$\pi\pi$ scattering at Large N_c

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QCD in the Large N_c limit

QCD simplifies in the Large N_c limit ('t Hooft limit)

$$N_c \to \infty \qquad \lambda = N_c G^2 \sim N_c \alpha_s = \text{constant}$$

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It keeps many non-perturbative properties:

- Asymptotic freedom

$$\beta(\lambda) := \mu \frac{d\lambda}{d\mu} = -\left(\frac{11}{3} - \frac{2}{3}\frac{N_f}{N_c}\right)\frac{\lambda^2}{8\pi^2}$$

- Spontaneous chiral symmetry breaking and confinement
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Large N_c has predictive power in the non-perturbative regime!

Meson observables at large N_c

Long-term goal: understand QCD at large N_c

- Resonances $\rightarrow$ Stable ($\Gamma \sim 1/N_c$)
- Exotic states (tetraquarks?)
- $K \rightarrow (\pi\pi)_{I=0,2}$ $\begin{cases} \text{Intrinsic QCD effects [Donini, et al. 2020]} \\ \text{Final state interactions} \end{cases}$

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This work: $\pi\pi$ scattering at large N_c from lattice simulations

- $N_f = 4$ (u, d, s c) $\to$ 7 channels (4 with s-wave)

$$15 \otimes 15 = \mathbf{84} \oplus 45 \oplus \overline{45} \oplus \mathbf{20} \oplus \mathbf{15} \oplus 15 \oplus \mathbf{1}$$

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D **C**

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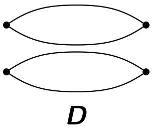

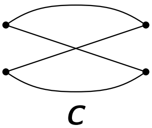

- Match to Chiral Perturbation Theory (ChPT) to constrain Low Energy Coupling (LECs)

$\pi\pi$ scattering in ChPT

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$$M_\pi a_0^{I=2} = -\frac{M_\pi^2}{16\pi F_\pi^2}\left[1 - \frac{16M_\pi^2}{F_\pi^2}L_{I=2} + \frac{M_\pi^2}{32\pi^2 F_\pi^2}\left(\frac{13}{4}\ln\frac{M_\pi^2}{\mu^2} - \frac{3}{4}\right)\right]$$

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$$L_{I=2} = L^{(0)} N_c + L^{(1)}_{I=2} + \dots$$
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At Large N_c the η' needs to be included in the EFT

$$M_{\eta'}^2 = M_\pi^2 + \frac{2N_f\chi_{top}}{F_\pi^2} \xrightarrow[\text{Large } N_c]{F_\pi^2\sim\mathcal{O}(N_c)} M_\pi^2 + \ldots$$ [Witten-Veneciano]

$\pi\pi$ scattering in Large N_c ChPT

Large N_c or U(N_f) ChPT [Kaiser, Leutwyler 2000]:

- Leutwyler counting scheme

$$\delta \sim \mathcal{O}(m_q) \sim \mathcal{O}(M_\pi^2) \sim \mathcal{O}(k^2) \sim \mathcal{O}(N_c^{-1}) \qquad \boldsymbol{\pi} = \begin{pmatrix} -\pi^0 + \eta' & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 + \eta' \end{pmatrix}$$

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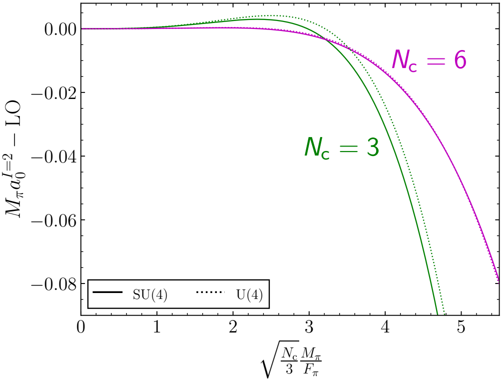
SU(4) vs U(4) ChPT

$I = 2$ channel

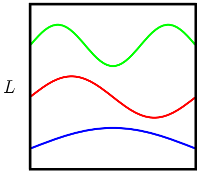


There is a tradeoff as N_c increases:

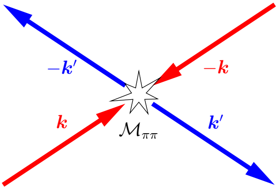
- Chiral logs change as $M_{\eta'}^2 \to M_\pi^2 + \mathcal{O}(N_c^{-1})$
- NNLO is suppressed ($\sim \mathcal{O}(N_c^{-1})$)

Scattering properties from the lattice

Finite-volume spectrum:
$\delta E_{\pi\pi} = E_{\pi\pi} - 2M_\pi$



Scattering properties:
$\mathcal{M}_{\pi\pi}$, $k \cot \delta_0$, a_0...



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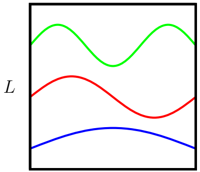

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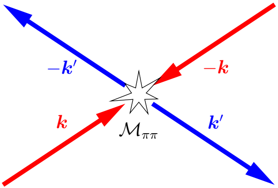

Lüscher's formalism [1986] $\longrightarrow$ $$k \cot \delta_0 = \frac{1}{\pi L} \mathcal{Z}\left(\frac{Lk}{2\pi}\right)$$

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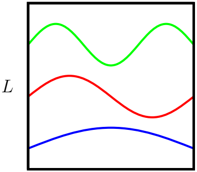

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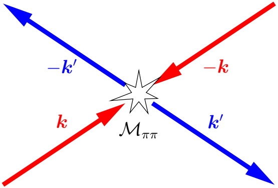

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Threshold expansion

$$\delta E_{\pi\pi} = -\frac{4\pi a_0}{M_\pi L^3}\left[1 + c_1\left(\frac{a_0}{L}\right) + c_2\left(\frac{a_0}{L}\right)^2 \underbrace{+ c_3\left(\frac{a_0}{L}\right)^3 + \frac{2\pi r_0 a_0}{L^3} + \frac{\pi a_0}{M_\pi^2 L^3}}_{\mathcal{O}(L^{-6})\ \text{[Hansen, Sharpe 2017]}} + \ldots\right]$$

Our ensembles

Configurations generated using HiRep [Del Debbio, et al. 2010]

- Iwasaki gauge action with $N_f = 4$ clover-improved Wilson fermions

Two regularizations for valence fermions:

- Unitary setup with improved Wilson fermions ($c_{sw} \neq 0$)
- Mixed-action setup at maximal twist

Summary of ensembles [Hernández, et al. 2019]

a $= 0.075$ fm $\rightarrow [N_c = 3 - 6] \times$ [4 or 5 values of M_π] $= 17$ ensembles
a $= 0.065$ fm $\rightarrow [N_c = 3] \times$ [2 values of M_π] $= 2$ ensembles
a $= 0.059$ fm $\rightarrow [N_c = 3] \times$ [2 values of M_π] $= 2$ ensembles

Energy spectrum from lattice simulations

Extract $\delta E_{\pi\pi}$ from lattice correlators

$$R(t) = \frac{\partial_0 C_{\pi\pi}}{C_\pi^2} \xrightarrow{t \gg 1} A_{\pi\pi} e^{-\delta E_{\pi\pi} t}$$

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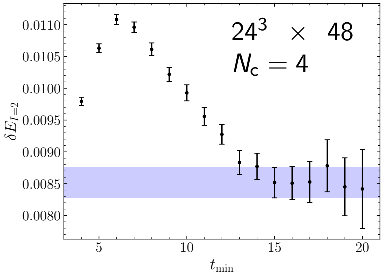

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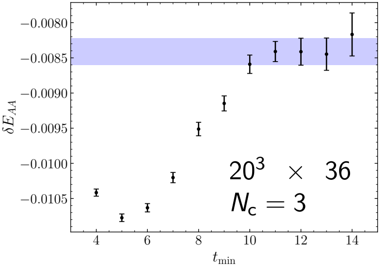

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Compare between both regularizations $\rightarrow$ Equal in the continuum

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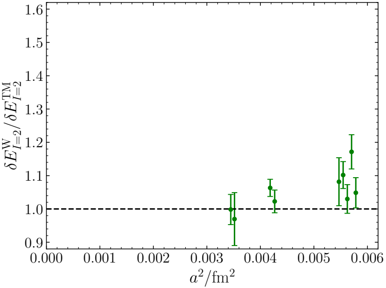

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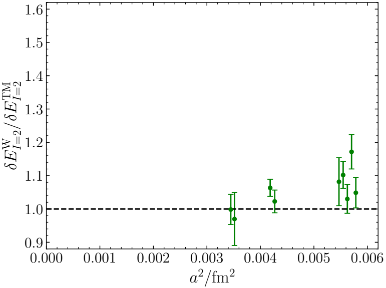

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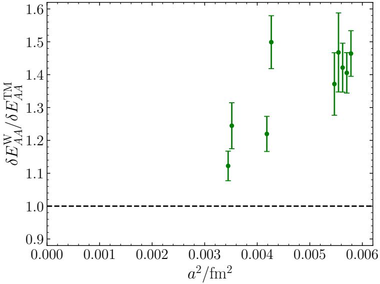

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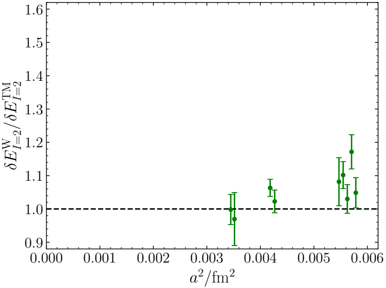

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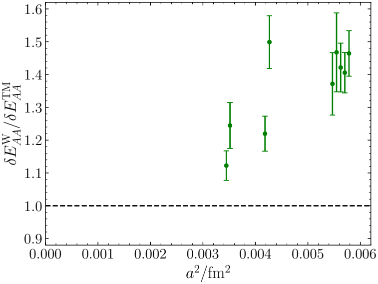

Large $\mathcal{O}(a^2)$ effects

☞ Need continuum extrapolation

AA-channel: Continuum extrapolation for $N_c = 0$

Continuum extrapolation of $k \cot \delta_0$ for $N_c = 3$ in 3 steps:

1. Extrapolation to $k/M_\pi = -0.08$ using Effective Range Expansion and $M_\pi^2 r_0 a_0 \in [-5, -1]$
2. Interpolation to $\xi = M_\pi^2/(4\pi F_\pi)^2 = 0.14$
3. Constrained continuum extrapolation

LO ChPT:
$M_\pi^2 r_0 a_0 = -3$

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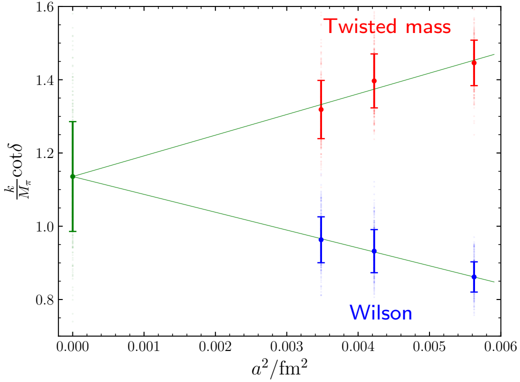

- ⋆ Large $\mathcal{O}(a^2)$ effects for both regularizations
- ⋆ Use TM fermions
- ⋆ Wilson-ChPT inspired parametrization

$$\mathcal{M}_{AA} = \mathcal{M}_{AA}^{\text{cont}} + a^2 W \xi$$

$[W \sim \mathcal{O}(N_c^0)]$

Fitting $M_\pi a_0$ to ChPT

Use threshold expansion to $\mathcal{O}(L^{-5})$ and do a simultaneous chiral and N_c fit of $M_\pi a_0$

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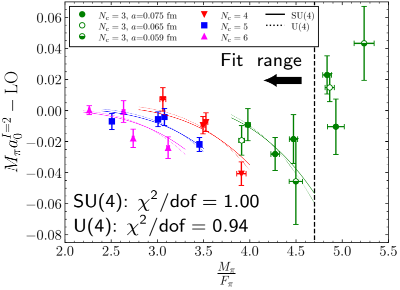

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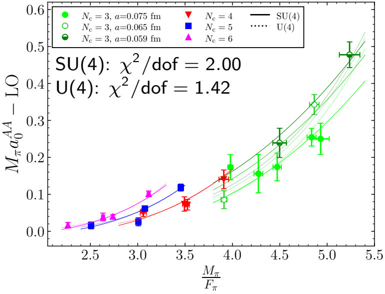

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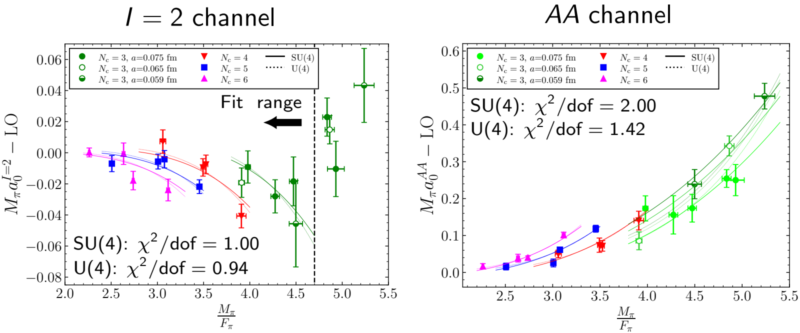

Match to ChPT to constrain LECs

SU(4) $\frac{L_{I=2}}{N_c} \times 10^3 = -0.11(4) - \frac{1.43(16)}{N_c}$ $\quad \frac{L_{AA}}{N_c} \times 10^3 = -1.08(13) + \frac{2.2(3)}{N_c}$

U(4) $\frac{L_{I=2}}{N_c} \times 10^3 = -0.10(7) - \frac{1.29(16)}{N_c}$ $\quad \frac{L_{AA}}{N_c} \times 10^3 = -0.6(4) + \frac{2.4(3)}{N_c}$

Lüscher's formalism

Check convergence of threshold expansion

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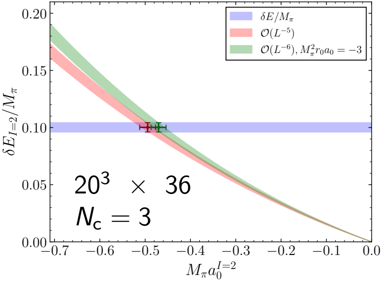

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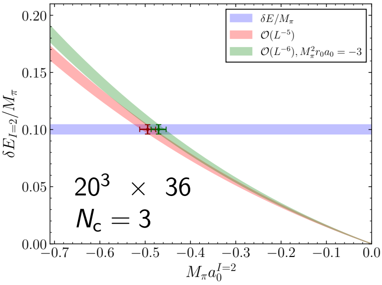

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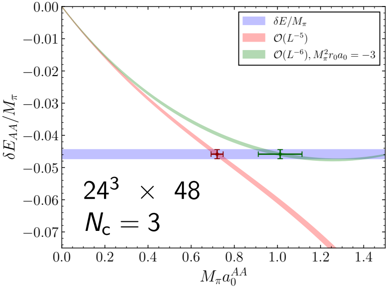


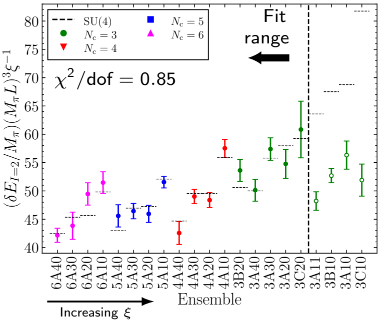
✓ Good convergence

✗ Not converging at $\mathcal{O}(L^{-6})$ for large ξ/small volume.

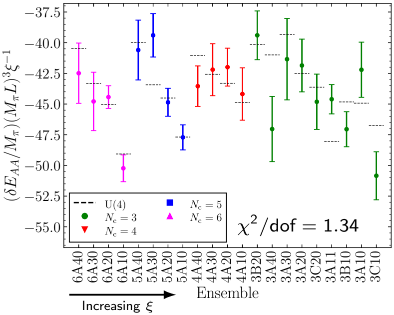
Fitting energy spectrum to ChPT [Preliminary]

Simultaneous chiral and N_c fit of energy spectrum

$I = 2$ channel



AA channel



U(4) $\frac{L_{I=2}}{N_c} \times 10^3 = -0.07(4) - \frac{1.4(2)}{N_c}$ $\frac{L_{AA}}{N_c} \times 10^3 = -0.9(2) + \frac{2.6(6)}{N_c}$

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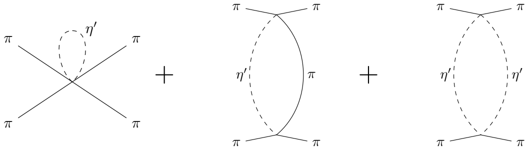
Thank you for your attention!

$\pi\pi$ scattering in Large N_c ChPT

We must consider the following loop diagrams + Mass renormalization



+ (u-channel)

$$\mathcal{T}_{AA}^{\mathrm{U}(N_f)} = \mathcal{T}_{AA}^{\mathrm{SU}(N_f)} + \frac{4M_\pi^4}{F_\pi^4 N_f}\left(1+\frac{2}{N_f}\right)\tilde{B}_{\eta'\pi}(q^2) - \frac{4M_\pi^4}{F_\pi^4 N_f^2}\tilde{B}_{\eta'\eta'}(q^2) + \frac{2M_\pi^4}{F_\pi^4}\left[K_{AA}^{(q^0)} + K_{AA}^{(q^2)}\left(\frac{q^2}{M_\pi}\right)^2 + \dots\right]$$

$$\mathcal{T}_{I=2}^{\mathrm{U}(N_f)} = \mathcal{T}_{I=2}^{\mathrm{SU}(N_f)} - \frac{4M_\pi^4}{F_\pi^4 N_f}\left(1-\frac{2}{N_f}\right)\tilde{B}_{\eta'\pi}(q^2) - \frac{4M_\pi^4}{F_\pi^4 N_f^2}\tilde{B}_{\eta'\eta'}(q^2) + \frac{2M_\pi^4}{F_\pi^4}\left[K_{I=2}^{(q^0)} + K_{I=2}^{(q^2)}\left(\frac{q^2}{M_\pi}\right)^2 + \dots\right]$$

Limits of U(N_f) ChPT

$M^2_{\eta'} \to M^2_\pi$ limit:

$$M_\pi a_0^{I=2} = -\frac{M_\pi^2}{16\pi F_\pi^2}\left[1 - \frac{16 M_\pi^2}{F_\pi^2} L_{I=2} + N_c^2 K_{I=2}^{(0)} \left(\frac{M_\pi^2}{F_\pi^2}\right)^2 + \frac{M_\pi^2}{8F_\pi^2\pi^2}\log\frac{M_\pi^2}{\mu^2}\right]$$

$$M_\pi a_0^{AA} = \frac{M_\pi^2}{16\pi F_\pi^2}\left[1 - \frac{16 M_\pi^2}{F_\pi^2} L_{AA} + N_c^2 K_{AA}^{(0)} \left(\frac{M_\pi^2}{F_\pi^2}\right)^2 - \frac{M_\pi^2}{8F_\pi^2\pi^2}\log\frac{M_\pi^2}{\mu^2}\right]$$

$M^2_{\eta'} \gg M^2_\pi$ to match SU(N_f) and U(N_f) LECs:

$$\left[L_{I=2}^{(1)}\right]_{SU(N_f)} = \left[L_{I=2}^{(1)}\right]_{U(N_f)} - \frac{1}{8N_f^2(4\pi)^2}(1 + N_f\lambda_0 - \lambda_0)$$

$$\left[L_{AA}^{(1)}\right]_{SU(N_f)} = \left[L_{AA}^{(1)}\right]_{U(N_f)} + \frac{1}{8N_f^2(4\pi)^2}(1 - N_f\lambda_0 - \lambda_0)$$

$$\lambda_0 = \log\frac{M^2_{\eta'} - M^2_\pi}{\mu^2}$$

First step of continuum extrapolation

Extrapolation to $k/M_\pi^2 = -0.08$ using Effective Range Expansion with $M_\pi^2 r_0 a_0 \in [-5, -1]$

LO ChPT:
$M_\pi^2 r_0 a_0 = -3$

$$\text{ERE: } \frac{k}{M_\pi} \cot \delta_0 = \frac{1}{M_\pi a_0} \left[1 + \frac{1}{2} (M_\pi^2 r_0 a_0) \left(\frac{k^2}{M_\pi^2} \right) \right]$$