

# Three pion interactions from the lattice

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Lattice 2021

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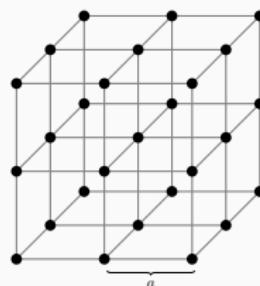


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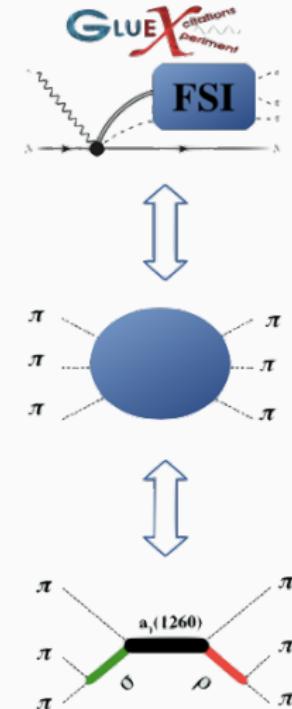


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# LIGHT HADRON SPECTRUM

- two-body scattering on the lattice at very mature stage
- two-body treatment of many states only sufficient for “special cases” → heavy  $m_\pi$ , etc.
- many unsolved three-body problems:
  - $N(1440) \ 1/2^+$  (**Roper**)
    - mass pattern doesn't match quark model
    - large BR to  $N\pi\pi$
  - $a_1(1260)$ 
    - decays to  $\rho\pi, \sigma\pi \rightarrow \pi\pi\pi$
    - resonant intermediate  $\rho, \sigma$
    - see Maxim's talk before this one

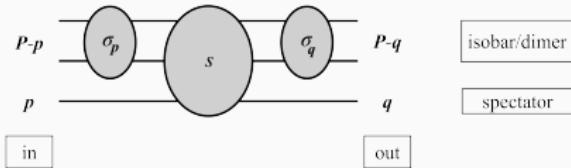


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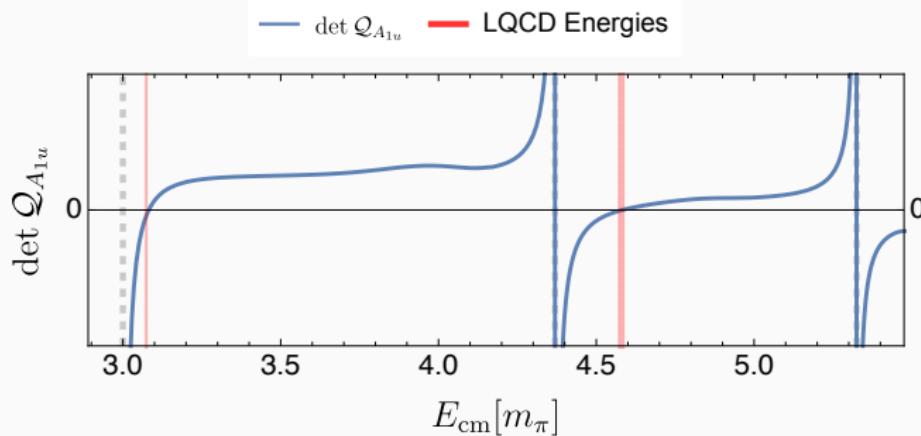
# THREE-BODY QUANTIZATION CONDITION (FVU)

finite-vol. energies  $\sqrt{s}$  satisfy:

determinant over spectator momenta  $p, q$ , angular momentum, etc.



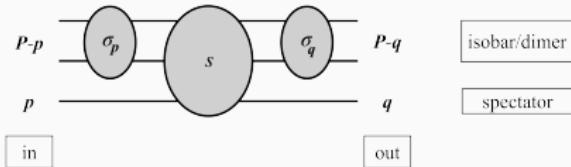
$$\det \mathcal{Q}_{\Lambda\mu} \equiv \det \left[ B(s) + C(s) - E_{L\eta} \left( \tilde{K}_2^{-1}(s) - \Sigma_2^L(s) \right) \right] = 0.$$



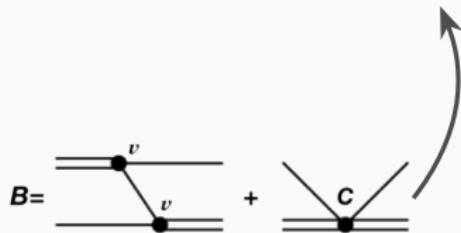
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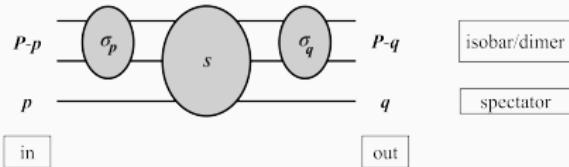
$B$ : one particle exchange

$C$ : isobar-spectator interaction

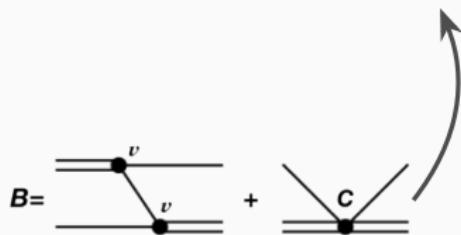
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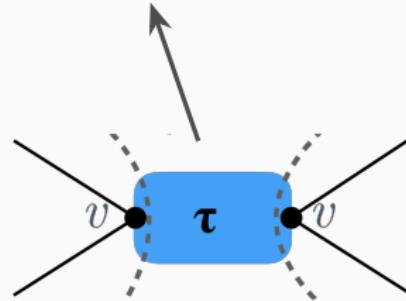
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2-body input  
 $(K$ -matrix, effective range, etc.)

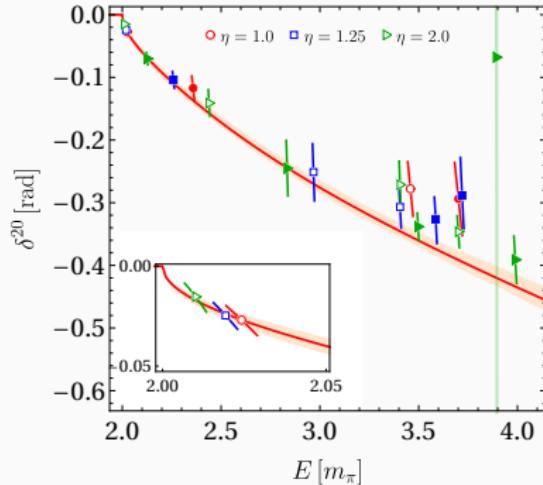
## GWU LATTICE DETAILS

- cubic ( $\eta = 1$ ) and lattices elongated in the  $z$ -direction ( $L_z = \eta L_{x,y}$ )
- $N_f = 2$  nHYP-smeared (clover) Wilson fermions at  
 $m_\pi \approx 315$  MeV ( $\eta = 1.0, 1.25, 2.0$ ) and  
 $m_\pi \approx 220$  MeV ( $\eta = 1.0, 1.17, 1.33$ )
- two-flavor simulations offer insight into importance of strange quark
  - e.g. strange quark content of  $\rho$  resonance: **1605.03993**
  - $3K^-$  : Alexandru et. al. **2009.12358**
- distillation/LapH for all quark propagation
- elongations help to sample scattering region (like moving frames),  
*much* cheaper than increasing cubic volume

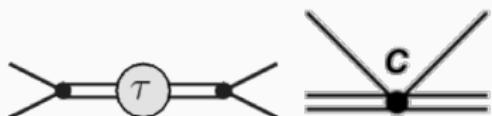
# $I = 3, 3\pi^+$ SCATTERING

$\pi^+\pi^+\pi^+$  scattering:

- simplest three-body channel to test formalism(s)
- two-body sub-channel:  $I = 2 \pi^+\pi^+$  (repulsive, no resonances)
- inputs to  $\infty$ -volume amplitude:
  - $I = 2 \pi\pi$  amplitude
  - three-body isobar-spectator interaction  $C$



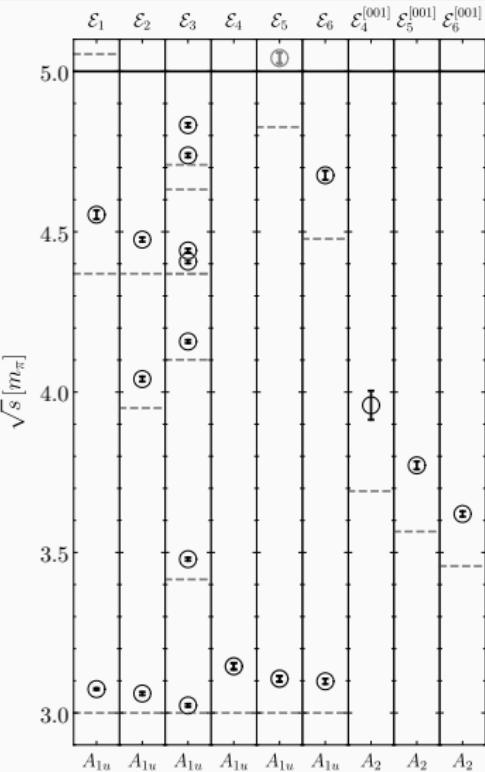
Mai et. al. 1908.01847



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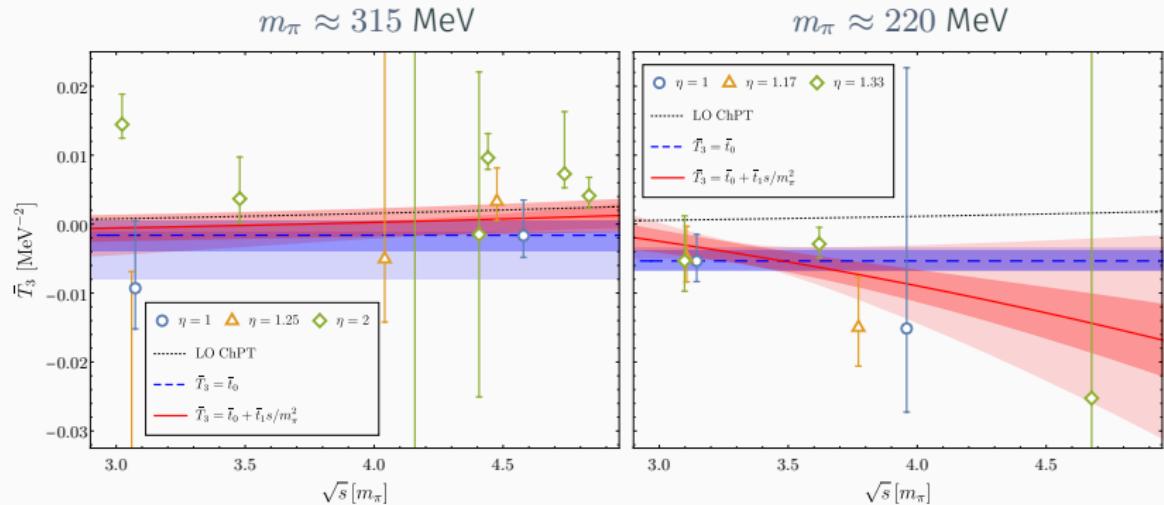
$\pi^+ \pi^+ \pi^+$  scattering:

- only  $3\pi^+$  operators ( $I = 3!$ ) in GEVP
- high precision ( $\sim 0.1\%$  statistical errors)
- time-dep. finite  $T$  wraparound important
- (non  $s$ -wave irreps not shown)



Culver et. al. 1911.09047, RB et. al. 2101.06144

# $3\pi^+$ CONTACT INTERACTION



real, three-body contact term:

$$\tilde{T}_3 = \frac{3}{2} \left( \frac{K^{-1}}{32\pi} \right)^{-1} \frac{C_0}{1 - C_0 E_{L\eta}^{-1} \left( \frac{K^{-1}}{32\pi} \right)^{-1}} \left( \frac{K^{-1}}{32\pi} \right)^{-1}$$

# COMPARING TO (SOME) RFT RESULTS

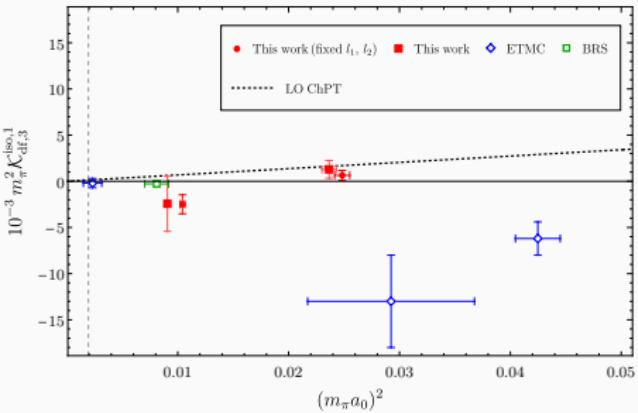
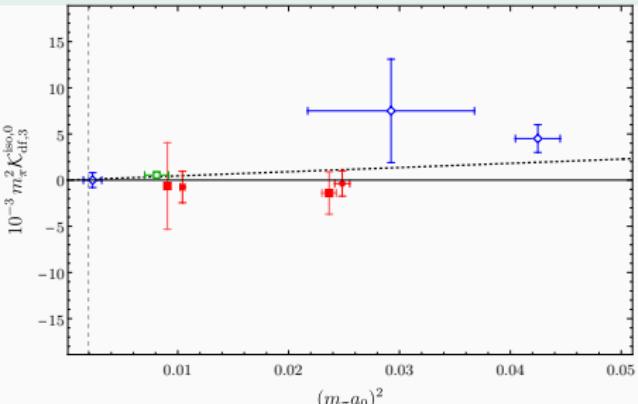
matching RFT and FVU at the level of the three-body amplitudes leads to\*

$$\mathcal{K}_{\text{df},3}^{\text{iso},0} \simeq 6(\bar{t}_0 + 9\bar{t}_1)$$

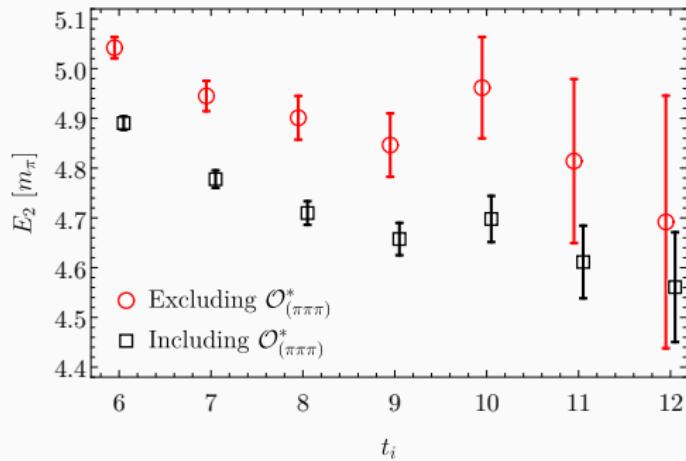
$$\mathcal{K}_{\text{df},3}^{\text{iso},1} \simeq 54\bar{t}_1$$

where the RFT 3-body term is

$$\mathcal{K}_{3,\text{df}} = \mathcal{K}_{\text{df},3}^{\text{iso},0} + \mathcal{K}_{\text{df},3}^{\text{iso},1} \left( \frac{s - 9m_\pi^2}{9m_\pi^2} \right)$$



## MORE COMPLICATED SYSTEMS: $a_1(1260)$



$$a_1(1260) \rightarrow \pi\rho, \pi\sigma \rightarrow 3\pi$$

cf.  $NN$  scattering: Tues talks from Green, Madanagopalan,  
Nicholson, Wagman

Mai et. al. 2107.03973

- much larger operator basis:  
 $\bar{q}q, \rho\pi, \sigma\pi, \pi\pi\pi$
- $\mathcal{O}_{(\pi\pi\pi)}^* \sim \pi(\mathbf{p})\pi(-\mathbf{p})\pi(0)$
- $\mathbf{p} = (0, 1, 1)$
- NI energy  $\approx 6.73m_\pi$
- comprehensive basis **crucial** to reliable spectrum determination

## MORE COMPLICATED SYSTEMS: $a_1(1260)$

Geometry	$\mathbf{P}$	$\Lambda$	$J^P (I^G = 1^-)$
Cubic	$\mathbf{P} = (0, 0, 0)$	$T_{1g}$	$1^+, 3^+, \dots$
$z$ -Elongated	$\mathbf{P} = (0, 0, 0)$	$A_{2g}$	$1^+, 3^+, \dots$
Cubic & $z$ -Elongated	$\mathbf{P} = (0, 0, n)$	$E_g$	$1^+, 2^+, 3^+, \dots$
		$A_2$	$0^-, 1^+, 2^-, 3^+, 4^-, \dots$
		$E$	$1^\pm, 2^\pm, 3^\pm, 4^\pm, \dots$
	$\vdots$	$\vdots$	$\vdots$

- $a_1(1260)$ :  $I^G(J^P) = 1^-(1^+)$
- partial wave mixing very severe in boosted frames
- e.g. at  $m_\pi L \approx 3.3$ , only 2 states below  $5m_\pi$  in cubic box (at rest)
- elongated boxes (i.e. multiple geometries) to the rescue!

Mai et. al. 2107.03973

## SUMMARY

- three-body (meson) spectra now realistically accessible from the lattice
- (unsurprisingly) more levels needed to constrain amplitudes than in 2-body scattering
- spectrum involving (coupled) resonant subchannels/isobars accessible
  - e.g.  $a_1(1260) \rightarrow \pi\rho, \pi\sigma \rightarrow 3\pi$
- connecting to  $\infty$ -volume amplitudes/poles much more complex than for 2-body
  - resonance poles accessible (Mai et. al. 2107.03973)
  - more ingredients than just 3-body spectrum needed for Dalitz plots (e.g. decay matrix elements: see talks by Müller and Romero-López tomorrow)

door is open now for more 3-body resonances to be studied over the next few years



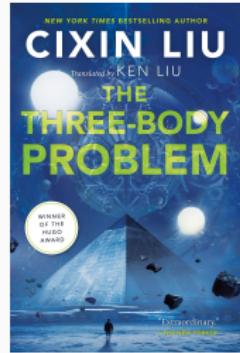
# THREE-BODY AMPLITUDES

- three-body amplitudes difficult even in infinite-volume
  - 8 kinematic variables
  - sub-channel dynamics
- three leading approaches:

**RFT** (Hansen, Sharpe, Blanton, Briceño, Romero-Lopez)  
relativistic, diagrammatic approach

**FVU** (Mai, Döring)  
relativistic, from unitarity

**NREFT** (Rusetsky, Peng et. al.)  
non-relativistic EFT

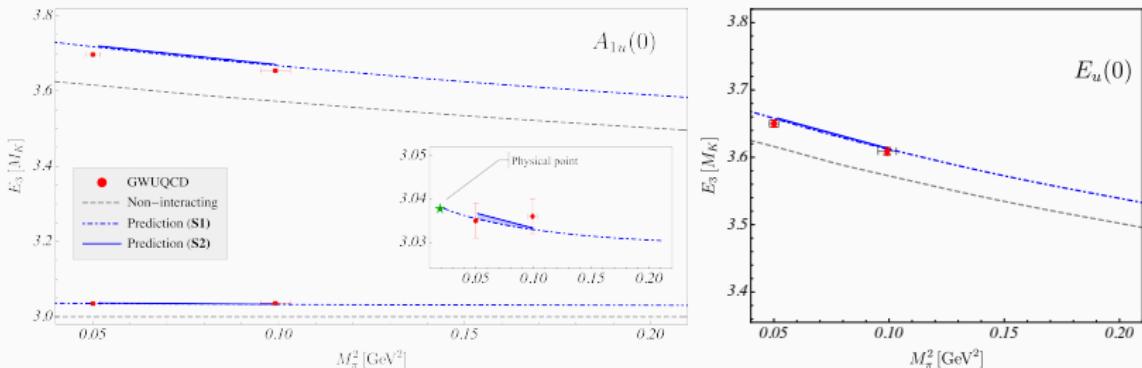


Reviews: Hansen, Sharpe 1901.00483, Mai, Döring, Rusetsky 2103.00577

Comparison of RFT & FVU amplitudes: Jackura et. al. 1905.12007

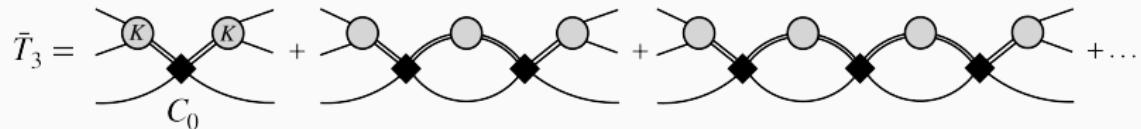
# WARMING UP: PREDICTING $3K^-$

- NLO ChPT for 2-body interactions:
  - S1:  $f_\pi$  extrapolated using physical point data and NLO chiral expressions
  - S2: meson decay constants determined on the lattice
- vanishing isobar-spectator interaction ( $C = 0$ )
- decent qualitative agreement\*: rough  $m_\pi$  dependence to  $\approx 315$  MeV
- no fitting of 2- or 3-body parameters, haven't extracted any  $\infty$ -vol physics yet



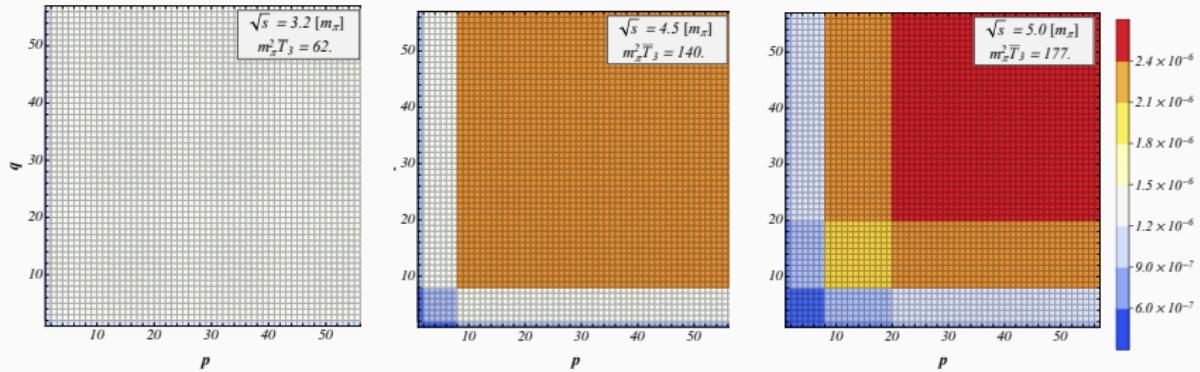
\*quenched strange quark

# THREE-BODY CONTACT INTERACTION



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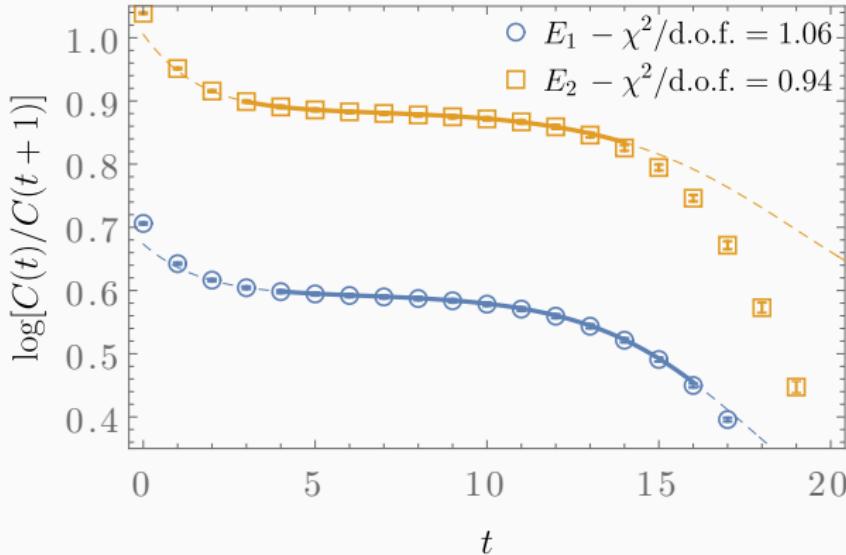
$C_0$  for fixed  $\bar{T}_3$ : (isotropic  $\bar{T}_3 \leftrightarrow$  anisotropic  $C$  and vice versa)



## LATTICE DETAILS

	$N_t \times N_{x,y}^2 \times N_z$	$\eta$	$a[\text{fm}]$	$N_{\text{cfg}}$	$am_\pi$
$\mathcal{E}_1$	$48 \times 24^2 \times 24$	1.00	0.1210(2)(24)	300	0.1931(4)
$\mathcal{E}_2$	$48 \times 24^2 \times 30$	1.25	—	—	0.1944(3)
$\mathcal{E}_3$	$48 \times 24^2 \times 48$	2.00	—	—	0.1932(3)
$\mathcal{E}_4$	$64 \times 24^2 \times 24$	1.00	0.1215(3)(24)	400	0.1378(6)
$\mathcal{E}_5$	$64 \times 24^2 \times 28$	1.17	—	—	0.1374(5)
$\mathcal{E}_6$	$64 \times 24^2 \times 32$	1.33	—	—	0.1380(5)

# FITTING THERMAL STATES



fit ansatz:

$$a_1 e^{-m_1 t} + a_2 e^{-m_2 t} + a_3 e^{-\Delta E t}$$

$$aN_t = 5.80(1)(12) \text{ fm}$$

$n$	$a_1$	$m_1$	$a_2$	$m_2$	$a_3$	$\Delta E$
1	0.810(4)	0.594(1)	0.151(4)	1.27(4)	0.00029(3)	0.1587(7)
2	0.752(4)	0.884(3)	0.211(3)	1.62(2)	0.005(4)	0.50(6)