{k\ell m;p\ell' m'}$

# $F_3$ collects 2-particle interactions

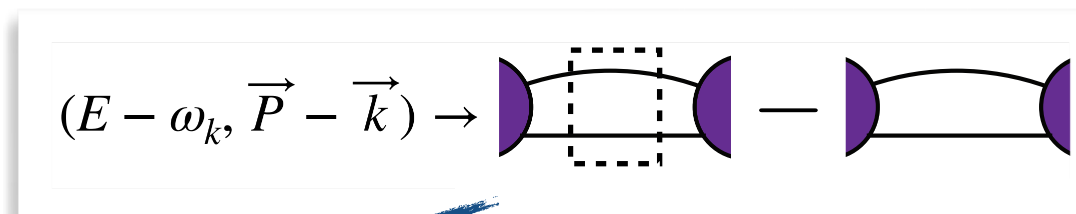
$$F_3 = \left[ \frac{\widetilde{F}}{3} - \widetilde{F} \frac{1}{(2\omega L^3 \mathcal{K}_2)^{-1} + \widetilde{F} + \widetilde{G}} \widetilde{F} \right]$$






- F & G are known geometrical functions, containing cutoff function H




$$\widetilde{F}_{p\ell' m'; k\ell m} = \frac{1}{2\omega_k L^3} \delta_{pk} \, H(\vec{k}) \, F_{\mathrm{PV}, \ell' m'; \ell m}(E - \omega_k, \vec{P} - \vec{k}, L)$$

$$\widetilde{G}_{p\ell' m'; k\ell m} = \frac{1}{2\omega_p L^3} \left( \frac{k^*}{q_p^*} \right)^{\ell'} \frac{4\pi Y_{\ell' m'}(\hat{k}^*) H(\vec{p}) H(\vec{k}) Y^*_{\ell m}(\hat{p}^*)}{(P - k - p)^2 - m^2} \left( \frac{p^*}{q_k^*} \right)^{\ell} \frac{1}{2\omega_k L^3}$$

# RFT 3-particle papers

**Max Hansen & SRS:**

**"Relativistic, model-independent, three-particle quantization condition,"**

**arXiv:1408.5933 (PRD) [HS14]**

**"Expressing the 3-particle finite-volume spectrum in terms of the 3-to-3 scattering amplitude,"**

**arXiv:1504.04028 (PRD) [HS15]**

**"Perturbative results for 2- & 3-particle threshold energies in finite volume,"**

**arXiv:1509.07929 (PRD) [HSPT15]**

**"Threshold expansion of the 3-particle quantization condition,"**

**arXiv:1602.00324 (PRD) [HSTH15]**

**"Applying the relativistic quantization condition to a 3-particle bound state in a periodic box,"**

**arXiv: 1609.04317 (PRD) [HSBS16]**

**"Lattice QCD and three-particle decays of Resonances,"**

**arXiv: 1901.00483 (Ann. Rev. Nucl. Part. Science) [HSREV19]**

**Raúl Briceño, Max Hansen & SRS:**

**"Relating the finite-volume spectrum and the 2-and-3-particle S-matrix for relativistic systems of identical scalar particles," arXiv:1701.07465 (PRD) [BHS17]**

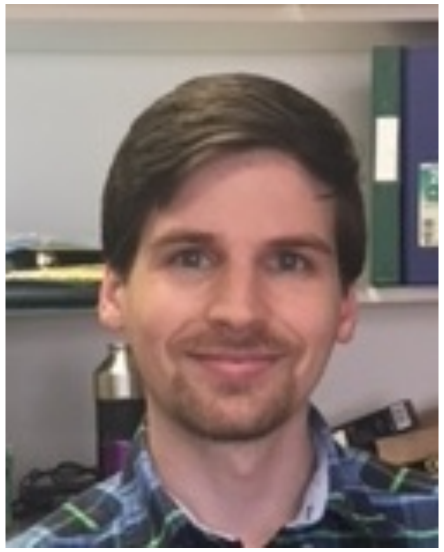
**"Numerical study of the relativistic three-body quantization condition in the isotropic approximation," arXiv:1803.04169 (PRD) [BHS18]**

**"Three-particle systems with resonant sub-processes in a finite volume," arXiv:1810.01429 (PRD 19) [BHS19]**

**SRS**

**"Testing the threshold expansion for three-particle energies at fourth order in $\varphi^4$ theory," arXiv:1707.04279 (PRD) [SPT17]**

**Tyler Blanton, Fernando Romero-López & SRS:**



**"Implementing the three-particle quantization condition including higher partial waves," arXiv:1901.07095 (JHEP) [BRS19]**

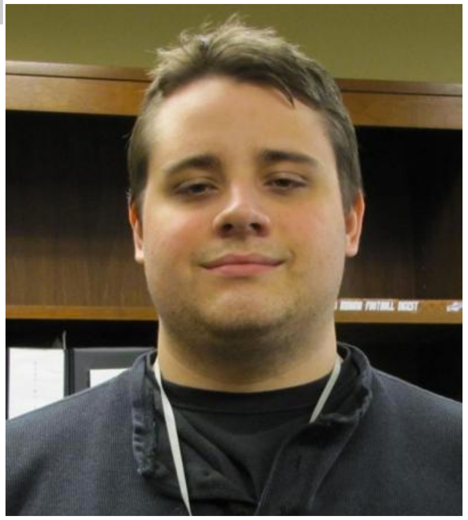
**"I=3 three-pion scattering amplitude from lattice QCD," arXiv:1909.02973 (PRL) [BRS-PRL19]**

Tyler Blanton, Raúl Briceño, Max Hansen, Fernando Romero-López, SRS:

"Numerical exploration of three relativistic particles in a finite volume including two-particle resonances and bound states", arXiv:1908.02411 (JHEP) [BBHRS19]

Raúl Briceño, Max Hansen, SRS & Adam Szczepaniak:

"Unitarity of the infinite-volume three-particle scattering amplitude arising from a finite-volume formalism," arXiv:1905.11188 (PRD)



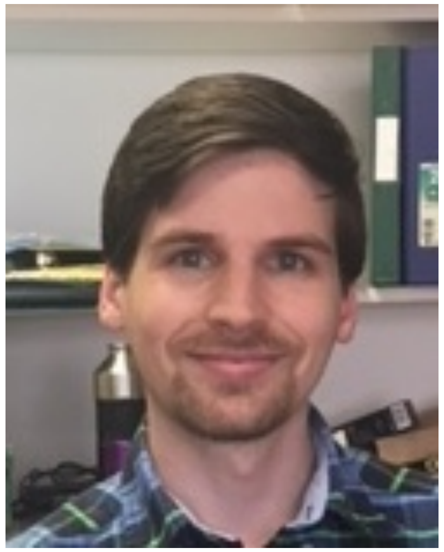
Andrew Jackura, S. Dawid, C. Fernández-Ramírez, V. Mathieu, M. Mikhasenko, A. Pilloni, SRS & A. Szczepaniak:

"On the Equivalence of Three-Particle Scattering Formalisms," arXiv:1905.12007 (PRD)

Max Hansen, Fernando Romero-López, SRS:

"Generalizing the relativistic quantization condition to include all three-pion isospin channels", arXiv:2003.10974 (JHEP) [HRS20]

"Decay amplitudes to three particles from finite-volume matrix elements," arXiv: 2101.10246 (JHEP)



**<u>Tyler Blanton & SRS:</u>**

**"Alternative derivation of the relativistic three-particle quantization condition,"**

**arXiv:2007.16188 (PRD) [BS20a]**

**"Equivalence of relativistic three-particle quantization conditions,"**

**arXiv:2007.16190 (PRD) [BS20b]**

**"Relativistic three-particle quantization condition for nondegenerate scalars,"**

**arXiv:2011.05520 (PRD)**

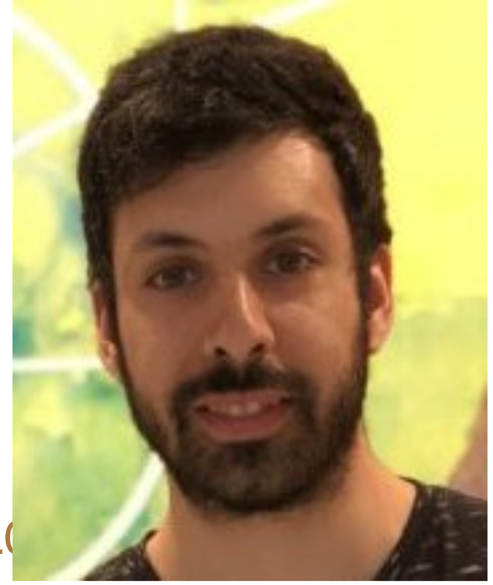
**"Three-particle finite-volume formalism for $\pi^+\pi^+K^+$ and related systems,"**

**arXiv:2105.12904 (PRD under review)**

**<u>Tyler Blanton, Drew Hanlon, Ben Hörz, Colin Morningstar, Fernando Romero-López & SRS</u>**

**"$3\pi^+$ & $3K^+$ interactions beyond leading order from lattice QCD,"**

**arXiv:2106.05590 (JHEP under review)**



# Other work

## ★ Implementing RFT integral equations

- A. Jackura et al., 2010.09820 [Solving s-wave RFT integral equations in presence of bound states]
- M.T. Hansen et al. (HADSPEC), 2009.04931, PRL [Calculating $3\pi^+$ spectrum and using to determine three-particle scattering amplitude]

## ★ Reviews

- A. Rusetsky, 1911.01253 [LATTICE 2019 plenary]
- M. Mai, M. Döring and A. Rusetsky, 2103.00577 [Review of formalisms and chiral extrapolations]

## ★ NREFT approach

- H.-W. Hammer, J.-Y. Pang & A. Rusetsky, 1706.07700, JHEP & 1707.02176, JHEP [Formalism & examples]
- M. Döring et al., 1802.03362, PRD [Numerical implementation]
- J.-Y. Pang et al., 1902.01111, PRD [large volume expansion for excited levels]
- F. Müller, T. Yu & A. Rusetsky, 2011.14178, PRD [large volume expansion for I=1 three pion ground state]
- F. Romero-López, A. Rusetsky, N. Schlage & C. Urbach, 2010.11715, JHEP [generalized large-volume exps]
- F. Müller & A. Rusetsky, 2012.13957, JHEP [Three-particle analog of Lellouch-Lüscher formula]

# Alternate 3-particle approaches

★ **Finite-volume unitarity (FVU) approach**

- M. Mai & M. Döring, 1709.08222, EPJA [formalism]
- M. Mai et al., 1706.06118, EPJA [unitary parametrization of $M_3$ involving R matrix; used in FVU approach]
- A. Jackura et al., 1809.10523, EPJC [further analysis of R matrix parametrization]
- M. Mai & M. Döring, 1807.04746, PRL [3 pion spectrum at finite-volume from FVU]
- M. Mai et al., 1909.05749,PRD [applying FVU approach to $3\pi^+$spectrum from Hanlon & Hörz]
- C. Culver et al., 1911.09047, PRD [calculating $3\pi^+$ spectrum and comparing with FVU predictions]
- A. Alexandru et al., 2009.12358, PRD [calculating $3K^-$ spectrum and comparing with FVU predictions]
- R. Brett et al., 2101.06144 [determining $3\pi^+$ interaction from LQCD spectrum]

★ **HALQCD approach**

- T. Doi et al. (HALQCD collab.), 1106.2276, Prog.Theor.Phys. [3 nucleon potentials in NR regime]