Decay amplitudes to three hadrons from lattice QCD

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In collaboration with: M. T. Hansen and S. R. Sharpe

Decay amplitudes to three hadrons from finite-volume matrix elements

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Disclaimer

This talk is based on the RFT three-particle formalism [Hansen, Sharpe]
Other approaches are FVU [Döring, Mai] and NREFT [Hammer, Pang, Rusetsky]

Analytic Expansions of Two- and Three-Particle Excited-State Energies	*Dr Dorota Maria Grabowska*	19:15 - 19:30
Three-particle quantization condition for nondegenerate particles	*Stephen Sharpe*	19:30 - 19:45
Three-hadron s- and d-wave interactions from lattice QCD	*Dr Andrew Hanlon*	19:45 - 20:00
Parameters of the a1(1260) resonance from lattice QCD	*Maxim Mai*	20:00 - 20:15
Three pion interactions from the lattice	*Ruairí Brett*	20:15 - 20:30
Three-particle finite-volume formalism for $\pi^+ \pi^+ K^+$ and related systems	*Tyler Blanton*	20:30 - 20:45
Infinite volume, three-body scattering formalisms in the presence of bound states	*Sebastian Dawid*	20:45 - 21:00

Introduction

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- We have a (relativistic) three-particle formalism for three-particle scattering [Hansen, Sharpe]
  - Recent extensions: nonidentical pions, nondegenerate particles. [Hansen, FRL, Sharpe] + [Blanton, Sharpe]
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- In the two-particle formalism this is the well-known Lellouch-Lüscher formalism
  - It has been applied successfully in $K \to \pi\pi$ processes [RBC/UKQCD, see plenary by C. Kelly]

Recap of the three-body formalism

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- **Finite-volume correlator:**

$$C_L^{\mathsf{M}}(E,\boldsymbol{P}) = \int_{-\infty}^{\infty} dx_0 \int_L d^3x \, e^{i(Ex^0 - \boldsymbol{P}\cdot\boldsymbol{x})} \langle 0|\mathrm{T}\sigma(x)\sigma^\dagger(0)|0\rangle \, ,$$

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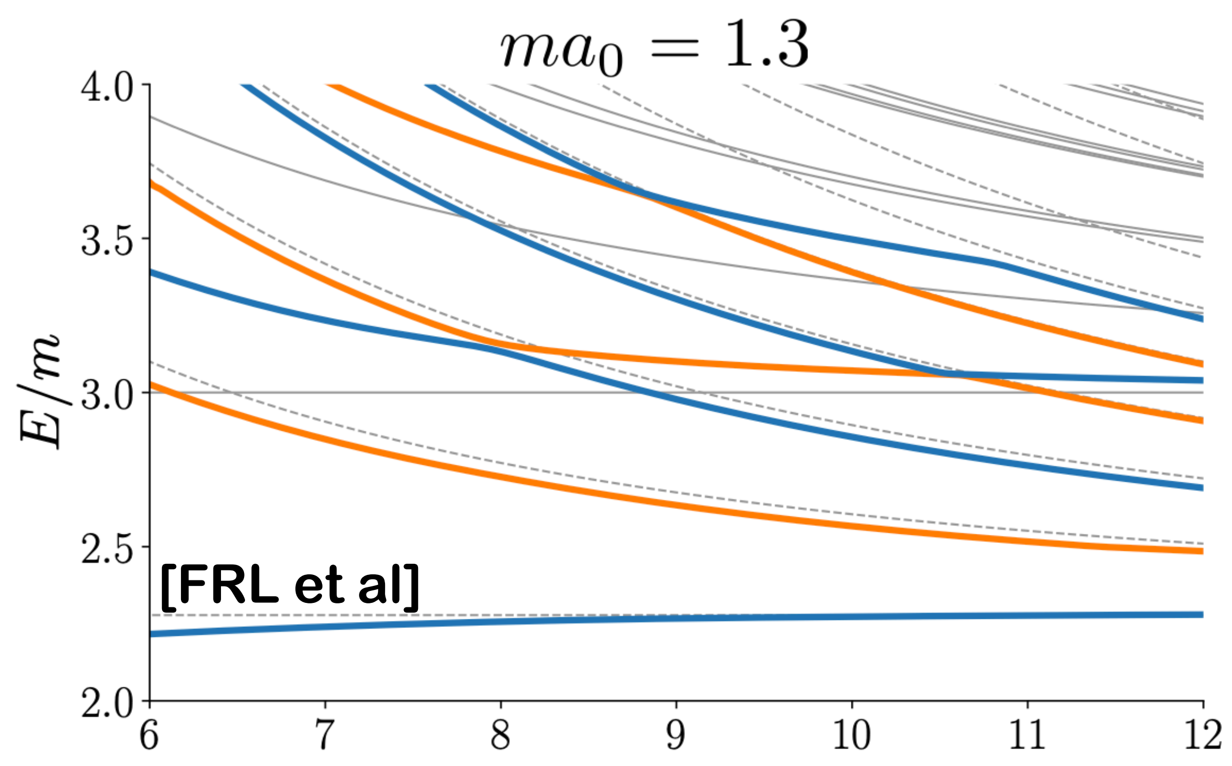

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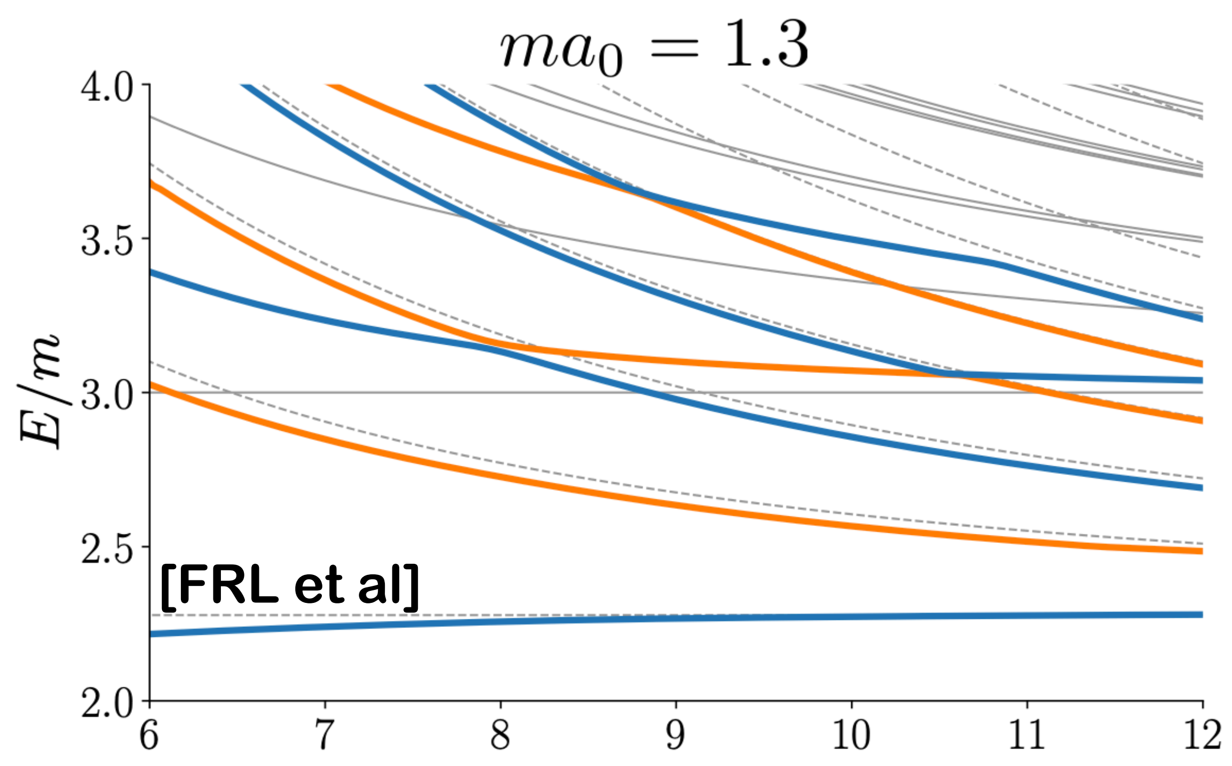
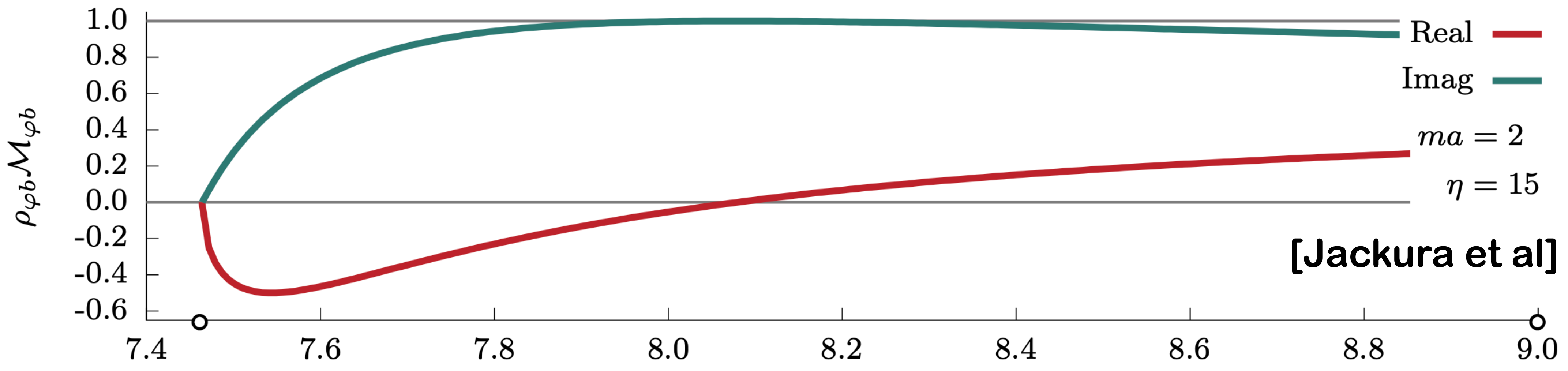


- **Step 2: three-particle scattering amplitude:**

$$\mathcal{M}_{3,L}^{(u,u)} = \mathcal{D}^{(u,u)} + \mathcal{L}_L^{(u)}\frac{1}{1+\mathcal{K}_{\mathrm{df},3}F_3}\mathcal{K}_{\mathrm{df},3}\mathcal{R}_L^{(u)}\,,$$



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- Tune the box size such that: $E_{3\phi}(L) \simeq M_K$

Formalism for identical scalars

- From the lattice, one can get the one-to-three finite-volume matrix element:

$$\langle E_n, \boldsymbol{P}, \Lambda\mu, L|\mathcal{H}_W(0)|K, \boldsymbol{P}, L\rangle\,.$$

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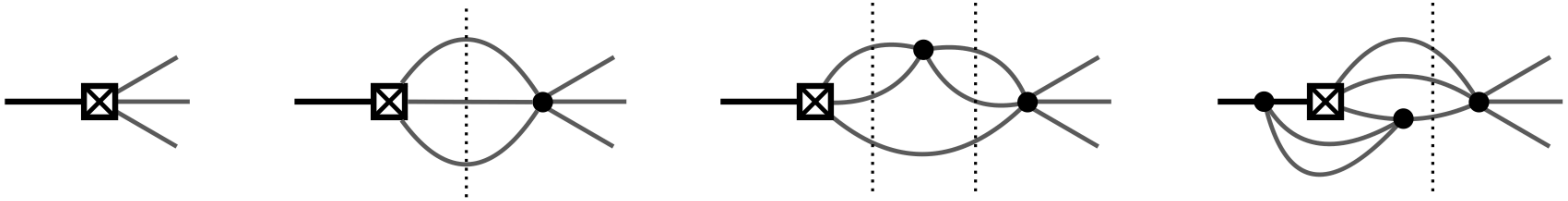

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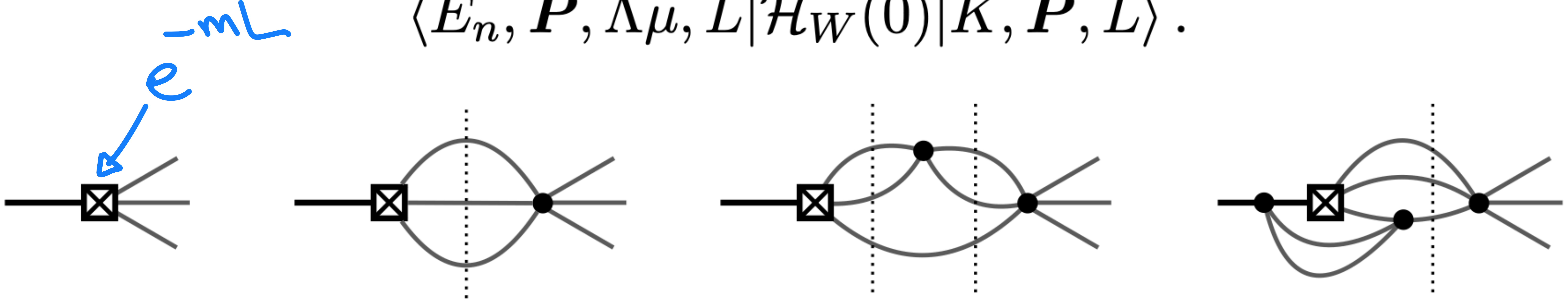

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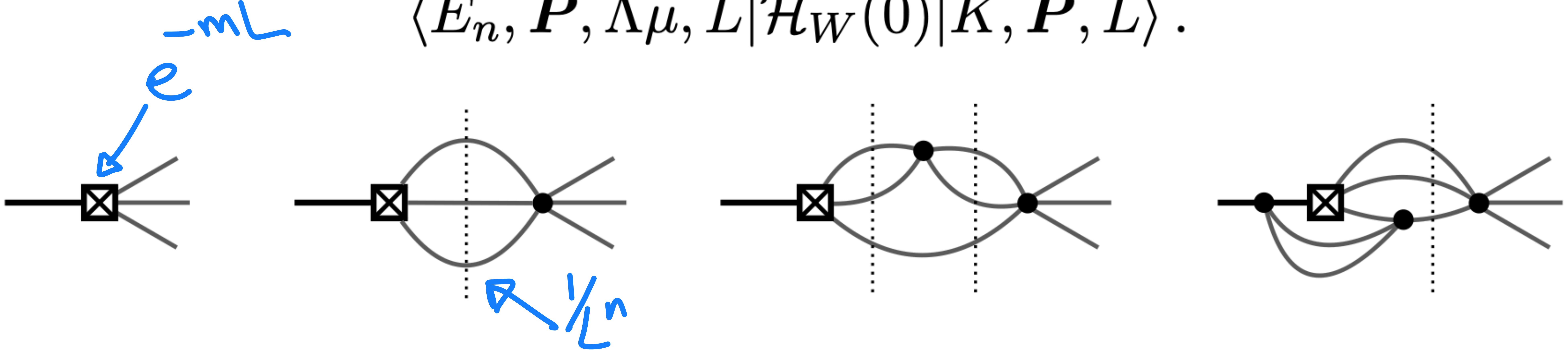

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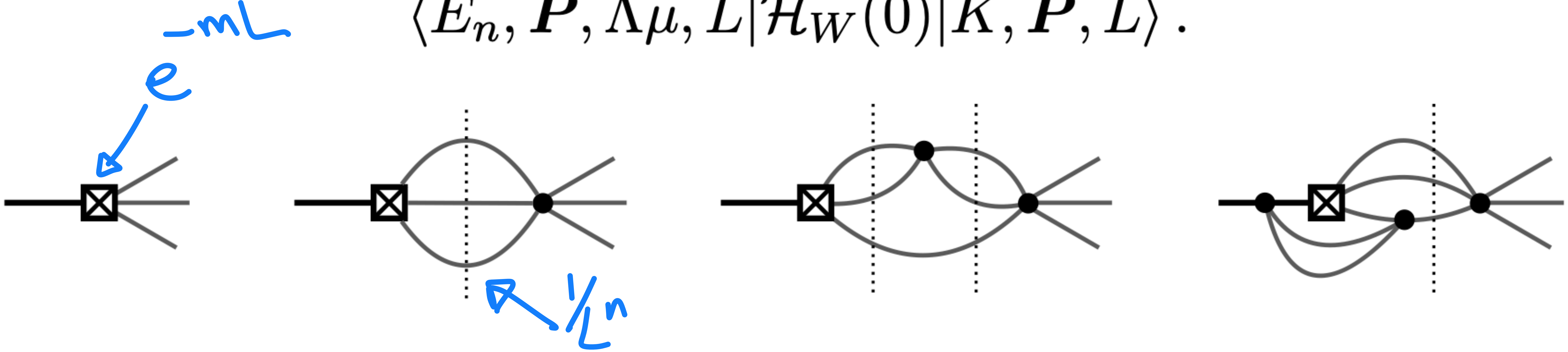

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$$T_{K3\pi} = \langle 3\pi, \mathrm{out} | \, \mathcal{H}_W(0) \, | K, \boldsymbol{P} \rangle \ ,$$

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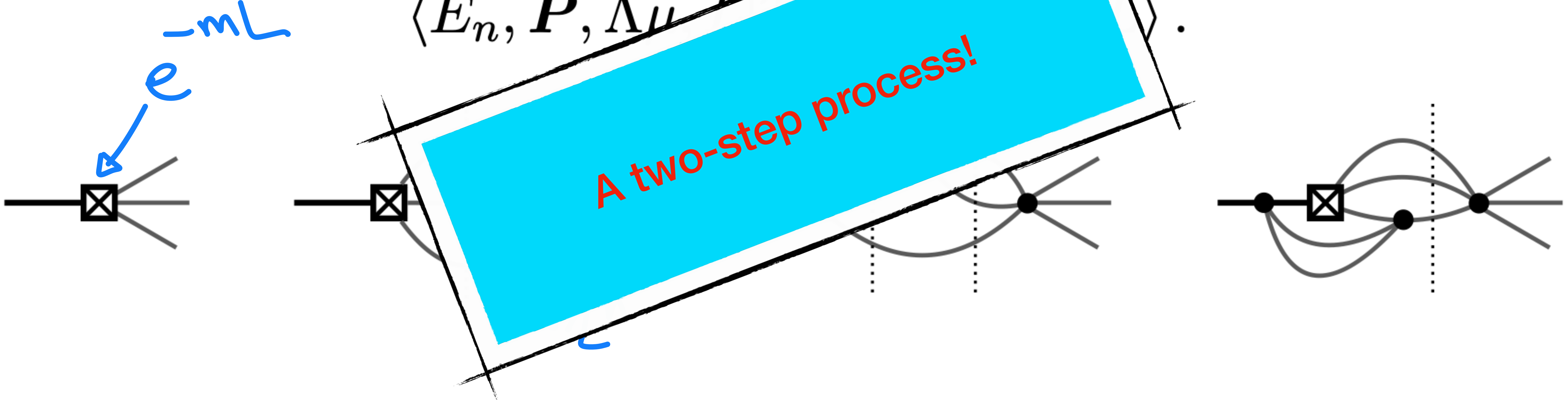

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- Use the measured matrix element to compute an intermediate quantity

$$\sqrt{2E_K(\boldsymbol{P})} L^3 \langle E_n, \boldsymbol{P}, \Lambda\mu, L|\mathcal{H}_W(0)|K, \boldsymbol{P}, L\rangle = v^\dagger A^{\mathrm{PV}}_{K3\pi}\,.$$

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- $A^{\text{PV}}_{K3\pi}$ **is however scheme dependent!** ⟶ need step 2!

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$$A^{\mathrm{PV}}_{K3\pi} = A^{\mathsf{iso}} + A^{(2)}\sum_i \Delta_i^2 + A^{(3)}\sum_i \Delta_i^3 + A^{(4)}\sum_i \Delta_i^4 + \mathcal{O}(\Delta^5)\,.$$

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- Need as many matrix elements as parameters!

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Similar to integral equations in three-to-three scattering! [Jackura et al, Hansen et al.]

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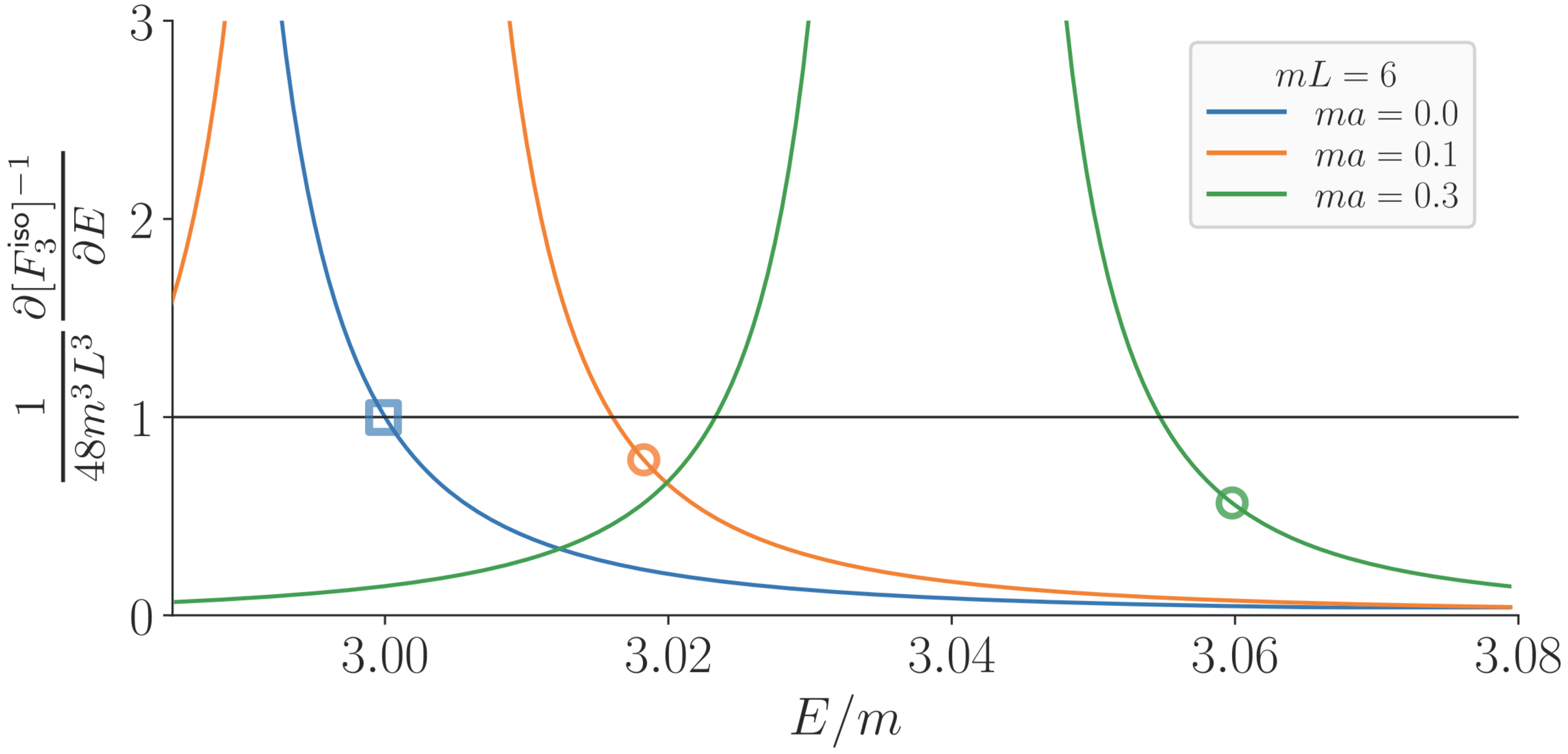

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Similar approximation to [Müller, Rusetsky]

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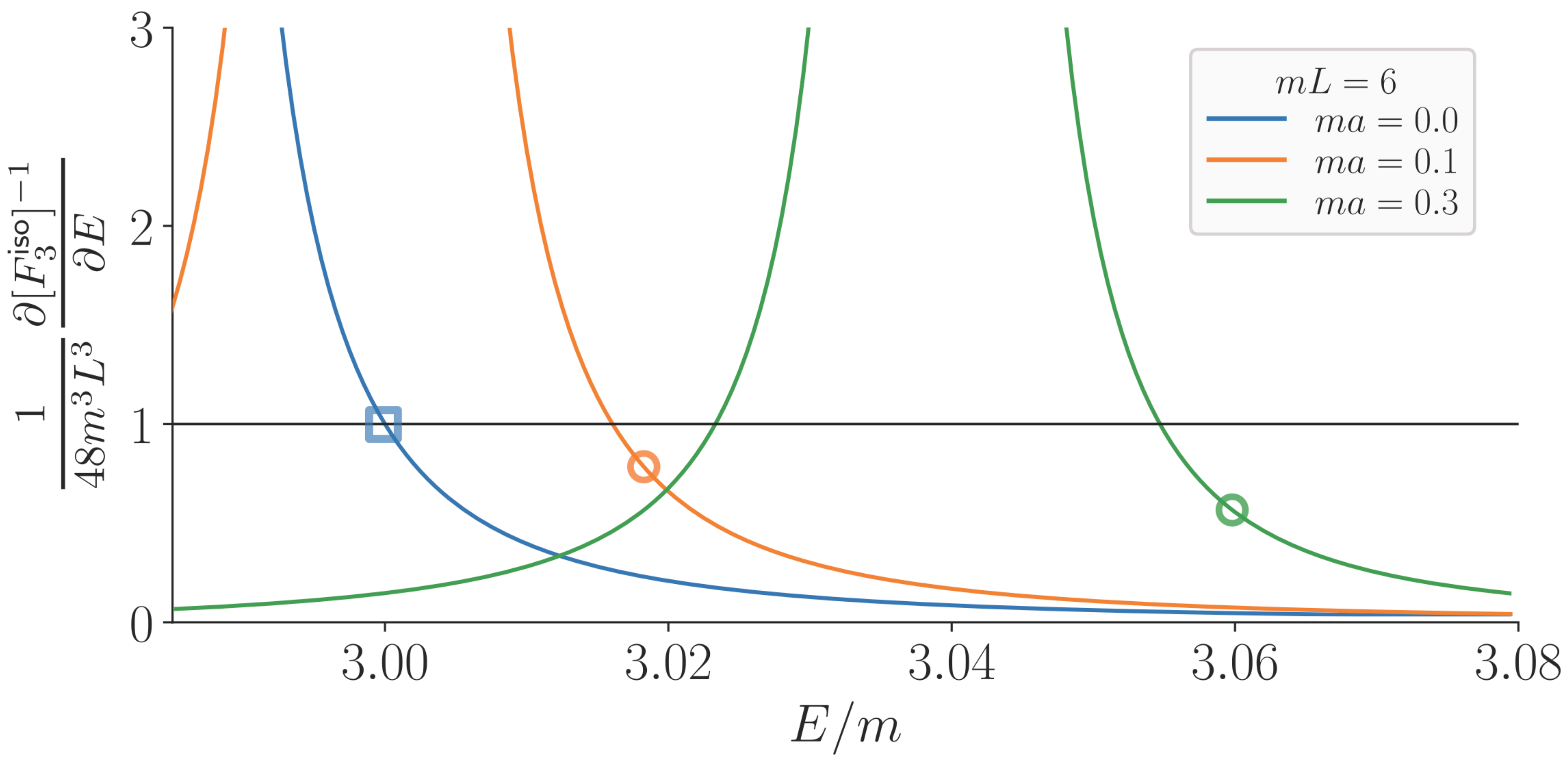

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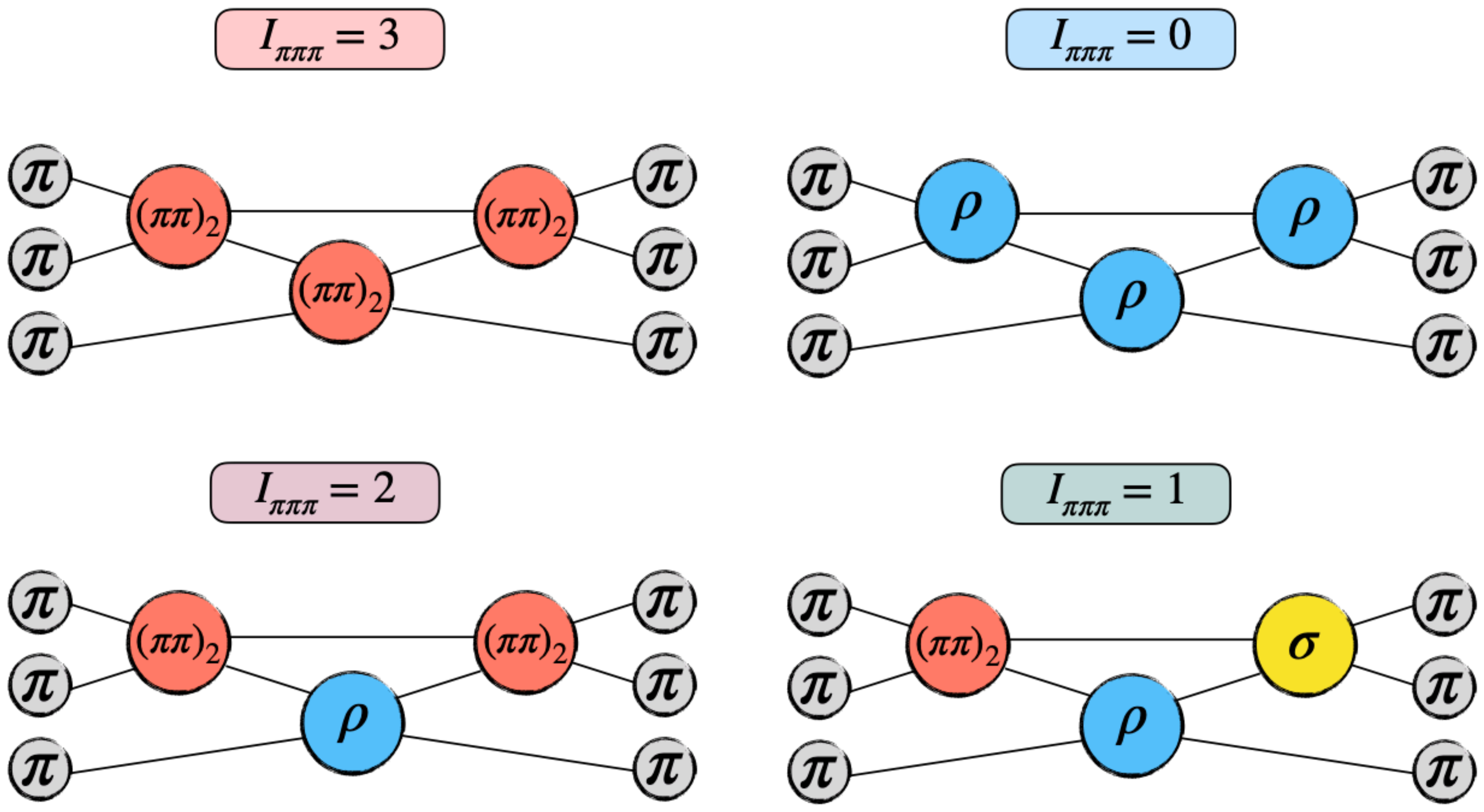

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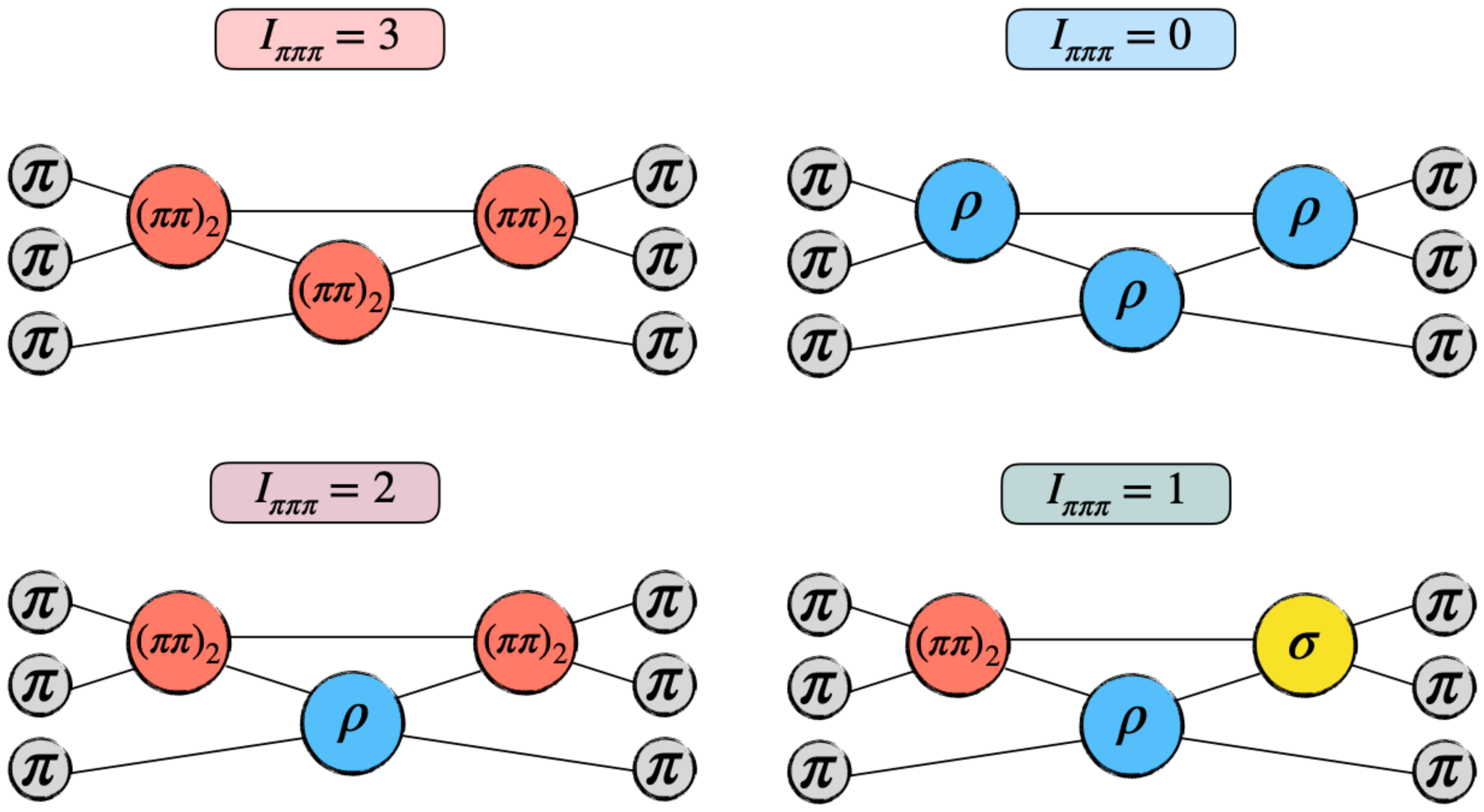

$$\det\left[1 - \mathbf{K}_{\mathrm{df},3}^{[I]}(E^\star)\,\mathbf{F}_3^{[I]}(E,\boldsymbol{P},L)\right] = 0$$

Three-pion decays

- Expressions for decays are formally very similar with additional flavor index:

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$$\sqrt{2E_K(\boldsymbol{P})}L^3\langle E_n^{\Lambda,[I]}, \boldsymbol{P}, I, I_3, \Lambda\mu, L|\mathcal{H}_W(0)|K, \boldsymbol{P}, L\rangle = \mathbf{v}^\dagger \mathbf{A}_{K3\pi}^{\mathrm{PV},[I]} \,.$$

$$\mathbf{R}_{\Lambda\mu}^{[I,I_3]}(E_n^{\Lambda,[I]}, \boldsymbol{P}, L) = \lim_{P_4 \to iE_n^{\Lambda,[I]}} -(E_n^{\Lambda,[I]} + iP_4)\mathbb{P}_{\Lambda\mu}^{[I,I_3]} \frac{(-i)}{1/\mathbf{F}_3^{[I]} - \mathbf{K}_{\mathrm{df},3}^{[I]}} \mathbb{P}_{\Lambda\mu}^{[I,I_3]} \equiv \mathbf{v}\,\mathbf{v}^\dagger \,.$$

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Isospin = 0,1,2	Isospin = 0	Isospin 1
$K\to\pi\pi\pi$	$\gamma^*\to\pi\pi\pi$	$\eta\to\pi\pi\pi$
CP violation	related to g-2	isospin-breaking effect

Summary and Outlook

- We have derived a formalism for three-particle decays.
- Formalism for identical particles similar to [Müller, Rusetsky]
- Extension to generic three-pion systems based on previous three-pion quantization condition
- Explicit expressions to treat $K \to \pi\pi\pi, \quad \gamma^* \to \pi\pi\pi, \quad \eta \to \pi\pi\pi\,.$
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Thanks!