Three-particle quantization condition for $\pi^+\pi^+K^+$ and related systems

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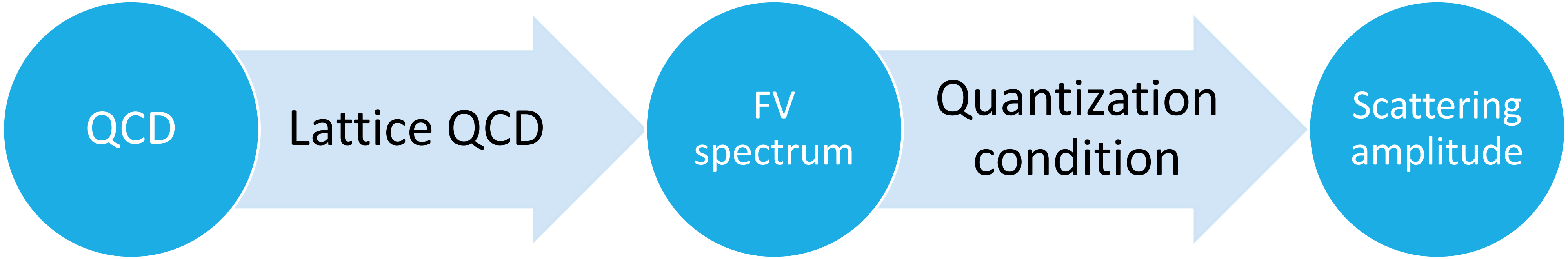
Based on work with Steve Sharpe

arXiv:2105.12094

Extracting scattering info from the lattice

Quantization condition (QC): Map from finite-volume (FV) spectrum to ∞-volume scattering amplitudes



QCD

Lattice QCD

FV spectrum

Quantization condition

Scattering amplitude

Current frontier: QCs for three-particle scattering (QC3s)

Original QC3 Derivation

Hansen & Sharpe (2014)

Rooted in a generic relativistic effective field theory (RFT)

- Involves complicated all-orders sum of Feynman diagrams

Simplifying assumptions used:

- Only scalar (spin-0) particles
- All particles identical
- No resonant 2-particle subchannels
- Lagrangian has global $\mathbb{Z}_2$ symmetry (no odd-legged vertices)

RFT Extensions

2→3 transitions	Briceño, Hansen, & Sharpe [arXiv:1701.07465]
2-particle resonances & bound states	Romero-López, Sharpe, TDB, Briceño, & Hansen [arXiv:1908.02411]
Isospin	Hansen, Romero-López, & Sharpe [arXiv:2003.10974]
Time-ordered PT (TOPT) approach	TDB & Sharpe [arXiv:2007.16188]
Equivalence of relativistic approaches	TDB & Sharpe [arXiv:2007.16190]
Nondegenerate scalars	TDB & Sharpe [arXiv:2011.05520]
2+1 states (e.g. $\pi^+\pi^+K^+$)	TDB & Sharpe [arXiv:2105.12094]

Steve's talk

this talk

QC3 for Identical Scalars (e.g. $3\pi^+$)

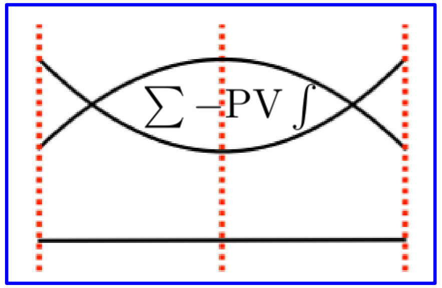
- Appears naturally in TOPT approach
- Asymmetric under particle interchange
- Not Lorentz invariant (in TOPT form)
- Related to $\mathcal{M}_3$ via integral equations

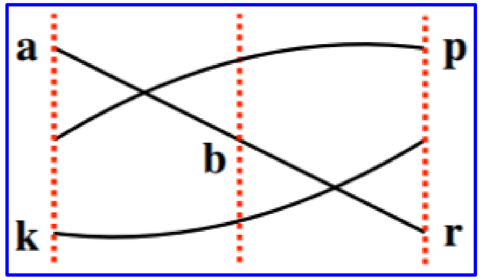
- Symmetric under particle interchange
- Lorentz invariant
- Related to $\mathcal{M}_3$ via integral equations

$$\det\left[1+\left(2\omega L^3\mathcal{K}_2+\mathcal{K}_{\mathrm{df},3}^{(u,u)}\right)\left(\widetilde{F}+\widetilde{G}\right)\right]=0 \quad\Longrightarrow\quad \det\left[F_3^{-1}+\mathcal{K}_{\mathrm{df},3}\right]=0$$

unknown

known





$$F_3=\frac{\widetilde{F}}{3}-\widetilde{F}\frac{1}{\widetilde{F}+\widetilde{G}+1/(2\omega L^3\mathcal{K}_2)}\widetilde{F}$$

QC3 for Nondegenerate Scalars (e.g. $D_s^+ D^0 \pi^-$)

Same as before, but with additional indices for spectator “flavor”

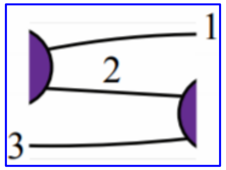
$$\det\left[1+\underbrace{\left(\widehat{\overline{\mathcal{K}}}_{2,L}+\widehat{\mathcal{K}}^{(u,u)}_{\mathrm{df},3}\right)}_{\text{unknown}}\underbrace{\left(\widehat{F}+\widehat{G}\right)}_{\text{known}}\right]=0 \quad\Longrightarrow\quad \det\left[\widehat{F}_3^{-1}+\widehat{\mathcal{K}}_{\mathrm{df},3}\right]=0$$

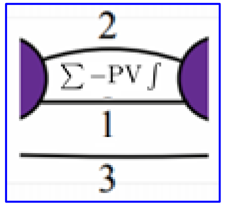
$$\widehat{F}_3=\frac{\widehat{F}}{3}-\widehat{F}\,\frac{1}{\widehat{F}+\widehat{G}+\widehat{\overline{\mathcal{K}}}^{-1}_{2,L}}\,\widehat{F}$$

$$\widehat{\overline{\mathcal{K}}}_{2,L}=\mathrm{diag}\left(\overline{\mathcal{K}}^{(1)}_{2,L},\,\overline{\mathcal{K}}^{(2)}_{2,L},\,\overline{\mathcal{K}}^{(3)}_{2,L}\right),\quad \overline{\mathcal{K}}^{(i)}_{2,L}\equiv 2\omega_i L^3\mathcal{K}^{(i)}_2$$

$$\widehat{\mathcal{K}}_{\mathrm{df},3}=\begin{pmatrix}[\mathcal{K}_{\mathrm{df},3}]_{k_1\ell m;p_1\ell'm'} & [\mathcal{K}_{\mathrm{df},3}]_{k_1\ell m;p_2\ell'm'} & [\mathcal{K}_{\mathrm{df},3}]_{k_1\ell m;p_3\ell'm'}\\ [\mathcal{K}_{\mathrm{df},3}]_{k_2\ell m;p_1\ell'm'} & [\mathcal{K}_{\mathrm{df},3}]_{k_2\ell m;p_2\ell'm'} & [\mathcal{K}_{\mathrm{df},3}]_{k_2\ell m;p_3\ell'm'}\\ [\mathcal{K}_{\mathrm{df},3}]_{k_3\ell m;p_1\ell'm'} & [\mathcal{K}_{\mathrm{df},3}]_{k_3\ell m;p_2\ell'm'} & [\mathcal{K}_{\mathrm{df},3}]_{k_3\ell m;p_3\ell'm'}\end{pmatrix}$$

$$\widehat{F}+\widehat{G}=\begin{pmatrix}\widetilde{F}^{(1)} & \widetilde{G}^{(12)}P_L & P_L\widetilde{G}^{(13)}\\ P_L\widetilde{G}^{(21)} & \widetilde{F}^{(2)} & \widetilde{G}^{(23)}P_L\\ \widetilde{G}^{(31)}P_L & P_L\widetilde{G}^{(32)} & \widetilde{F}^{(3)}\end{pmatrix}$$





QC3 for “2+1” Systems (e.g. $\pi^+\pi^+K^+$)

Same QC3 form, except with 2 × 2 flavor structure & symmetry factors

- Details aren’t obvious though; had to derive from scratch

$$\det\left[1+\left(\widehat{\overline{\mathcal{K}}}_{2,L}+\widehat{\mathcal{K}}^{(u,u)}_{\mathrm{df},3}\right)\left(\widehat{F}+\widehat{G}\right)\right]=0 \quad\Longrightarrow\quad \det\left[\widehat{F}_3^{-1}+\widehat{\mathcal{K}}_{\mathrm{df},3}\right]=0$$

unknown known

$$\widehat{\overline{\mathcal{K}}}_{2,L}=\mathrm{diag}\left(\overline{\mathcal{K}}^{(1)}_2,\ \frac{1}{2}\overline{\mathcal{K}}^{(2)}_2\right)$$

$$\widehat{F}_3=\frac{\widehat{F}}{3}-\widehat{F}\frac{1}{\widehat{F}+\widehat{G}+\widehat{\overline{\mathcal{K}}}^{-1}_{2,L}}\widehat{F}$$

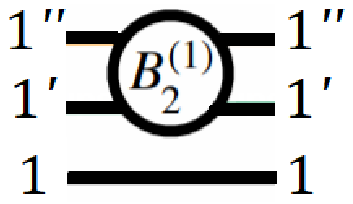
$$\widehat{\mathcal{K}}_{\mathrm{df},3}=\begin{pmatrix}[\mathcal{K}_{\mathrm{df},3}]_{k_1\ell m;p_1\ell' m'} & [\mathcal{K}_{\mathrm{df},3}]_{k_1\ell m;p_2\ell' m'}/\sqrt{2}\\ [\mathcal{K}_{\mathrm{df},3}]_{k_2\ell m;p_1\ell' m'}/\sqrt{2} & [\mathcal{K}_{\mathrm{df},3}]_{k_2\ell m;p_2\ell' m'}/2\end{pmatrix}$$

$$\widehat{F}+\widehat{G}=\begin{pmatrix}\widetilde{F}^{(1)}+\widetilde{G}^{(11)} & \sqrt{2}P_L\widetilde{G}^{(12)}\\ \sqrt{2}\widetilde{G}^{(21)}P_L & \widetilde{F}^{(2)}\end{pmatrix}$$

Understanding exchange symmetry (ID)

We used to think that TOPT required asymmetric building blocks to yield simple geometric series, but that's not the case!

Identical (ID) case: flavors {1,1′, 1′′}



$$C_{3,L}^{\mathrm{ID}} - C_{3,L}^{\mathrm{ID},(0)} = A' \frac{iD}{3!} A + A' \frac{iD}{3!} i\mathcal{M}_{23,L,\mathrm{ID}}^{\mathrm{off}} \frac{iD}{3!} A$$

$$\mathcal{S}_{\mathrm{ID}} = 1 + P^{(11')} + P^{(11'')}$$

$$P_{\{\boldsymbol{p}\};\{\boldsymbol{k}\}}^{(11')} \equiv \delta_{\boldsymbol{p}_1 \boldsymbol{k}_{1'}} \delta_{\boldsymbol{p}_{1'} \boldsymbol{k}_1} \delta_{\boldsymbol{p}_{1''} \boldsymbol{k}_{1''}}$$

$$\mathcal{S}_{\mathrm{ID}} \frac{D}{3!} \mathcal{S}_{\mathrm{ID}} = D_F^{\mathrm{ID}} + D_G^{\mathrm{ID}}$$

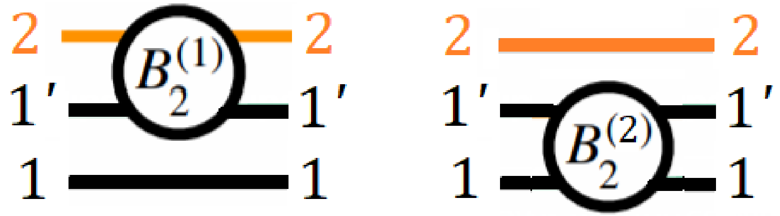
$$\begin{aligned} \mathcal{M}_{23,L,\mathrm{ID}}^{\mathrm{off}} &\equiv \mathcal{S}_{\mathrm{ID}} \overline{\mathcal{M}}_{2,L,\mathrm{off}}^{(1)} \mathcal{S}_{\mathrm{ID}} + \mathcal{M}_{3,L,\mathrm{ID}}^{\mathrm{off}} \\ &= \left(\mathcal{S}_{\mathrm{ID}} \overline{\mathcal{B}}_{2,L}^{(1)} \mathcal{S}_{\mathrm{ID}} + \mathcal{B}_3^{\mathrm{ID}} \right) \frac{1}{1 - \frac{iD}{3!} i \left(\mathcal{S}_{\mathrm{ID}} \overline{\mathcal{B}}_{2,L}^{(1)} \mathcal{S}_{\mathrm{ID}} + \mathcal{B}_3^{\mathrm{ID}} \right)} \\ &= \mathcal{S}_{\mathrm{ID}} \left(\overline{\mathcal{B}}_{2,L}^{(1)} + \frac{\mathcal{B}_3^{\mathrm{ID}}}{9} \right) \frac{1}{1 - i \left(D_F^{\mathrm{ID}} + D_G^{\mathrm{ID}} \right) i \left(\overline{\mathcal{B}}_{2,L}^{(1)} + \frac{\mathcal{B}_3^{\mathrm{ID}}}{9} \right)} \mathcal{S}_{\mathrm{ID}} \end{aligned}$$

new form
(symm.)

old form
(asymm.)

Understanding exchange symmetry (2+1)

2+1 case: flavors {1,1′, 2}



$$C_{3,L}^{2+1} - C_{3,\infty}^{2+1,(0)} = A' \frac{iD}{2} A + A' \frac{iD}{2} i\mathcal{M}^{\text{off}}_{23,L,2+1} \frac{iD}{2} A$$

$$\begin{aligned}\mathcal{M}^{\text{off}}_{23,L,2+1} &\equiv \mathcal{S}_{11'} \overline{\mathcal{M}}^{(1)}_{2,L,\text{off}} \mathcal{S}_{11'} + \overline{\mathcal{M}}^{(2)}_{2,L,\text{off}} + \mathcal{M}^{\text{off}}_{3,L,2+1} \\ &= \left(\mathcal{S}_{11'}\overline{\mathcal{B}}^{(1)}_{2,L}\mathcal{S}_{11'} + \overline{\mathcal{B}}^{(2)}_{2,L} + \mathcal{B}_3\right) \frac{1}{1 - \frac{iD}{2} i \left(\mathcal{S}_{11'}\overline{\mathcal{B}}^{(1)}_{2,L}\mathcal{S}_{11'} + \overline{\mathcal{B}}^{(2)}_{2,L} + \mathcal{B}_3\right)} \\ &= \mathcal{S}_{11'}\left(\overline{\mathcal{B}}^{(1)}_{2,L} + \frac{1}{4}\overline{\mathcal{B}}^{(2)}_{2,L} + \frac{1}{4}\mathcal{B}_3\right) \frac{1}{1 - i\left(D_F^{2+1} + D_G^{2+1}\right)\left(\overline{\mathcal{B}}^{(1)}_{2,L} + \frac{1}{4}\overline{\mathcal{B}}^{(2)}_{2,L} + \frac{1}{4}\mathcal{B}_3\right)} \mathcal{S}_{11'}\end{aligned}$$

symm. form

asymm. form

$$\mathcal{S}_{11'} = 1 + P^{(11')}$$

$$P^{(11')}_{\{\boldsymbol{p}\};\{\boldsymbol{k}\}} \equiv \delta_{\boldsymbol{p}_1 \boldsymbol{k}_{1'}} \delta_{\boldsymbol{p}_{1'} \boldsymbol{k}_1} \delta_{\boldsymbol{p}_2 \boldsymbol{k}_2}$$

$$\mathcal{S}_{11'} \frac{D}{2} \mathcal{S}_{11'} = D_F^{2+1} + D_G^{2+1}$$

Consistency Check: $2d + 1$ Limit

As a consistency check, we took the "$2d + 1$" limit $m_1 = m_2 \neq m_3$ of the completely nondegenerate QC3 and projected onto the symmetric irrep

- <u>Example:</u> $(\pi^+\pi^0)_{I=2} D_s^{\pm}$ is identical to $\pi^+\pi^+ D_s^{\pm}$ in isosymmetric QCD

$$\widehat{\widetilde{\mathcal{K}}}^{2d+1}_{\mathrm{df},3,+} = \begin{pmatrix} \widetilde{\mathcal{K}}^{(11)}_{\mathrm{df},3;\mathrm{nd}} + \widetilde{\mathcal{K}}^{(12)}_{\mathrm{df},3;\mathrm{nd}} & \sqrt{2}\widetilde{\mathcal{K}}^{(13)}_{\mathrm{df},3;\mathrm{nd}}\mathbb{P}_e \\ \sqrt{2}\mathbb{P}_e\widetilde{\mathcal{K}}^{(31)}_{\mathrm{df},3;\mathrm{nd}} & \widetilde{\mathcal{K}}^{(33)}_{\mathrm{df},3;\mathrm{nd;ee}} \end{pmatrix} = \widehat{\mathcal{K}}^{2+1}_{\mathrm{df},3}$$

This "$2d + 1 \rightarrow 2 + 1$" trick does not work if additional channels are involved

- <u>Example:</u> $\pi^+\pi^+K^+$ is related by isospin to both $\pi^+\pi^0K^+$ and $\pi^+\pi^+K^0$

Threshold Expansions for $\mathcal{K}_{\text{df},3}$

Identical case ($3\pi^+$): $\mathcal{K}_{\text{df},3;\text{ID}} = \mathcal{K}^{\text{iso},0}_{\text{df},3;\text{ID}} + \mathcal{K}^{\text{iso},1}_{\text{df},3;\text{ID}}\Delta + \mathcal{O}(\Delta^2)$

2+1 case ($\pi^+\pi^+K^+$): $\mathcal{K}_{\text{df},3;2+1} = \mathcal{K}^{\text{iso},0}_{\text{df},3;2+1} + \mathcal{K}^{\text{iso},1}_{\text{df},3;2+1}\Delta + \mathcal{K}^{B,1}_{\text{df},3;2+1}\Delta_3^S + \mathcal{K}^{E,1}_{\text{df},3;2+1}\tilde{t}^S_{33} + \mathcal{O}(\Delta^2)$

Nondegenerate case ($D_s^+D^0\pi^-$):

$$\mathcal{K}_{\text{df},3;\text{nd}} = \mathcal{K}^{\text{iso},0}_{\text{df},3;\text{nd}} + \mathcal{K}^{\text{iso},1}_{\text{df},3;\text{nd}}\Delta + \mathcal{K}^{A,1}_{\text{df},3;\text{nd}}(\Delta_1^S - \Delta_2^S) + \mathcal{K}^{B,1}_{\text{df},3;\text{nd}}\Delta_3^S$$
$$+ \mathcal{K}^{C,1}_{\text{df},3;\text{nd}}(\tilde{t}^S_{11} + \tilde{t}^S_{22} - 2\tilde{t}^S_{12}) + \mathcal{K}^{D,1}_{\text{df},3;\text{nd}}(\tilde{t}^S_{13} - \tilde{t}^S_{23}) + \mathcal{K}^{E,1}_{\text{df},3;\text{nd}}\tilde{t}^S_{33} + \mathcal{O}(\Delta^2)$$

$$s \equiv E^{*2}\,, \quad s_i \equiv (p_j + p_k)^2, \quad s'_i \equiv (p'_j + p'_k)^2\,, \quad t_{ij} \equiv (p_i - p'_j)^2$$

$$M_\Sigma \equiv m_1 + m_2 + m_3$$

$$\Delta \equiv \frac{s - M_\Sigma^2}{M_\Sigma^2}\,, \quad \Delta_i \equiv \frac{s_i - (m_j + m_k)^2}{M_\Sigma^2}\,, \quad \Delta'_i \equiv \frac{s'_i - (m_j + m_k)^2}{M_\Sigma^2}\,, \quad \tilde{t}_{ij} \equiv \frac{t_{ij} - (m_i - m_j)^2}{M_\Sigma^2}$$

$$0 \le \Delta_i, \Delta'_i, -\tilde{t}_{ij} \le \Delta$$

Conclusions

We now have a QC for each type of single-channel three-scalar system ($\pi^+\pi^+\pi^+$, $\pi^+\pi^+K^+$, $D_s^+D^0\pi^-$, etc.)

<u>Future goals:</u>

- Implementing the “2+1” QC3 on $\pi^+\pi^+K^+$ and $K^+K^+\pi^+$ LQCD data
- Generalize QC3 to include multiple channels, spin
- 4-particle QC

Thanks for listening! I’m happy to take any questions